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Renormalization Group as a Probe for Dynamical Systems

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Abstract. The use of renormalization group (RG) in the analysis of nonlinear dynamical problems has been pioneered by Goldenfeld and co-workers [1]. We show that perturbative renormalization group theory of Chen *et al* can be used as an effective tool for asymptotic analysis for various nonlinear dynamical oscillators. Based on our studies [2] done on two-dimensional autonomous systems, as well as forced non-autonomous systems, we propose a unified methodology — that uses renormalization group theory — for finding out existence of periodic solutions in a plethora of nonlinear dynamical systems appearing across disciplines. The technique will be shown to have a non-trivial ability of classifying the solutions into limit cycles and periodic orbits surrounding a center. Moreover, the methodology has a definite advantage over linear stability analysis in analyzing centers.

1. Introduction

In the early 1990s Chen, Goldenfeld and Oono proposed a new method based on renormalization group (RG) [1, 3] to analyze nonlinear dynamical problems. Their method was able to extract the asymptotic behaviour of solutions to differential equations. In quantum field theory (QFT), it's quite well known that RG equations [4] have the peculiar capability of improving the "global" nature of functions obtained in the perturbation theory (which is essentially "local"). The RG flow equations may be interpreted as the following: any physical quantity $Q(\alpha, \beta, \mu)$ should have no dependence whatsoever, on the renormalization point μ which is arbitrarily chosen and is absent from the original Lagrangian of the problem, i.e.

$$\frac{d\mathcal{Q}}{d\mu} = 0. \tag{1}$$

The concept of such a floating renormalization point was first introduced in a classic paper by Gell-Mann and Low [5]. Chen et al relying heavily on analogies and ideas drawn from RG method for QFT, formulated their own brand of RG which can be used effectively to analyze dynamical systems. They showed how RG can predict the asymptotic behaviour of solutions to various nonlinear differential equations and in the process demonstrated that RG is equivalent to various singular perturbation techniques used in nonlinear dynamics like the multiple scale analysis, boundary layer technique, asymptotic matching, WKB method etc. However the chief advantage of RG over traditional perturbative techniques lies in the fact that it does not require any *ad hoc* assumptions, rather it simply uses a naive perturbative expansion. The mathematical reasons as to why the RG method of Chen et al works has been discussed by Kunihiro [6]. The study of nonlinear differential equations in a two dimensional dynamical system is of considerable interest to researchers across disciplines. And since RG method of Goldenfeld *et al* extracts the asymptotic behavior of solutions to differential equations, makes it particularly handy while analyzing periodic solutions to nonlinear dynamical problems which will be the focus of this study.

2. The Methodology

How does one apply the RG principle to a problem in dynamics? We begin by observing the fact that any periodic solution in 2-dimensions can be expressed as a Fourier series with amplitude A and phase of the lowest harmonic θ determining the amplitude and phase of the higher order ones. The amplitude and phase are the quantities that will feature in the renormalization *flow* equations. The renormalization procedure takes advantage of the fact that for a given phase path (see Fig. 1), the initial condition can be taken at any point, τ , along the path without affecting the final outcome at t i.e. x(t) remains the same and independent of τ . This condition ultimately leads to the RG flow equations. In this respect it is akin to the Bogoliubov-Krylov method [7]. But as mentioned earlier, the advantage is that RG uses a naive perturbation theory;



Figure 1. The initial condition can be at any point on the path

and we do not need to anticipate scales (as in multiple scales method) or make an assumption about slowly varying amplitudes and phases (Bogoliubov-Krylov).

Suppose we have an ordinary differential equation of the form: $\ddot{x} + \omega^2 x = \epsilon F(x, \dot{x})$. If one attempts to solve the equation using an expansion, $x(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$, it results in breakdown of the perturbation theory at times t such that $\epsilon(t - t_0) > \mathcal{O}(1)$ due to the presence of secular terms. Thus a naive expansion of the dynamical variable leads, in the asymptotic limit, to an unphysical answer. If t be the time at which we want to know x(t) and t_0 the initial time, then the perturbative answer x(t) diverges when $t - t_0 \to \infty$. This is completely similar to divergence in field theories where a physical quantity (e.g. two point correlation function) diverges as the renormalization cutoff $\Lambda \to \infty$. If we are dealing with physical variables, we must always have quantities that are finite and while this is achieved in field theory by constructing running coupling constants, it is done for the differential equation by introducing an arbitrary time scale τ and letting the amplitude and phase depend on τ . To regularize the perturbation

series, RG technique uses the arbitrary time τ to split $t - t_0$ as $(t - \tau) + (\tau - t_0)$ and absorbing the terms containing $\tau - t_0$ into the respective *renormalized* counterparts A and θ of A_0 and θ_0 . At the end of the process one arrives at the RG-flow-equations for A and θ :

$$\frac{dA}{d\tau} = f(A,\theta) \tag{2}$$

$$\frac{d\theta}{d\tau} = g(A,\theta) \tag{3}$$

For autonomous systems, f and g are generally function of A alone. We propose to use flow equations (2) and (3) to differentiate between oscillators which are of the *center* variety and *limit cycles*. The center type oscillation consists of a continuous family of closed orbits in phase space, each orbit being determined by its own initial condition. This implies that the amplitude A gets fixed, once the initial condition is set. This must lead to,

$$\frac{dA}{d\tau} = 0. \tag{4}$$

This statement is exact and is not tied to any perturbation theory argument. On the other hand for the limit cycle ,

$$\frac{dA}{d\tau} = f(A),\tag{5}$$

such that the flow equation (5) has a fixed point. The fixed point corresponding to where $f(A^*) = 0$ ($A^* \neq 0$), has to be stable for the limit cycle to be stable and A^* gives the radius of the limit cycle. And if $A^* = 0$ is a fixed point of equation (5), then we have either a focus or a node.

This simple prescription, though not proved rigorously, appeals to one's intuition when one notes that (i) A = 0 means the assumed periodic solution has zero amplitude and hence hints at an attractor (a node or a focus), (ii) $f^{(A)} = 0 \forall A \ge 0$ hints at a family of non-isolated periodic orbits surrounding the fixed point and so a center solution is implied, and (iii) vanishing of $dA/d\tau$ at $A = A^* \ne 0$ logically indicates that an isolated periodic orbit of amplitude A^* happens to be surrounding the fixed point.

The calculation of f(A) requires the use of perturbation theory. Application of perturbation theory is possible only if one can locate a linear center (the basic periodic state) about which to perturb. Locating a center can sometimes be straightforward, for example: $\dot{x}_1 = x_2$, $\dot{x}_2 = -\partial V/\partial x_1$, where V is a general anharmonic potential, $V = x_1^2/2 + \lambda_1 x_1^3/3 + \lambda_1 x_1^4/4$, where $(x_1, x_2) = (0, 0)$ is a linear center around which perturbation theory can be done; similarly for the Van der Pol oscillator $\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0$, the origin $(x = 0, \dot{x} = 0)$ is a center for $\epsilon = 0$. For the Lotka-Volterra population dynamics model — $\dot{x}_1 = x_1 - x_1 x_2$, $\dot{x}_2 = -x_2 + x_1 x_2$ the origin is a saddle but the other fixed point, (1, 1), is a center. In such cases shifting the origin to the center is the first step in the process of determining f(A).

A more complicated situation can arise in few cases such as the Belushov-Zhabotinsky reaction [8, 9] system (discussed later). In that case, a transfer of origin to the fixed point has to be followed by a proper setting of parameters to make the origin a center which is the starting point of our perturbative method. This raises the problem that the given dynamical system may not have a relevant parameter at all, e.g. the well known paradigm for a limit cycle,

$$\dot{z} = (1+i)z - \beta |z|^2 z, \tag{6}$$

where z = x + iy is the complex variable and $\beta > 0$. The only fixed point is the origin and it is an unstable focus for all β . However, we can overcome this difficulty by considering the more

general system

$$\dot{z} = (\alpha_1 + i\alpha_2)z - \beta |z|^2 z. \tag{7}$$

The origin is now a stable focus for $\alpha_1 < 0$, unstable focus for $\alpha_1 > 0$ and a center for $\alpha_1 = 0$. It is this center about which one can set up a perturbation theory. The perturbative determination of f(A) and g(A) thus involves the following steps:

- (i) Find all the fixed points of the system and identify the linear centers.
- (ii) If there are no linear centers, look to extend the parameter space and see if a linear center can be located as the parameters are changed.
- (iii) For every linear center, thus located, we need to check the existence of a limit cycle by perturbatively constructing f(A) and g(A).
- (iv) If linear center is absent even after extension of the parameter space this methodology can't handle the problem.

3. Examples

As a test case let us consider a general 2D-dynamical equation represented by,

$$\ddot{x} + x = -k\dot{x} - \epsilon f(x, \dot{x}) + F \cos \Omega t, \tag{8}$$

where $f(x, \dot{x})$ is a nonlinear function of x and \dot{x} . For simplicity we will first consider only the autonomous cases, i.e. where F = 0. We will deal with non-autonomous case later on.

3.1. Duffing Oscillator

In order to elucidate the RG methodology we take a fairly simple and well studied nonlinear oscillator, the Duffing oscillator $(f(x, \dot{x}) = x^3))$, as our starting point and do the RG calculation in detail. The equation of motion of this damped nonlinear oscillator is given by,

$$\ddot{x} + k\dot{x} + \omega^2 x + \lambda x^3 = 0. \tag{9}$$

We immediately see that a linear center exists for $k = \lambda = 0$ and around this limit we can do our perturbative calculation. We naively expand x as $x = x_0 + kx'_1 + \lambda x_1 + k^2x'_2 + \lambda^2x_2 + k\lambda x''_2 + \cdots$. Then substituting this into Eq.(9), we obtain the following equations at different orders of λ and k:

$$\ddot{x}_0 + \omega^2 x_0 = 0, (10)$$

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^3, \tag{11}$$

$$\ddot{x}_1' + \omega^2 x_1' = -\dot{x}_0. \tag{12}$$

We set the initial condition as $x(t = 0) = A_0$ and $\dot{x}(t = 0) = 0$. We then write the solution of Eq.(10) as $x_0 = A_0 \cos \omega t$, so that x_0 picks up the initial condition, $x_0(t = 0) = A_0$. Hence for the subsequent orders, the initial condition becomes $x_i(t = 0) = \dot{x}_i(t = 0) = 0$ for all $i \ge 1$. Using the zeroth order solution we rewrite Eq.(11) and Eq.(12) as:

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{A_0^3}{4} \left(\cos 3\omega t + 3\cos \omega t \right), \tag{13}$$

$$\ddot{x}_1' + \omega^2 x_1' = \omega A_0 \sin \omega t. \tag{14}$$

Keeping in mind the initial conditions, we can immediately write the solutions to the above equations,

$$x_1 = -\frac{3A_0^3}{8\omega} t \sin \omega t + \frac{A_0^3}{32\omega^2} (\cos 3\omega t - \cos \omega t),$$
(15)

$$x_1' = -\frac{A_0}{2}t\cos\omega t + \frac{A_0}{2\omega}\sin\omega t.$$
(16)

Upto this order, the displacement of the oscillator can be written as,

$$x(t) = A_0 \cos \omega t - \frac{3\lambda A_0^3}{8\omega} (t - \tau + \tau) \sin \omega t + \frac{\lambda A_0^3}{32\omega^2} (\cos 3\omega t - \cos \omega t) - \frac{kA_0}{2} (t - \tau + \tau) \cos \omega t + \frac{kA_0}{2\omega} \sin \omega t.$$
(17)

Here we have split the interval 0 to t as 0 to τ and τ to t in accordance with our prescription for renormalization. In order to remove the divergences, we introduce two renormalization constants, $\mathcal{Z}_1(0,\tau)$ and $\mathcal{Z}_2(0,\tau)$, in the following manner:

$$A_0 = A_0(t_0 = 0) = A(\tau)\mathcal{Z}_1(0,\tau), \tag{18}$$

$$0 = \theta_0(t_0 = 0) = \theta(\tau) + \mathcal{Z}_2(0, \tau).$$
(19)

The renormalization constants have the expansions,

$$\mathcal{Z}_1(0,\tau) = 1 + a_1\lambda + a_1'k + \cdots, \tag{20}$$

$$\mathcal{Z}_2(0,\tau) = b_1 \lambda + b_1' k + \cdots, \qquad (21)$$

so that the constants a_i and b_i can be chosen order by order to remove divergences arising at each order. One needs to note here that the presence of nonlinear perturbations means that A_0 and θ_0 do no longer remain constants of motion. We now absorb the $\tau - t_0 (= 0)$ containing terms into the renormalized counterparts A and θ of A_0 and θ_0 so that both are now functions of τ . We can now rewrite Eq.(17) as

$$x(t) = A\cos(\omega t + \theta) + (a_1\lambda + a'_1k)A\cos(\omega t + \theta) - (b_1\lambda + b'_1k)A\sin(\omega t + \theta) - \frac{3\lambda A^3}{8\omega}(t - \tau + \tau)\sin(\omega t + \theta) + \frac{\lambda A^3}{32\omega^2} \Big(\cos 3(\omega t + \theta) - \cos(\omega t + \theta)\Big) - \frac{kA}{2}(t - \tau + \tau)\cos(\omega t + \theta) + \frac{kA}{2\omega}\sin(\omega t + \theta).$$
(22)

Then choosing $a'_1 = kA\tau/2$, $a_1 = 0$, $b'_1 = 0$ and $b_1 = -3\lambda\tau/8\omega$, one gets rid of the $\tau - 0$ containing terms and we write Eq.(22) as

$$x(t,\tau) = A\cos(\omega t + \theta) - \frac{3\lambda A^3}{8\omega}(t-\tau)\sin(\omega t + \theta) + \frac{\lambda A^3}{32\omega^2} \Big(\cos 3(\omega t + \theta) - \cos(\omega t + \theta)\Big) - \frac{kA}{2}(t-\tau)\cos(\omega t + \theta) + \frac{kA}{2\omega}\sin(\omega t + \theta).$$
(23)

Since τ does not appear in the original problem it should not be in the final solution. Therefore we impose the condition that the solution has to be independent of τ i.e. $(\partial x/\partial \tau)_t = 0$, for any t and this yields (to the lowest order) the flow equations:

$$\frac{dA}{d\tau} = -\frac{kA}{2},\tag{24}$$

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$$\frac{d\theta}{d\tau} = \frac{3\lambda A^2}{8\omega},\tag{25}$$

which integrate to $A(t) = A_0 e^{-k\tau/2}$ and $\theta = \theta_0 + \frac{3\lambda A^2}{8\omega}\tau$. Finally we set $\tau = t$ to get rid of remaining τ -dependence, to get

$$x(t) = A\cos\left[\Omega t + \theta_0\right] + \frac{\lambda A^3}{32\Omega^2} \left(\cos 3(\Omega t + \theta_0) - \cos(\Omega t + \theta_0)\right) + \frac{kA}{2\Omega}\sin(\Omega t + \theta_0), \quad (26)$$

where $\Omega = \omega + 3\lambda A^2/8\omega$.

When there is no damping term, i.e. k = 0, we have the conservative anharmonic oscillator,

$$\ddot{x} + \omega^2 x + \lambda x^3 = 0, \tag{27}$$

for which the fixed point (0,0) in the $x - \dot{x}$ plane (*i.e.* x - y plane) is a center. And as expected the flow equation in this case reduces to,

$$\frac{dA}{d\tau} = 0, \tag{28}$$

meaning we have a center. The periodic solution correct to first order in λ is given by

$$x(t) = A\cos\Omega t + \frac{\lambda A^3}{32\Omega^2} \left[\cos 3\Omega t - \cos\Omega t\right] + \mathcal{O}(\lambda^2) \qquad , \tag{29}$$

where Ω is the corrected frequency given by the expression:

$$\Omega = \omega + \frac{3\lambda A^2}{8\omega} + \mathcal{O}(\lambda^2).$$
(30)

The standard results [10] for the oscillator have, thus, been correctly captured and we find that the emergence of $x = \dot{x} = 0$ as a center is confirmed by the fact that $dA/d\tau = 0$.

3.2. Van der Pol Oscillator

As our second example we turn to the well known paradigm for limit cycle — the Van der Pol oscillator i.e. $f(x, \dot{x}) = \epsilon \dot{x} (x^2 - 1)$ and F = 0 in Eq. (8). Thus the equation of motion is,

$$\ddot{x} + \epsilon \dot{x} \left(x^2 - 1 \right) + \omega^2 x = 0. \tag{31}$$

We look at it as a second order dynamical system: $\dot{x} = y$, $\dot{y} = -\epsilon y(x^2 - 1) - \omega^2 x$, in order to identify the fixed points. Clearly the only fixed point of the system is at the origin which is a stable focus for $\epsilon < 0$ and unstable focus for $\epsilon > 0$. For $\epsilon = 0$ Eq. (31) is simply the well known equation for a simple harmonic oscillator and thus we have located a center about which we can base the perturbation expansion around. Expanding x as $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$, we obtain at different orders of ϵ ,

$$\ddot{x}_0 + \omega^2 x_0 = 0, (32)$$

$$\ddot{x}_1 + \omega^2 x_1 = -\dot{x}_0 (x_0^2 - 1).$$
(33)

We work with initial condition $x = A_0$ at t = 0 and $\dot{x} = 0$ at t = 0. Proceeding in an identical manner as we did for the Duffing oscillator we can arrive at the following flow equations:

$$\frac{dA}{d\tau} = \frac{\epsilon A}{2} \left(1 - \frac{A^2}{4} \right) + \mathcal{O}(\epsilon^2), \qquad (34)$$

$$\frac{d\theta}{d\tau} = 0 + \mathcal{O}(\epsilon^2). \tag{35}$$

The flow equations obtained are in accordance with what our prescription suggests has to be the structure of the RG flow in case of a limit cycle solution. One can immediately see that the flow has a stable fixed point at $A^2 = 4$ which in fact is the usual answer for Van der Pol limit cycle for small ϵ . The solution up to this order of perturbation is given by

$$x(t) = A\cos(\omega t + \theta) - \frac{\epsilon A^3}{96\omega^2} \left(\sin 3(\omega t + \theta) - 3\sin(\omega t + \theta)\right) + \mathcal{O}(\epsilon^2).$$
(36)

Its well known that in order to obtain the correct results for the Van der Pol oscillator one must anticipate the hidden multiple scale in the problem and include it in the perturbation expansion. But RG starts with a simple expansion and the hidden multiple scales are automatically identified. To illustrate this point we need to calculate the flow equations up to the next higher order. The resultant flow equation turns out to be,

$$\frac{dA}{d\tau} = \frac{\epsilon A}{2} \left(1 - \frac{A^2}{4} \right) + \mathcal{O}(\epsilon^3), \tag{37}$$

$$\frac{d\theta}{d\tau} = -\frac{\epsilon^2}{8} \left(1 - \frac{A^4}{32}\right) + \mathcal{O}(\epsilon^3).$$
(38)

One can immediately see that the usual multiple scales $T_1 = \epsilon t$, $T_2 = \epsilon^2 t$ and so on, used in the analysis of Van der Pol oscillator appear naturally in the flow equations.

3.3. Belushov-Zhabotinsky reaction

An similar approach is effective for the so called Belushov-Zhabotinsky reaction system (chlorine dioxide-iodine-malonic acid reaction). A relatively recent two variable model [8, 9] based on the fact that the concentrations of the intermediaries $I^{-}(x)$ and $ClO_{2}^{-}(y)$ vary on a much faster time scale than ClO_{2} , I_{2} and *Malonic acid* is given by,

$$\dot{x} = a - x - \frac{4xy}{1 + x^2},\tag{39}$$

$$\dot{y} = bx \left(1 - \frac{y}{1+x^2} \right). \tag{40}$$

The constants 'a' and 'b' are parameters which depend on the rate constants and the approximately constant concentrations of the other reactants. First we note that there is only one fixed point x = a/5 and $y = 1 + x^2 = 1 + a^2/25$. And thus our first step is to shift the origin to $(a/5, 1 + a^2/25)$ i.e. use the shifted variables X(=x + a/5), $Y(=y + 1 + a^2/25)$. A linear stability analysis of the resulting system about the fixed point X = Y = 0 reveals that it's a center for some $b^* = 3a/5 - 25/a$. Further that the origin is an unstable focus for $b < b^*$ and stable for $b > b^*$. As discussed we need to fix the parameters to proceed. So we pick a value of 'a' and choose $b = b^* - \delta$, where $\delta \ll b^*$. One carries out a perturbation analysis for the variables X and Y by assuming that the amplitude is small for small δ . The amplitude flow works out to be

$$\frac{dA}{d\tau} = -\frac{a}{5}\delta\Omega A + \frac{\Omega A^3}{\left(1 + \frac{a^2}{25}\right)^2} \left[\frac{3a^4}{125} - 3a^2 - 315 + \frac{1875}{a^2}\right],\tag{41}$$

where $\Omega^2 = a \left(1 + \frac{a^2}{25}\right) \left(\frac{3a}{5} - \frac{25}{a}\right)$. From the flow equation we can conclude that a limit cycle exists for positive values of δ . It is apparent that as we measure the value of 'a' for which limit cycles can exist, there is a cyclic-fold bifurcation at $a = a_c \simeq \sqrt{191.43}$ — obtained by setting the expression inside square bracket to zero.



Figure 2. The figure shows the phase diagram of the glycolytic oscillator given by Eqs. (42) and (43). The curve (solid line) separates the figure into two regions. For parameters in the shaded region, one gets limit cycle while unshaded region corresponds to parameters giving rise to a stable focus. Linear stability analysis predicts center for parameters on the curve.

3.4. Glycolytic oscillator

We now turn to a rather interesting example which clearly illustrates the usefulness of shifting of origin. Further it requires a determination of the locus of Hopf bifurcation points, to set up the perturbation theory, in order to locate the limit cycle. This example is drawn from biology – the glycolytic oscillator. Selkov [11] gave a simple mathematical model describing this oscillator as a 2-dimensional dynamical system. The variable x is the concentration of ADP (adenosine diphosphate) and y that of F6P (fructose-6-phosphate). The dynamics is given by

$$\dot{x} = -x + (a + x^2)y, \tag{42}$$

$$\dot{y} = b - (a + x^2)y,$$
 (43)

where 'b' is the rate of fructose production and 'a' the rate at which fructose decomposes (converts to ADP). It should be noted here that the presence of ADP catalyzes this conversion and hence 'a' is augmented to ' $a + x^2$ '. The only fixed point of the system is at x = b, $y = b/(a + b^2)$. It turns out to be a stable focus for a certain parameter range and an unstable focus for certain others. The crossover from stable to unstable focus occurs on a curve which is a locus of points in the *a-b* plane where a 'Hopf bifurcation' occurs i.e. the fixed point for those values of (a, b) is a center. The curve is given by $2a = \sqrt{1 + 8b^2} - (1 + 2b^2)$ and is shown in Fig. 2. We shift the origin to the fixed point and use the new coordinates X, Y given by X = b - xand $Y = \frac{b}{a+b^2} - y$. To use perturbation theory, we chose (a, b) close to the boundary. Setting $b = \sqrt{3/8}$ (the turning point of the curve), we take $a = 1/8 - \delta$ to consider a point inside the boundary but close to it. Clearly, δ is small and positive. To $\mathcal{O}(\delta)$, the equation of motion reads

$$\dot{X} = \frac{1}{2}(X+Y) + \mathcal{N}(X,Y),$$
(44)

$$\dot{Y} = -\frac{3}{2}X - \frac{Y}{2} - \mathcal{N}(X, Y),$$
(45)

where

$$\mathcal{N}(X,Y) = \delta(3X - Y) + \sqrt{\frac{3}{8}}X(X + Y) + X^2Y.$$
(46)

 \mathcal{N} has to be expanded in amplitude and the parameter δ . On solving the above equation we



Figure 3. Limit cycle in glycolytic oscillator for a = 0.124, $b = \sqrt{0.375}$ and $\delta = 0.001$.

find the amplitude flow,

$$\frac{dA}{d\tau} = 2\delta A - \frac{3A^3}{8} + \mathcal{O}(\delta^2).$$
(47)

Thus the amplitude emerges to go as $\delta^{1/2}$ for small δ . The frequency changes from the zeroth order value of $\frac{1}{\sqrt{2}}$ according to the flow

$$\frac{d\theta}{d\tau} = -\frac{\delta}{\sqrt{2}} + \frac{A^2}{4\sqrt{2}} + \mathcal{O}(\delta^2).$$
(48)

The stable fixed point $A^2 = 16\delta/3$ gives us the size of the limit cycle for $\delta \ll 1$. A typical small- δ orbit obtained numerically, is shown in Fig. 3 which bears out the correctness of the above flow.

3.5. Advantage over linear stability analysis

Lets reconsider the glycolytic oscillator given by eqs. (42) and (43). As discussed earlier according to linear stability analysis we have a loci of Hopf bifurcation points on the solid curve in Fig. 2. So for parameter values corresponding to the ones on the curve (e.g. a = 1/8, $b = \sqrt{3/8}$), we are supposed to have a center (periodic solution). But a simple numerical check establishes otherwise; it turns out for parameter values on the curve the solution is actually a *focus*. It is well known that linear stability analysis can predict wrongly the existence of a center [12] when the fixed point in question is actually a focus. However our RG method can clearly distinguish between an attractor and a center. If one looks at the amplitude flow equation (47), we notice for $\delta = 0$ it reduces to,

$$\frac{dA}{d\tau} = -\frac{3A^3}{8}.\tag{49}$$

Which according to our prescription suggests that the fixed point in question is an stable attractor and not an center. So we see that the linearized version of a nonlinear dynamical system may not reproduce qualitatively correct picture of the phase portrait near a fixed point because due to the linearization, the fixed point is shielded from full bombardment of the non-linear terms. On the other hand RG manages to capture the true nature of the fixed point.

3.6. Non-autonomous system

The RG technique can also be used to deal with non-autonomous systems. A very simple example $(f(x, \dot{x}) = 0 \text{ and } F \neq 0 \text{ in Eq.}(8))$ will suffice to show how. We consider a damped driven oscillator given by,

$$\ddot{x} + \omega^2 x + k\dot{x} = F\cos\Omega t,\tag{50}$$

which we rewrite as,

$$\ddot{x} + \Omega^2 x = -k\dot{x} + F\cos\Omega t + (\Omega^2 - \omega^2)x.$$
(51)

We treat k, F and $\Omega^2 - \omega^2$ as small to perturb about a center $(k = F = \Omega^2 - \omega^2 = 0)$. Proceeding as explained earlier, to the first order in all these small parameters, we obtain:

$$\frac{dA}{d\tau} = -\frac{kA}{2} - \frac{F\sin\Theta}{2\Omega}$$
(52)

$$\frac{d\Theta}{d\tau} = -\frac{F\cos\Theta}{2\Omega A} + \Delta\omega, \qquad (53)$$

where $\Delta \omega \equiv \omega - \Omega$. Since we are dealing with a forced oscillator, i.e. Ω is maintained externally, it cannot change, implying $d\Theta/d\tau = 0$. Also, existence of a periodic solution requires that $dA/d\tau = 0$. Therefore, setting both Eqs. (52) and (53) to zero and solving we find that the fixed point corresponds to the amplitude $A = f/[k^2+4(\Delta \omega)^2]^{1/2}$ (where $f = F/\Omega$) and the phase $\theta = \tan^{-1}[-k/2(\Delta \omega)]$. This is exactly in accordance with the literature of forced oscillators. The stable non-zero fixed point in the evolution of A corresponds to a limit cycle in accordance with what we have claimed has to happen. So in case of non-autonomous systems one needs to set both the flow equations to zero and solve for the fixed points in order to extract information about the periodic solution. This process can also be repeated for cases when $f(x, \dot{x}) \neq 0$ and obtain the correct results.

3.7. Oscillator without any linear terms

As we have pointed out earlier in this article, for the RG method to work one needs to find a linear center (a basic periodic state) about which to perturb. We have also showed that in a few cases if initially there isn't a linear center in the problem one can suitably expand the parameter space or introduce new parameters to create one. But now we ask the question what if the dynamical system does not have a linear part nor is it amenable for parameter expansion via which one can locate a center. For example consider the oscillator: $\ddot{x} + \lambda x^3 = 0$ or $\ddot{x} + \beta \lambda x \dot{x} + \lambda^2 x^3 = 0$. Such cases can't be handled by the prescription we have laid down here nevertheless a little tweaking of the method can handle a class of such problems. To illustrate our point lets consider the so called Riccati equation given by,

$$\ddot{x} + \beta \lambda x \dot{x} + \lambda^2 x^3 = 0. \tag{54}$$

The trick we use here is motivated from the so called "equivalent linearization" method of nonlinear dynamics. We rewrite the above equation as,

$$\ddot{x} + \beta \lambda x \dot{x} + \lambda^2 \alpha \langle x^2 \rangle x + \lambda^2 \left(x^3 - \alpha \langle x^2 \rangle x \right) = 0$$

or
$$\ddot{x} + \Omega^2 x = -\beta \lambda x \dot{x} - \lambda^2 \left(x^3 - \alpha \langle x^2 \rangle x \right)$$
(55)

where $\langle x^2 \rangle$ is the average of $x(t)^2$ over a cycle. In the above equation α is a constant number which we need to find out. If we ignore the term in parenthesis, we have an equivalent linear oscillator with frequency

$$\Omega^2 = \alpha \lambda \langle x^2 \rangle. \tag{56}$$

So now we have a linear system about which to perturb and we can carry out the normal RG calculation. A detailed account of the problem and calculations can be found in the reference [13]. The RG calculation yields the flow equations,

$$\frac{dA}{d\tau} = 0, \tag{57}$$

$$\frac{d\theta}{d\tau} = \lambda \left(\frac{3}{4} - \frac{\alpha}{2} - \frac{\beta^2}{12}\right) \frac{A^2}{2\Omega}.$$
(58)

And the above flow equations correctly predict the salient features of this oscillator [13].

4. Conclusion

In concluding we must revisit the method described here and discuss some of its merits and demerits. We have shown here that the RG method can handle a wide variety of nonlinear dynamical problems in 2-dimensions. It can further differentiate between different kinds of periodic solutions, namely centers and limit cycles. Further where as linear stability analysis fails to distinguish between an attractor and a center in a few cases our RG method successfully distinguishes the two. The major advantage of RG however is the fact that it does not require any *apriori ad hoc* assumptions like the other singular perturbation techniques. It starts with a simple perturbative expansion assuming only the existence of a small parameter in the problem and is able to capture various features of the dynamics. For example the multiple scales involved in a dynamics need not be pre-identified as is done in multiple scale analysis rather the scales appear naturally in the RG calculation.

However being perturbative this method has its limitations. As we have already discussed, for a given 2-D system, application of RG depends on our ability to identify a suitable linear center (periodic state) about which to perturb. While in many cases identifying a linear center is straightforward, there are situations where one needs to employ certain tricks to do so. We have discussed here a few ways in which one can identify a linear center in the problem. But if in a problem one is not able to identify such a basic periodic state about which to perturb we can't proceed with the RG prescription. Nevertheless a wide range of dynamical problems can be handled with our prescription. And it has been our intention here to illustrate our method in a manner so that people from various disciplines may find it useful while analyzing periodic solution in two dimensions.

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