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Algebraic Geometry and Elliptic Integrals Approach for Calculation of the Propagation Time of a Signal Between GPS Satellites with Account of General Relativity Theory

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Abstract. The approach for calculating the propagation time of a signal between GPS satellites will be summarized, based on the proposed new theoretical approach in several previous publications, as well as the perspectives for future development of the theory. Topics include: 1. Basic notions of inter-satellite communications. 2. Shapiro delay formulae in General Relativity Theory - basic formalism and the necessity to extend the formalism by taking into account the satellite motion on a plane elliptic or space-distributed elliptic orbit. 3. Basic facts about the disturbed motion in celestial mechanics and the necessity to incorporate it in the theory of inter-satellite communications, accounting for General Relativity Effects. 4. Propagation time of a signal, emitted by a satellite on a plane and also space-distributed elliptical orbit in terms of zero-order elliptic integrals and respectively of higher order integrals. Proof of the real-valuedness of the propagation time for all cases as one of the criteria for the correctness of the theoretical approach. 5. New analytical algorithms for calculation of zero-order elliptic integrals in the Legendre form. Relation to two representations in the Weierstrass form. 6. The new formalism of intersecting four-dimensional null cones and the resulting physical notions of the (intersecting)) space-time interval (with the property of being positive, negative or equal to zero) and the (intersecting) geodesic distance (being only positive, because is related to the distance, travelled by light or radio signals). Proof of these properties in the general case and in some partial cases. New numerical estimate $E_{\mathrm{lim}} > 45.002510943228$ [deg], above which the space-time interval is positive and thus inter-satellite communications between satellites on one plane elliptical orbit are possible. The angular distance of 45[deg] is typical for the disposition of 8 satellites on one orbit in the Russian satellite constellation $GLONASS$, so it might be claimed that such a configuration is favourable from the point of view of inter-satellite communications (with account of GRT effects).

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1. Introduction

Autonomous navigation is one of the most important ingredients of satellite communications. The essence of such a concept is that a satellite system such as the Global Navigation Satellite System ($GNSS$), consisting of 30 satellites and orbiting the Earth at a height of 23616 km should be able to function properly without exchanging any signals with ground stations in the course of 6 months (180 days). This means that $GNSS$ should be inter-operable with other navigational satellite systems such as GPS (Global Positioning System) and the Russian system $GLONASS$ (Globalnaya Navigazionnaia Sputnikovaia Sistema) (see the monographs [1] and [2] for a brief introduction into the characteristics of these systems). In other words, satellites on different orbital planes should be able to exchange signals between each other and in such a way, "inter-satellite communications" are realized. Since the exchange of signals takes place in the near-Earth gravitational field (and generally in the gravitational field of the Solar system), the creation of a theory for such processes should take into account the General Relativity Theory effects. In brief, this theory is based on the following two important facts:

1. An electromagnetic, radio or light signal (including a laser signal) is propagating on the null light cone, which can be obtained after setting up the infinitesimal metric element equal to zero $ds^2 = 0$. The metric element ds^2 will be given later in the paper and is chosen to be typical for the near-Earth space. A general theory of the signal propagation in the gravitational field is given in the contemporary monograph [3].

2. From a qualitative and also physical point of view, the significance of the null cone light equation is that it gives the opportunity to calculate the propagation time of the signal - the well-known Shapiro delay formulae, known yet from 1964 [4] and widely applied in various investigations. The Shapiro delay formulae has an important physical meaning - due to the action of the gravitational field, which "curves" the trajectory of the signal, the propagation time increases with a small logarithmic correction. Although very small (at the order of picoseconds), it plays an important role in satellite ranging, presently achieved with great precision.

It should be stressed that the application of General Relativity Theory in the theory of signal propagation between GPS satellites began in the pioneering works of Neil Ashby [5] and [6], where the s.c. increment of the coordinate time Δt was calculated as

$$\int\limits_{path} \Delta t = \int\limits_{path} \left[1 - \frac{(V - \Phi)}{c^2} + \frac{v^2}{2c^2} \right] d\tau \quad , \tag{1}$$

representing a (path) integral over the path of the atomic clock (on board of the satellite), V and Φ are potentials of the Earth and v is the orbital velocity of the satellite. In fact, as noted in [5], "the rate of coordinate time is determined by atomic clocks at rest at infinity, but the rate of the GPS coordinate time, however, is closely related to the International Atomic Time (TAI), which is a time scale computed on the basis of inputs from hundreds of primary time standards, hydrogen masers, and other clocks from all over the world." Because of these path-dependent effects (especially for the case of the inter-satellite communications (ISC) to be considered in this paper), comprising also time dilation (apparent slowing of moving clocks) and frequency shifts due to gravitation, the proper time cannot be used when a signal is being transmitted between two satellites. However, this statement needs to be clarified. If the definition of the atomic time τ is used and also formulae (1), it can be written also [5]

$$d\tau = \frac{ds}{c} = \left[1 + \frac{(V - \Phi)}{c^2} - \frac{v^2}{2c^2} \right] dt \quad , \tag{2}$$

from where it can be concluded that the proper atomic time of fictitious atomic clocks at rest and in a local inertial frame agrees with the coordinate time up to the accuracy of $O(\frac{1}{c^2})$ terms. By definition, an accuracy of $O(\frac{1}{c^2})$ or $O(\frac{1}{c})$ means that terms of the order $\frac{1}{c^2}$ or $\frac{1}{c}$ are accounted in the corresponding expressions. For example, up to $O(\frac{1}{c})$ terms and taking into account the metric element ds^2 in its most general form

$$ds^2 = g_{00}c^2dt^2 + 2g_{0j}cdtdx^j + g_{ij}dx^i dx^j = 0 \quad , \tag{3}$$

the increment of the coordinate time Δt can be found first as a solution of a quadratic algebraic equation and subsequently - a differential equation [7]

$$\Delta t = \pm \frac{1}{c} \int_A^B \frac{1}{\sqrt{-g}} . \sqrt{\left(g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j} + \frac{1}{c} \int_A^B \frac{g_{0j}}{-g_{00}} \, dx^j \quad . \tag{4}$$

The coordinate time now is related to the propagation of a light or radio signal from a space-point A to the space-point B and therefore represents the propagation time $\Delta t = T_B - T_A$ with T_B and T_A being the times of reception and emission. Two important conclusions can be made: 1. Since time in satellites is measured by means of atomic clocks and atomic time agrees with the propagation (coordinate) time up to terms of a certain order in $\frac{1}{c}$, it is a measurable quantity. 2. The presence of the differentials dx^i enables to find such a parametrization of the space coordinates, which might not necessarily be related to the signal propagation path and the endpoints A and B. In the present investigation, the coordinates x, y and z will be related to the elliptic motion of the satellite and the endpoints A and B will mark the initial position of the satellite on the orbit (when it emits the signal) and the point B-the final position of the satellite, when the signal is percepted. The reason why the reference system is chosen to be attached to the satellite trajectory is that the s.c. orbital parameters of all the satellite configurations GPS, $GLONASS$, $Galileo$ and $BeiDou$ are known with great accuracy. Further this will be demonstrated. In spite of the fact that the satellite motion will be very important in calculating the propagation time of the signal, it will be proved by simple numerical calculations that the obtained propagation time is typical for the light or radio signals and is not directly related to the celestial time of motion of the satellite. This means that the propagation time is determined by the type of equation used for its calculation (in the case this is the null cone equation) and not by the type of the spatial reference system.

Two cases will be considered further.

1st case This is the most simple case of plane elliptical motion of the satellite on an ellipse, parametrized as

$$x = a(\cos E - e) \quad , \quad y = a\sqrt{1-e^2} \sin E \quad , \tag{5}$$

where E is the eccentric anomaly angle (characterizing the position of the satellite on the orbit) and a and e are respectively the semi-major axis and the eccentricity of the ellipse. This is the problem, investigated in a series of publications. Although in a sense the notion about the "plane elliptical motion" is an idealized one, any further development of the theory (see the proposal in the next "2-nd case") will be based on the applied approach. An interesting fact, which will be proved is that the obtained from the null cone equation propagation time has the dimension of seconds. This is a nontrivial fact, since the propagation time is expressed by a combination of elliptic integrals of the first, second and the third kind, so it is not evident at the beginning that the elliptic integrals are real-valued expressions. This should be so, because time (in the case-the propagation time) as a physical quantity cannot be imaginary. Moreover, further it will be shown that for the case of space-distributed orbits, the corresponding higher-order (second and fourth) order elliptic integrals are imaginary, but since they enter imaginary expressions for the propagation time, it will be again a real-valued quantity.

2. Perturbed motion in celestial mechanics and its relation to the problem about signal propagation

2-nd case. This is the more complicated case of space-distributed (elliptical) orbits, characterized by the full set of 6 Keplerian parameters $(M, a, e, \Omega, I, \omega)$ (called in some monographs such as [8] "the contact elements" or "orbital elements"), where $M = \sqrt{\frac{GM_{\oplus}}{a^3}}$ is the mean motion and

Ω, I, ω are analogous of the Eulerian angles (known from theoretical and celestial mechanics) which will be defined further. Following the same mathematical formalism and assuming that the position of the satellite is determined only by means of the true anomaly angle f (i.e. the other orbital parameters do not change), it will be proved in this paper that the propagation time will depend on more complicated elliptic integrals of the second and fourth order (the order of the elliptic integral is determined by the degree of the integration variable in the nominator of the under-integrand expression). This result is important both from physical and mathematical point of view, because a complete theory how the propagation time changes under the changes $(\delta M, \delta a, \delta e, \delta\Omega, \delta I, \delta\omega)$ of the orbital elements is still lacking. These changes are caused by gravitational perturbations by celestial bodies, tidal forces and etc. A well-known theoretical method in celestial mechanics is the method of "osculating conics", the essence of which is that the orbital elements are assumed to depend on time $\{C_i(t) = (M(t), a(t), e(t), \Omega(t), I(t), \omega(t)),$ $i = 1, 2, 3......, 5, 6)\}$ [8]. It is assumed that the body under consideration moves along a conic, which is osculating (i.e. tangent) to the actual physical trajectory and so it is slowly evolving. In other words, if the orbital vector $\overrightarrow{r}$ and the velocity vector $\overrightarrow{v}$ depend on time and on the orbital parameters $C_i(t)$

$$\overrightarrow{r} = \overrightarrow{r}(t, C_1(t), C_2(t), C_3(t), C_4(t), C_5(t), C_6(t)) \quad ,$$

$$\frac{\partial \overrightarrow{r}}{\partial t} = \overrightarrow{v}(t, C_1(t), C_2(t), C_3(t), C_4(t), C_5(t), C_6(t)) \tag{6}$$

and the parameter t plays the role of time in the perturbed equation of motion

$$\frac{\partial^2 \overrightarrow{r}}{\partial t^2} + \frac{GM}{r^3}\overrightarrow{r} + \sum_{i=1}^{6} \frac{\partial \overrightarrow{v}}{\partial C_i} . \frac{dC_i}{dt} = F_i \quad , \tag{7}$$

then a constraint called " Lagrange constraint" is imposed [8]

$$\sum_{i=1}^{6} \frac{\partial \overrightarrow{r}}{\partial C_i} . \frac{dC_i}{dt} = 0 \qquad . \tag{8}$$

The last equality means that locally the physical trajectory, defined by $\{C_i(t), \quad i = 1, 2, 3......, 5, 6)\}$, will coincide with the unperturbed orbit, followed by the body if the perturbations were to cease instantaneously. An important consequence of the theory is the nonlinear system of ordinary differential equations in the Euler-Gauss form [8] with respect to $\frac{dM(t)}{dt}, \frac{da(t)}{dt}, \frac{de(t)}{dt}, \frac{d\Omega(t)}{dt}, \frac{dI(t)}{dt}$ and $\frac{d\omega(t)}{dt}$. If the equations include the s.c. "disturbing gravitational potential" (accounting for the perturbations of outer celestial bodies), then the equations are called "Lagrange planetary equations" [8], [9], [10], [11].

The above formalism constitutes the fundamentals of the celestial mechanics theory of disturbing motion, but the estimates in [5] for the perturbed radius-vector, perturbed eccentric anomaly (based on the celestial mechanics monograph of Richard Fitzpatrick, ch.11.6, [12]), perturbed semi-major axis and perturbed energy are used to estimate the fractional frequency shift

$$\frac{\delta \overline{f}}{\overline{f}} = -\frac{v^2}{2c^2} - \frac{GM_E}{c^2 r} + \frac{V'}{c^2} \quad , \tag{9}$$

where V' is the perturbed energy. The estimate is based, however, only on the evaluation of the separate contributions in the perturbed potential and not on any path integration, which in principle can show how the parameters change. In other words, the change of each one of the parameters is not shown to what change in $\frac{\delta \overline{f}}{\overline{f}}$ will correspond. In order to avoid further confusion, we have denoted here the frequency by $\overline{f}$, to distinguish it from the true anomaly angle f, used to denote the position of a satellite on a space-distributed orbit. In fact, the calculation of the propagation time in this paper and the elliptic integrals of second and fourth order will be related only to the true anomaly angle f as a variable of integration. Thus any disturbances in the motion of the satellite will not be taken into account.

Since evidently because of the perturbations of all the 6 Keplerian parameters, it will be natural to assume that the variation of the propagation time δT along all the perturbed orbital elements $(\delta M, \delta a, \delta e, \delta\Omega, \delta I, \delta\omega)$ will give independent contributions

$$\delta T = \frac{\partial T}{\partial M}\delta M + \frac{\partial T}{\partial a}\delta a + \frac{\partial T}{\partial e}\delta e + \frac{\partial T}{\partial \Omega}\delta\Omega + \frac{\partial T}{\partial I}\delta I + \frac{\partial T}{\partial \omega}\delta\omega \quad . \tag{10}$$

3. Keplerian parameters for space-distributed orbits of the satellite configurations *GPS*, *GLONASS*, *BeiDou* and *Galileo*

If a theoretical model is created, the development of a numerical model will be greatly facilitated by the fact that the Keplerian parameters are well-known for the satellite configurations GPS, $GLONASS$ [13], Galileo (European satellite system) and $BeiDou$ (Chinese satellite system). For example, the satellites of the Galileo constellation are situated on three orbital planes with nine-equally spaced operational satellites in each plane. The Galileo satellites are in nearly circular orbits with semi-major axis of 29600 km and a period of about 14 hours [14] and an inclination of the orbital planes 56 degrees. For comparison, the Russian Global Navigation Satellite System $GLONASS$, launched in 1982, consists of 21 satellites in three orbital planes. Each satellite operates in nearly circular orbits with semi-major axis of 25510 km, and the satellites within the same orbital plane are equally spaced by 45 degrees. In the following sections, this fact will be important when discussing the theoretical approach of the space-time interval on two intersecting four-dimensional null cones. Each orbital plane has an inclination angle of $64.8^0 \pm 0.3^0$ (see also [1]), which is more than the inclination angle 56^0 of the orbital planes of the Galileo satellites. The longitude of ascending nodes $\Delta\Omega$ between the orbital planes and the argument of latitude difference Δu are well-known and are correspondingly equal to $\Delta\Omega = 120^0$ and $\Delta u = 45^0$. The satellites of the $GLONASS$ constellation within one orbit are situated at an angular distance of 45^0 (8 satellites on one orbit). The eccentricity of the orbit in [1] is estimated to be $e < 0.01$, but according to some other date it can be $e = 0.02$. A $GLONASS$ satellite completes an orbit in approximately 11 hours 15 minutes ± 5 sec - less than the orbital period of 14 hours for the Galileo satellite. Consequently, the three characteristic angles of rotation - the eccentric anomaly E, the mean anomaly $M = n(\tau - t)$ and the true anomaly f should be different for the different satellite systems and can be found from the corresponding tables.

The data about the GPS satellites can be found in many monographs, but they are briefly summarized for example in [15] - 6 orbital planes with $4-5$ satellites per orbit, semi-major axis $a = 26400\ km$, eccentricity at most $e = 0.02$, but further we shall use the more precise data from the PhD dissertation [16]-$a = 26560.25169632944\ km$ and $e = 0.01323881349526$, the inclination of the orbit I is equal to $I = 55.5^0$ [15] or $I = 0.9614884100802\ rad$ [16].The mean anomaly $M = -0.3134513508155\ rad$ [16] will be used in order to determine the eccentric anomaly angle E, related to the position of the satellite on the orbit and also to the true anomaly f, expressed through E by means of the formulae

$$\tan\frac{f}{2} = \sqrt{\frac{1-\cos f}{1+\cos f}} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \quad . \tag{11}$$

The data about the longitude of the ascending node Ω and the argument of perigee ω can also be found in [16], but they will not be used for the concrete calculations in this paper.

4. Description of the most general theoretical model and the general system of algebraic equations

Since the orbital parameters of the satellites from all the satellite constellations are well-known, the following interesting research problem can be formulated: suppose that a light or radio signal is being send from a satellite belonging to one satellite constellation with orbital parameters $(M_1, a_1, e_1, \Omega_1, I_1, \omega_1)$ to another satellite (from the same or from another constellation) on a space-distributed orbit with another orbital parameters $(M_2, a_2, e_2, \Omega_2, I_2, \omega_2)$. Then how can the propagation time of the signal be calculated? This problem is more complicated than it might seem because during the time of flight of the photon or radio signal, the orbital parameters of the second satellite change.Consequently, the curved signal trajectory and the signal, emitted from the first satellite should be "correlated" with the movement of the second satellite. So if one imagines that after its emission the signal is "propagating" on the first null-cone (with origin at the point of emission), then at the moment of reception the signal is propagating not on the first null-cone, but on a second null-cone, defined with origin at the point of reception. So these two points will be constantly changing, but the phenomena of emission and subsequent reception of the signal mathematically will mean that the two four-dimensional null-cones (in terms of the two sets of space-time coordinates T_1, x_1, y_1, z_1 and T_2, x_2, y_2, z_2) will intersect. If the two sets of coordinates are expressed through the changing orbital parameters, then evidently one obtains two seven-dimensional null cones (the coordinate times of emission T_1 and T_2 are the additional seventh coordinates) in terms of the variables $(dT_1, dM_1, da_1, de_1, d\Omega_1, dI_1, d\omega_1)$ and $(dT_2, dM_2, da_2, de_2, d\Omega_2, dI_2, d\omega_2)$, which should intersect. Correspondingly, the algebraic equations of the two four-dimensional null cones in terms of the two sets of four-dimensional coordinates (T_1,x_1, y_1, z_1) and (T_2, x_2, y_2, z_2) will be

$$ds_1^2 = -c^2 \left(1 + \frac{2V_1}{c^2}\right)(dT_1)^2 + \left(1 - \frac{2V_{1.}}{c^2}\right)\left((dx_1)^2 + (dy_1)^2 + (dz_1)^2\right) = 0 \quad , \tag{12}$$

$$ds_2^2 = -c^2 \left(1 + \frac{2V_2}{c^2}\right)(dT_2)^2 + \left(1 - \frac{2V_{2.}}{c^2}\right)\left((dx_2)^2 + (dy_2)^2 + (dz_2)^2\right) = 0 \quad . \tag{13}$$

These two null cones will not intersect, because their coordinates are independent. However, one may intersect these null cones with the 14-dimensional hyperplane, defined as the differential of the Euclidean distance R_{AB}

$$dR_{AB}^2 = d(x_1 - x_2)^2 + d(y_1 - y_2)^2 + d(z_1 - z_2)^2 \quad . \tag{14}$$

Now the problem about finding the propagation times T_1 and T_2 is formulated as an algebraic geometry problem of the intersection of two seven-dimensional null cones (12) and (13) (if written in terms of the variables $(dT_1, dM_1, da_1, de_1, d\Omega_1, dI_1, d\omega_1)$ and $(dT_2, dM_2, da_2, de_2, d\Omega_2, dI_2, d\omega_2)$) with the $14-$dimensional hyperplane (14), written in terms of all the two sets of variables. It may be noted that since the Euclidean distance R_{AB}^2 between the satellites is changing, the distance R_{AB} depends on the following variables

$$R_{AB} = R_{AB}(T_1, \Gamma_i^{(1)}, T_2, \Gamma_i^{(2)}) \quad , \tag{15}$$

where the following new notations are introduced

$$\Gamma_i^{(1)} = (M_1, a_1, e_1, \Omega_1, I_1, \omega_1) \quad , \quad \Gamma_i^{(2)} = (M_2, a_2, e_2, \Omega_2, I_2, \omega_2) \quad , i = 1, 2, ...6 \quad , \tag{16}$$

$$X_j^{(1)} = (x_1, y_1, z_1) \quad , \quad X_j^{(2)} = (x_2, y_2, z_2) \quad , \quad j = 1, 2, 3 \quad . \tag{17}$$

The notations $\Gamma_i^{(1)}$ and $\Gamma_i^{(2)}$ are the previous notations $C_i(t)$ in (8), but here we do not necessarily suppose that they depend on the "perturbed time". The notations (16) and (17) enable us to write equation (12) in the following concise form

$$-c^2\left(1+\frac{2V_1}{c^2}\right)\left[\sum_{i=1}^{6}\frac{\partial T_1}{\partial\Gamma_i^{(1)}}d\Gamma_i^{(1)}\right]^2+\left(1-\frac{2V_{1.}}{c^2}\right)\sum_{j=1}^{3}\left[\sum_{i=1}^{6}\frac{\partial X_j^{(1)}}{\partial\Gamma_i^{(1)}}d\Gamma_i^{(1)}\right]^2=0 \quad . \tag{18}$$

It may easily be proved that

$$(dT_1)^2=\left[\sum_{i=1}^{6}\frac{\partial T_1}{\partial\Gamma_i^{(1)}}d\Gamma_i^{(1)}\right]^2$$

$$=\sum_{i=1}^{6}\left(\frac{\partial T_1}{\partial\Gamma_i^{(1)}}\right)^2\left(d\Gamma_i^{(1)}\right)^2+2\sum_{\substack{i,k=1\\ i\neq k}}^{6}\left(\frac{\partial T_1}{\partial\Gamma_i^{(1)}}\right)\left(\frac{\partial T_1}{\partial\Gamma_k^{(1)}}\right)d\Gamma_i^{(1)}d\Gamma_k^{(1)} \quad . \tag{19}$$

An analogous formulae is valid also for the expression $\left[\sum\limits_{i=1}^{6}\frac{\partial X_j^{(1)}}{\partial\Gamma_i^{(1)}}d\Gamma_i^{(1)}\right]^2$. Thus expression (18) can be written as a six-dimensional quadratic surface with respect to the variables $d\Gamma_i^{(1)}$ $i=1,2,...6$

$$-c^2\left(1+\frac{2V_1}{c^2}\right)\left[\sum_{i=1}^{6}\left(\frac{\partial T_1}{\partial\Gamma_i^{(1)}}\right)^2\left(d\Gamma_i^{(1)}\right)^2+2\sum_{\substack{i,k=1\\ i\neq k}}^{6}\left(\frac{\partial T_1}{\partial\Gamma_i^{(1)}}\right)\left(\frac{\partial T_1}{\partial\Gamma_k^{(1)}}\right)d\Gamma_i^{(1)}d\Gamma_k^{(1)}\right]$$

$$+\left(1-\frac{2V_{1.}}{c^2}\right)\left[\sum_{i=1}^{6}\sum_{j=1}^{3}\left(\frac{\partial X_j^{(1)}}{\partial\Gamma_i^{(1)}}\right)^2\left(d\Gamma_i^{(1)}\right)^2+2\sum_{\substack{i,k=1\\ i\neq k}}^{6}\sum_{j=1}^{3}\left(\frac{\partial X_j^{(1)}}{\partial\Gamma_i^{(1)}}\right)\left(\frac{\partial X_j^{(1)}}{\partial\Gamma_k^{(1)}}\right)d\Gamma_i^{(1)}d\Gamma_k^{(1)}\right]=0 \quad . \tag{20}$$

The same type of an equation can be written with respect to equation (13) for the second propagation time T_2. Similarly, the equation (14) with respect to the 14−dimensional hyperplane can be written as

$$dR_{AB}^2=\sum_{j=1}^{3}2\left(X_j^{(1)}-X_j^{(2)}\right)\sum_{i=1}^{6}\left[\frac{\partial X_j^{(1)}}{\partial\Gamma_i^{(1)}}d\Gamma_i^{(1)}-\frac{\partial X_j^{(2)}}{\partial\Gamma_i^{(2)}}d\Gamma_i^{(2)}\right] \quad , \tag{21}$$

where the differential dR_{AB}^2 should be found from

$$dR_{AB}^2=\sum_{l=1}^{2}\frac{\partial R_{AB}^2}{\partial T_l}\sum_{i=1}^{6}\frac{\partial T_{(l)}}{\partial\Gamma_i^{(l)}}d\Gamma_i^{(l)}+\sum_{l=1}^{2}\sum_{i=1}^{6}\frac{\partial R_{AB}^2}{\partial\Gamma_i^{(l)}}d\Gamma_i^{(l)} \quad . \tag{22}$$

If in the system of equations (20)-(22) the differentials $d\Gamma_i^{(l)}$ are considered to be independent (meaning that the constraint (8) can no longer be applied), then a complicated system of nonlinear differential equations is obtained with respect to $\frac{\partial X_j^{(l)}}{\partial\Gamma_i^{(2)}}$, $\frac{\partial T_{(l)}}{\partial\Gamma_i^{(l)}}$ and $\frac{\partial R_{AB}^2}{\partial T_l}$. In fact, equations (20)-(22) are related to the s.c. intersection theory of algebraic surfaces and they are more peculiar and difficult to deal with in comparison with the examples, given in most of the basic monographs on algebraic geometry (see for example [17] and [18]). In the cited monographs, examples of conics are given [18], also intersection of conics [17]. Basic knowledge about algebraic varieties and intersection theory can be found also in [17], the basic notions of algebraic curves and algebraic geometry in a clear and accessible style are presented in [20], also in [21].

5. The purpose of the present paper

The present paper is a part of a series of papers [22], [23], [24] and [25], each of which represents a part of a systematic approach, aiming at creating a new theoretical formalism for the description of satellite communications between two satellites on different space-distributed orbit, accounting for the effects of GRT. In this final form, the problem is rather complicated and so can be treated, if several other more simple problems are solved. The purposes of this review are several:

1. To present a more general description of the problem, both from physical and mathematical aspects, which have not been given in the previous publications. This has already been performed in the previous sections.
2. To present the mathematical derivation of the propagation time of a signal in terms of zero-order elliptic integrals of the first, second and the third kind for the most simple case of a signal, emitted by a moving along a plane elliptical orbit [24]. This case may seem highly unrealistic, but here the main problem is to justify the assumed elliptical parametrization of the space coordinates in the null cone equation. In order to avoid any doubts that the obtained propagation time is really related to the signal propagation and not to the movement of the satellite, an additional (not previously presented) section has been added with some concrete numerical estimates, based on typical parameters for the GPS orbits, given in the PhD thesis [16].
3. To derive the signal propagation time for the more complicated case of a signal emission by a satellite on a spatially-distributed orbit, characterized by the full set of 6 Keplerian parameters [24]. The obtained propagation time is expressed again in terms of elliptic functions, but they are more complicated, since are of higher order (second and fourth). This case is again not realistic from a physical point of view, because a satellite in its orbital motion may deviate tens and even hundreds of meters due to various disturbing forces, even solar pressure and etc. But although not quite realistic, this method is necessary to be developed, because it constitutes one of the basic elements in the developed in the publications [22], [23] method of intersecting four-dimensional null cones (in terms of the coordinates T, x, y, z in the initially given null-cone metric).
4. In the papers [22], [23] the method of intersecting null cones has been studied on the base of the equations, which are in fact the simple version of the equations (18) - (22) from the previous section. The most simple case of plane elliptical motion of the satellite, described by the changing eccentric anomaly angles E_1 and E_2 have been chosen, and the two propagation times T_1 and T_2. Although such a simplified model might be considered physically unrealistic, most amazingly it helped us to establish some interesting physical consequences. The most important one is that the intersection of the two null cones with the four-dimensional hyperplane defines a distance R_{AB}^2, which is called a space-time interval. It can be positive, negative or equal to zero. Although initially R_{AB}^2 was defined as an Euclidean distance, this result is not amazing, because the two null cones are notions from General Relativity Theory, where space-like and time-like vectors (respectively, outside or inside isotropic null cones) and curves are quite natural (see ch. 4 and ch. 6 of the known monograph [26]). Since the real propagation of light or radio signals in GRT is related to time geodesics and to large-scale distances, a new interpretation has been given of the notion of the "geodesic distance"- the space-time interval becomes the positive geodesic distance, if the s.c. "condition for inter-satellite communications" (expressed from the

null cone equation) is substituted into the expression for the space-time interval.

There is also one direct consequence from the proposed formalism of two intersecting null cones, which is important for the inter-satellite communications between the satellites from the $GLONASS$ constellation. It has been proved in [22] and [23] that the limiting value of the eccentric anomaly angle $E_{\lim} = 45.002510943228$ [deg], above which (below) the space-time interval dR_{AB}^2 becomes positive (negative) corresponds to the angular distance of 45^0 of disposition of the satellites from the $GLONASS$ constellation within one plane orbit and eccentricity of the orbit $e = 0.02$.

There are a number of other interesting problems, which will be commented in the conclusion of this paper.

6. The Shapiro delay formulae in station–satellite and inter-satellite communications - review of some basic physical facts

We shall begin with some trivial and simple examples in [27], which will help us to understand the physical essence of the phenomena of Shapiro time delay of a signal. In the case of the trivial metric

$$-g_{00} = 1 \ \ , \ g_{11} = g_{22} = g_{33} = 1 \ \ , \ g_{\mu\nu} = 0 \text{ for } \mu \neq \nu \ \ , \ \mu, \nu = 0, 1, 2...3 \tag{23}$$

one can obtain the simple expression

$$ds^2 = -c^2(1 - \frac{v^2}{c^2})dt^2 = -c^2 d\tau^2 \ \ , \tag{24}$$

which implies the time dilation of a moving atomic clock (on board of a satellite) relative to a clock at rest. Note also that t is the time, related to the motion of the satellite (v respectively is the velocity).

In order to obtain the Shapiro time delay, one has to introduce a gravitational potential U in the temporal and spatial parts of the metric

$$-g_{00} = 1 - \frac{2U}{c^2} \ \ , \ g_{0j} = 0 \ \ , \ g_{ij} = \left(1 + \frac{2U}{c^2}\right)\delta_{ij} \ \ , \ i, j = 1, 2, 3 \ \ . \tag{25}$$

In other words, there is no Shapiro delay in the case of the metric (23).

Now taking into account expression (4) for the coordinate time Δt (which is in fact also the propagation time of the signal), it can be expressed as [7]

$$\Delta t \approx \frac{1}{c}\int\limits_{path}\sqrt{\frac{g_{ij}}{-g_{00}}dx^i dx^j} \approx \frac{1}{c}\int\limits_{path}\left(1 + \frac{2U}{c^2}\right)\sqrt{\delta_{ij}dx^i dx^j} \tag{26}$$

$$= \frac{\rho}{c} + \frac{1}{c^3}\int\limits_{path} 2U d\rho \ \ , \tag{27}$$

where ρ is called "geometric path"(in the thesis [16] it was called "the slant range") and the second term is the Shapiro delay term. The calculation in (26)-(27) in [7] however presumed that the second contribution $\frac{2U}{c^2}$ in g_{00} in the metric (25) was neglected.

This will turn out to be the correct approach, but the mathematical justification is more evident from the calculation in [16] for the case of the metric (25)

$$\Delta t \approx \frac{1}{c}\int\limits_{path}\sqrt{\frac{g_{ij}}{-g_{00}}dx^i dx^j} \tag{28}$$

$$= \frac{1}{c^2} \int\limits_{path} \sqrt{\frac{1-\frac{2U}{c^2}}{1+\frac{2U}{c^2}} \delta_{ij} dx^i dx^j} \approx \int\limits_{path} d\rho - \frac{1}{c^3} \int\limits_{path} 2U d\rho \quad . \tag{29}$$

The result is the same as in (27), but the sign before $\frac{1}{c^3} \int\limits_{path} 2U d\rho$ is negative. So if the gravitational potential is negative, the second Shapiro delay term will increase the propagation time, as should be. Expression (29) was derived by decomposing the under-integral expression $\sqrt{\frac{1-\frac{2U}{c^2}}{1+\frac{2U}{c^2}}}$ in the s.c. weak limit, when $\frac{2U}{c^2} \ll 1$. Further the weak limit will be used in order to obtain the new expression for the propagation time in terms of the eccentric anomaly angle E, but the general case without using this approximation will be briefly considered. The weak-field limit has an important physical meaning, clarified in the PhD thesis [30]: if $\overrightarrow{R}$ is a vector, related to the signal trajectory with respect to a Earth-centered system, $\overrightarrow{r}_A$ and $\overrightarrow{r}_B$ are vectors, related to the points of emission and reception of the signal and $\overrightarrow{r} = \overrightarrow{r}_A - \overrightarrow{r}_B$, then the tangent line dR to the signal trajectory $\overrightarrow{R}$ deviates from the straight line cdt by $O(\frac{1}{c^2})$ terms. This can be written as $\parallel dr \parallel = \parallel dR \parallel + O(\frac{1}{c^2})$ and follows also from the null cone equation [30]

$$cdt = \left(1 + 2\frac{G_{\oplus}M_E}{rc^2}\right) \parallel dR \parallel \quad , \tag{30}$$

where $G_{\oplus}M_E$ is the s.c. geocentric gravitational constant. Further its numerical value will be given in order to estimate numerically the Shapiro delay term. After integration, the Shapiro delay formulae can be found

$$\Delta t = t_B - t_A = T_{rec} - T_{emis} = \int\limits_{0}^{R_{AB}} \left(1 + 2\frac{G_{\oplus}M_E}{rc^2}\right) \frac{dR}{c} \tag{31}$$

$$= \frac{R_{AB}}{c} + 2\frac{G_{\oplus}M_E}{c^3} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right) \tag{32}$$

$$= \frac{\mid \overrightarrow{r}_B(t_B) - \overrightarrow{r}_A(t_A) \mid}{c} + 2\frac{G_{\oplus}M_E}{c^3} \ln\left(\frac{r_A(t_A) + r_B(t_B) + R_{AB}}{r_A(t_A) + r_B(t_B) - R_{AB}}\right) \quad . \tag{33}$$

This is the final formulae, frequently applied in many papers, can be found also in the review paper [31] by Sovers, Fanselow and Jacobs on $VLBI$ radio interferometry. The calculation is presented also in the mentioned papers [30], [16], [7], [27] and others. Note one apparent inconsistency of the above formulae (33) - it depends on the Euclidean distance $R_{AB} = \mid \overrightarrow{r}_B(t_B) - \overrightarrow{r}_A(t_A) \mid$between the points of emission $r_A(t_A)$ and reception $r_B(t_B)$ of the signal, which are taken at two different moments of time t_A and t_B. However, r_B will be known after the propagation time is calculated, so equation (33) represents a complicated transcendental equation with respect to the reception time t_B.

The Shapiro delay formulae (32) first was widely applied in the initial publications [5], [6] and others, which were dedicated to the problem about communications between stations on the Earth and satellites, accounting for GRT effects. Numerical estimates for the Shapiro delay term are also known [7] - for the signal between a geostationary satellite with orbital radius 42164 km with a clock on the equator at the same longitude, the Shapiro path delay is -27 ps (picoseconds, 1 $ps = 10^{-9}$ sec). Note that the minus sign is in agreement with the minus sign in front of the second (Shapiro) term in (29).

Later on, in the series of three publications [32], [33] and [34] a theoretical modelling has been performed for inter-satellite communications for the purposes of the $GRAIL$ (Gravity Recovery and Interior Laboratory) and the $GRACE$ (Gravity Recovery and Climate Experiment) space missions-each of these missions represent a system of two spacecrafts at a distance around 200 $miles$ and orbiting around either the Earth, the Moon or Mars. Precise ranging between the two spacecrafts in $GRAIL$ or $GRACE$ is achieved by applying General Relativity Theory.

7. Is it necessary to apply the light-like geodesic equations for finding the propagation time?

In the paper [32] it was asserted that the points of emission and reception A and B are joined by a time-like geodesics, but yet the Shapiro delay equation (33) was used and was called a "light-like" equation and the geodesic equation has not been used. The reason, from a theoretical point of view, is a theorem, proved in paragraph 38 in the known monograph by Fock [28]:

Theorem: Let

$$L = \sqrt{2F} = \sqrt{2\left(\frac{1}{2}g_{\alpha\beta}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp}\right)} \tag{34}$$

is the Lagrange function for the variational problem, when the extremum of the integral

$$s = \int_{p_1}^{p_2} L dp \tag{35}$$

is to be found and p is a space parameter along a curved line (It should be noted that in the proof in [28] it was not presumed that p is parameter along a geodesic line). Let also p is chosen so that the Lagrange equation of motion is satisfied

$$\frac{dF}{dp} = 0 \implies F = const \implies \frac{d}{dp}\frac{\partial F}{\partial \dot{x}_\alpha} - \frac{\partial F}{\partial x_\alpha} = 0 \quad , \tag{36}$$

which has the first integral

$$\dot{x}_\alpha \frac{\partial F}{\partial \dot{x}_\alpha} - F = F = const \quad , \quad \dot{x}_\alpha \equiv \frac{dx_\alpha}{dp} \quad . \tag{37}$$

Then the geodesic line equations $(\alpha, \beta, \nu = 0, 1, 2, 3)$

$$\frac{d^2 x_\nu}{dp^2} + \Gamma^\nu_{\alpha\beta}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp} = 0 \tag{38}$$

are compatible with the null-cone equation

$$F = g_{\alpha\beta}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp} = 0 \tag{39}$$

and in (38) $\Gamma^\nu_{\alpha\beta}$ are the affine connection components for a given metric $g_{\alpha\beta}$.

In other words, since the first integral (37) for the variational problem $s = \int_{p_1}^{p_2} L dp$ is valid also for the partial case $F = 0$, from the fulfillment of the null-cone equation (39) follows the fulfillment of the geodesic system of equations (39).

This theorem in the monograph by Fock has a great importance for the present investigation. If the propagation time T is found from the null-cone equation, it is not necessary to find the propagation time from the complicated nonlinear system of equations (38). In particular, this is the justification for choosing the parameter p not along the curved trajectory of the signal, but for example along the elliptic orbit of the satellite-both for the cases of plane motion of the satellite (characterized by the eccentric anomaly angle E), motion along a space-distributed orbit (characterized by the single parameter f - the true anomaly angle or the whole multitude of parameters $(M, a, e, \Omega, I, \omega)$, characterizing any disturbing motion.

Lastly, it should be mentioned that the relation between the null cone equation and the geodesic equations is a topic, not very well investigated in General Relativity Theory. Some issues will be mentioned in the conclusion part. But now one such issue can be mentioned as an example. After the variational problem is solved and the geodesic equations (38) are obtained, p becomes the parameter along the geodesic line. However, because of the condition $F = 0$ in (39), one may choose any other parameter p_1, not related to the geodesic line, i.e. one may write $F = g_{\alpha\beta}\frac{dx_\alpha}{dp_1}\frac{dx_\beta}{dp_1} = 0$ as well. This is the case, which is related to the further investigation. But $F = 1$ will be another integral of the system (38) and then one can write only $F = g_{\alpha\beta}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp} = 1$ [29] and not $g_{\alpha\beta}\frac{dx_\alpha}{dp_1}\frac{dx_\beta}{dp_1} = 0$.

8. New results for the propagation time in terms of elliptic functions-signal, emitted by a satellite on a plane elliptical orbit

8.1. The general case without any approximations

Since we have proved in the preceding section that an arbitrary parametrization of the space coordinates in the null cone equation

$$ds^2 = -c^2\left(1+\frac{2V}{c^2}\right)(dT)^2 + \left(1-\frac{2V}{c^2}\right)\left((dx)^2+(dy)^2+(dz)^2\right) = 0 \tag{40}$$

can be chosen, we shall investigate the most simple example of a satellite, moving along a plane elliptical orbit, parametrized by (5) $x = a(\cos E - e)$, $y = a\sqrt{1-e^2}\sin E$, $z = 0$. From the null cone equation after expressing dT, subsequent integration and taking into account, that the satellite velocity can be expressed as

$$\mathbf{v} = \sqrt{v_x^2+v_y^2} = \frac{na}{(1-e\cos E)}\sqrt{1-e^2\cos^2 E} \tag{41}$$

($n = \sqrt{\frac{G_\oplus M_\oplus}{a^3}}$ is the mean motion), the propagation time can be written as [22], [24]

$$T = \int \frac{\mathbf{v}}{c}\sqrt{\frac{(c^2-2V)}{(c^2+2V)}}\,dE = \frac{a}{c}\int\sqrt{\frac{a_1y^3+a_2y^2+a_3y}{b_1y^3+b_2y^2+b_3y+b_4}}dy \quad , \tag{42}$$

where the numerical constants a_1, a_2, a_3, b_1, b_2, b_3, b_4 can easily be calculated, y is the variable $y = 1 - e\cos E$. Integrals of the type (42) are not abelian ones (see the monograph by Prasolov and Solovyev [35]), because abelian integrals are related to algebraic curves $F(x, y) := x^2 - P(y) = 0$, where $P(y)$ is an algebraic polynomial. In the case for the integral (42), the under-integral expression is a rational function and not a polynomial. In the monograph [36], where various analytical solutions of most complicated integrals are presented, no analytical solutions of integrals of the type (42) can be found.

8.2. The weak field approximation $\frac{2V}{c^2} \ll 1$-theoretical and numerical considerations

In the book [28] two approximations are proposed for the metric (40)

$$U \ll c^2 \quad , \tag{43}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2 \ll c^2 \tag{44}$$

so that the metric (40) differs insignificantly from the metric

$$ds^2 = -c^2\left(1 + \frac{2V}{c^2}\right)(dT)^2 + \left((dx)^2 + (dy)^2 + (dz)^2\right) \quad . \tag{45}$$

However, it is not stated that the imposition of such assumptions is not absolutely necessary, and the metric (40) is considered to be the more precise metric. From this point of view, the problem about the calculation of the propagation time in the form of the complicated integral (42) is correctly stated. Moreover, the inequality $\frac{U}{c^2} \ll 1$ is valid only to a certain approximation, and it is important to find the precise approximation for its validity.

For the parameters of the GPS orbit - the typical radius $r_s = 26561$ $[km]$ of the GPS orbit, in the review paper [37] the constant

$$\beta = \frac{2V}{c^2} = \frac{2G_{\oplus}M_{\oplus}}{c^2 r_s} \ll 1 \tag{46}$$

β was calculated to be 0.334×10^{-9} [24] with the velocity of light taken to be $c = 299792458$ $[\frac{m}{\sec}]$ and the mass of the Earth is taken approximately to be $M_E \approx 5.97 \times 10^{24}$ $[kg]$. The geocentric gravitational constant $G_{\oplus}M_{\oplus}$ (obtained from the analysis of laser distance measurements of artificial Earth satellites) was taken to be equal to $G_{\oplus}M_{\oplus} = (3986004.405 \pm 1) \times 10^8$ $[\frac{m^3}{\sec^2}]$. The value of $G_{\oplus}M_{\oplus}$ can vary also in another range from $G_{\oplus}M_{\oplus} = 3986056.75236 \times 10^8$ $[\frac{m^3}{\sec^2}]$ to the value $G_{\oplus}M_{\oplus} = 3987999.07898 \times 10^8$ $[\frac{m^3}{\sec^2}]$ due to the uncertainties in measuring the Newton gravitational constant $G_{\oplus}$. One of the latest values for $G_{\oplus}$ from deep space experiments was reported in the paper [38] to be $(6.674 + 0.0003).10^{-11}$ $[\frac{m^3}{kg.\sec^2}]$.

Now let us point out one another numerical fact, which is interesting from the point of view of the preceding result. In the book [1] it was pointed out that at an altitude of 20184 km, due to the difference in the gravitational potential, the satellite atomic time runs faster by 45 $\mu\sec/d$ (microseconds per day, 1 $\mu\sec = 10^{-6}$ sec). Consequently, for one second the atomic time will run faster by $0.5208333.10^{-9}$ [sec]. In the following sections it will be proved that the dominant part in the propagation time dT (i.e. the part without the Shapiro delay term, which is very small) will be $dT = 0.0281341332790419$ [sec]. The corresponding to this propagation time interval dT atomic time is $d\tau = 0.0146531934.10^{-9}$ [sec] and can easily be found from 45 $\mu\sec/d$. The ratio of the two time intervals is equal to

$$\frac{d\tau}{dT} = \frac{0.0146531934.10^{-9}}{0.0281341332790419} = 0.52083329721.10^{-9} \ . \tag{47}$$

The very small atomic time interval compared to the propagation interval means that the atomic time can serve as a standard for measuring the propagation time, because it will be able to detect changes even at the nanosecond level.

It should be noted that this value is of the order of 10^{-9}, but it is a little greater than the value $\beta = \frac{2V}{c^2} = 0.334 \times 10^{-9}$

$$\frac{2V}{c^2 r} = \frac{2GM}{c^2 a(1 - e\cos E)} = \frac{\beta}{(1 - e\cos E)} = \frac{0.334.10^{-9}}{(1 - e\cos E)} \ll 1 \quad . \tag{48}$$

So due to the changing gravitational potential in our model ($r = a(1 - e\cos E)$ and $\frac{1}{(1-e\cos E)} > 1$)

$$\frac{2V}{c^2 r} > \beta \sim 10^{-9} \quad . \tag{49}$$

8.3. Propagation time in terms of a sum of elliptic integrals of the first, second and the third kind in the weak field approximation

This section presents the new results in the papers [22], [24]. From the null cone equation (40) and taking into account the weak-field approximation $\frac{2V}{c^2} \ll 1$, after decomposing the expression (29) into a sum and keeping only the first two sums, one can obtain

$$T = \int \frac{\mathbf{v}}{c}\sqrt{\frac{(1-\frac{2V}{c^2})}{(1+\frac{2V}{c^2})}}\,dt \approx \int \frac{\mathbf{v}}{c}(1-\frac{2V}{c^2})dt = I_1 + I_2 \tag{50}$$

$$= \frac{a}{c}\int \sqrt{1-e^2\cos^2 E}\;dE - \frac{2G_{\oplus}M_{\oplus}}{c^3}\int \sqrt{\frac{1+e\cos E}{1-e\cos E}}dE \quad . \tag{51}$$

The formulae in fact gives the propagation time T of the signal, emitted by the satellite at some initial position (given by the eccentric anomaly angle E_{init}), and the final point (given by E_{fin}) of reception of the signal by another satellite.

The coefficient $\frac{a}{c}$ as a ratio of the large semi-major axis of the orbit and the velocity of light $c = 299792458$ $[\frac{m}{\text{sec}}]$ has a dimension $[m/\frac{m}{\text{sec}}] = [\text{sec}]$. The second coefficient $\frac{2G_{\oplus}M_{\oplus}}{c^3}$ has a corresponding dimension $[\frac{m^3}{\text{sec}^2} : \frac{m^3}{\text{sec}^3}] = [\text{sec}]$, the two integrals $\int \sqrt{1-e^2\cos^2 E}$ and $\int \sqrt{\frac{1+e\cos E}{1-e\cos E}}dE$ after the integration are numbers and represent elliptic integrals of the second kind (first term in (51)) and a sum of elliptic integrals of the first and the third kinds respectively (second term in (51)). The general theory of elliptic integrals and curves is given in the monographs [35] and [43], more specific mathematical problems on a higher level are treated in [39], [41] and [42]. Elliptic integrals of first, second and third kinds are defined in the monographs [35], [44] and [45], but the corresponding definitions are briefly summarized also in the paper [24]. Various approaches for analytical calculation are given in the old book [36].

Let us now express the two integrals in (51) in terms of the various types of elliptic integrals. For the purpose, the first integral can be rewritten as

$$T_1 = \int_0^E \sqrt{1-e^2\cos^2 E}dE = -\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\overline{E}} \sqrt{1-e^2\sin^2 \overline{E}}d\overline{E} \quad , \tag{52}$$

where it was accounted that $\cos E = \sin(\frac{\pi}{2} - E)$ and also $k = e < 1$ (since e is the eccentricity of the orbit) represents the modulus of the elliptic integral.. The integral (52) represents an elliptic integral of the second kind [35].

In order to calculate the second integral $T_2 = -\frac{2G_{\oplus}M_{\oplus}}{c^3}\int \sqrt{\frac{1+e\cos E}{1-e\cos E}}dE$ in (51), let us perform the substitution

$$\sqrt{\frac{1+e\cos E}{1-e\cos E}} = \overline{y} \tag{53}$$

and introduce also the notations

$$-k^2 = \widetilde{k}^2 = \frac{1-e}{1+e} = q \quad , \quad \frac{\overline{y}}{\widetilde{k}} = y \quad , \quad \left(\frac{\overline{y}}{k}\right)^2 = \left(\frac{\overline{y}}{\widetilde{k}}\right)^2 = \widetilde{y} \quad , \quad \widetilde{y} = -\widetilde{\widetilde{y}} \quad . \tag{54}$$

Then the integral T_2 can be represented as a sum of two integrals, i.e. $T_2 = I_2^{(A)} + I_2^{(B)}$. The first integral is

$$I_2^{(A)} = \frac{4GM}{c^3}\frac{1}{\widetilde{k}\; i\sqrt{1-e^2}\; i}\int \frac{d\widetilde{\widetilde{y}}}{\sqrt{\widetilde{\widetilde{y}}\left(\widetilde{\widetilde{y}}+1\right)\left(\widetilde{\widetilde{y}}+\frac{1}{\widetilde{k}^4}\right)}} \tag{55}$$

is of zero - order and of the first kind and is in the s.c. Weierstrass form (third-order polynomial under the square root in the under-integral expression). The modulus of this elliptic integral can be calculated if by a suitable linear variable transformation of $\widetilde{y}$ the integral is transformed to an integral in the s.c. Legendre form $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, associated with the elliptic curve $y^2 = (1-x^2)(1-k^2x^2)$ (k is the modulus of the elliptic integral).

It is important to note that the integral is a real-valued expression, because the propagation time should be real. This fact again confirms the of the applied approach of choosing the space coordinates parametrization to coincide with the orbit of the satellite.

The second integral $I_2^{(B)}$ in the expression for $T_2 = I_2^{(A)} + I_2^{(B)}$ can also be written in the form of a real-valued expression

$$I_2^{(B)} = \frac{4GM}{c^3q^2}\frac{1}{\sqrt{1-e^2}}\int \frac{d\widetilde{y}}{(\widetilde{y}-\frac{1}{q})\sqrt{\widetilde{y}(\widetilde{y}+1)(\widetilde{y}+\frac{1}{q^2})}} \quad . \tag{56}$$

This is an elliptic integral of the third kind.

8.4. Numerical calculation of the propagation time of a signal, emitted by a satellite on a plane elliptical orbit

8.4.1. Numerical calculation of the first six iterations of the eccentric anomaly angle E We shall perform some numerical calculations of the propagation time, based on the derived formulae (51). The numerical data about the parameters of the GPS orbit are taken from the PhD dissertation [16] and are known with great precision. From all the parameters, listed below only the first three will be used in the calculation, performed by the online program web2.0 scientific calculator [46]

semi-major axis a 26560.25169632944 $[km]$,
eccentricity e 0.01323881349526 ,
mean anomaly M -0.3134513508155 $[rad]$,
inclination I 0.9614884100802 $[rad]$,
longitude of the ascending node Ω -0.4495096737336 $[rad]$,
argument of perigee ω -3.001488651204 $[rad]$.

Most important is to find the eccentric anomaly angle E from the Kepler equation $M = E - e\sin E$. The calculation will be based on formulaes (2.54) and (2.55) in Ch.2 of the monograph [11].

The iterative sequence of formulaes are

$$E_{(i+1)} = M + e\sin E_{(i)} \quad , \quad i = 0,1,2,....., \tag{57}$$

where for the first approximation it is assumed $E_0 = M = -0.3134513508155$ $[rad]$ The first three iterative solutions are given according to the following formulaes:

$$E_{(1)} = M + e\sin M \quad , \tag{58}$$

$$E_{(2)} = M + e\sin E_{(1)} = M + e\sin(M + e\sin M) \quad , \tag{59}$$

$$E_{(3)} = M + e\sin E_{(2)} = M + e\sin[M + e\sin(M + e\sin M)] \quad . \tag{60}$$

Thus from the formulaes for $E_{(3)}$, $E_{(5)}$ and $E_{(6)}$ the following numerical values can be found

$$E_{(3)} = M + e\sin E_{(2)} = -0.31758547588467897473 \quad [rad] \quad , \tag{61}$$

$$E_{(4)} = M + e\sin E_{(3)} = -0.31758548401096719083 \quad [rad] \quad , \tag{62}$$

$$E_{(5)} = M + e\sin E_{(4)} = -0.31758548411317083102\ [rad] \quad , \tag{63}$$

$$E_{(6)} = M + e\sin E_{(5)} = -0.31758548411445499592\ [rad] \quad . \tag{64}$$

The approximation $E_{(5)}$ up to the ninth digit is identical with $E_{(4)}$ and $E_{(6)}$ up to the eleventh digit is identical with $E_{(5)}$. The approximation $E_{(6)}$ also up to the seventh digit is identical with $E_{(3)}$, given by (61). However, if the approximate value for $E_{(3)}$ is compared with the value for $E_{(6)}$, the coincidence is only up to the fifth digit.

It is therefore necessary to check what is the impact of the approximations in the eccentric anomaly E on the approximations of the time of propagation of the signal.

8.4.2. Numerical calculation of the first $O(\frac{1}{c})$ correction in the propagation time The corresponding elliptic integral (52) of the second kind for the eccentric anomaly approximation $E_{(3)}$ is

$$T_1^{(E_3)} = \int_0^{E_{(3)}} \sqrt{1 - e^2\cos^2 E}\, dE = -0.317557268125933936045 \quad . \tag{65}$$

If we compare this value with the value (61) $E_{(3)} = -0.31758547588467897473\ [rad]$, then it can be noted that the integration changes the value of $E_{(3)}$ after the fourth digit after the decimal dot.

The same integral with $E_{(6)}$ as an upper integration limit is

$$T_1^{(E_6)} = \int_0^{E_{(6)}} \sqrt{1 - e^2\cos^2 E}\, dE = -0.317558568963886638536 \quad . \tag{66}$$

This value is different from the value (65) after the fifth digit, so it is a better approximation. However, if compared with $E_{(6)} = -0.3175854841144549959 29\ [rad]$, the integration in (66) changes the value of $E_{(6)}$ after the fourth digit after the decimal dot.

The calculation of the first $O(\frac{1}{c})$ time correction $\frac{a}{c}T_1$ for the values of the GPS orbit and for the eccentric anomaly $E_{(3)}$ gives the following numerical value

$$\frac{a}{c}T_1^{(E_3)} = -0.0281341332790419\ \text{[sec]} \quad . \tag{67}$$

Correspondingly, the first $O(\frac{1}{c})$ time correction $\frac{a}{c}T_1^{(E_6)}$ for the eccentric anomaly $E_{(6)}$ is

$$\frac{a}{c}T_1^{(E_6)} = -0.0281342485273829\ \text{[sec]} \quad . \tag{68}$$

So the two $O(\frac{1}{c})$ time corrections (67) and (68) are identical up to the sixth digit after the decimal dot. Consequently, the important conclusion is that the eccentric anomaly at the sixth approximation level can ensure microsecond stability of the $O(\frac{1}{c})$ time correction. This means also that the third approximation $E_{(3)}$ can be reliably used, at least at the microsecond level.

One more argument in support of the microsecond approximation is that for a satellite, moving along circular orbit (when $e = 0$) the above expression acquires the form:

$$\frac{a}{c}t_{1(circular)}^{(E_3)} = \frac{a}{c}E_{(3)} = -0.028136517830159266\ \text{[sec]} = -28136.517830159266\ [\mu\,\text{sec}] \quad . \tag{69}$$

Evidently, the coordinate time for the circular orbit differs from the coordinate time for the GPS elliptic orbit only after the fifth digit after the decimal dot.

8.4.3. Numerical calculation of the Shapiro delay term (the $O(\frac{1}{c^3})$ time correction) Now let us calculate numerically the whole under-integral expression in formulae (51) for the propagation time, taking into account the 25−th digit after the decimal dot

$$\frac{a}{c}\sqrt{1-e^2\cos^2 E}-\frac{2GM}{c^3}.\sqrt{\frac{1+e\cos E_{(3)}}{1-e\cos E_{(3)}}}=0.0885884561709019072629880 \tag{70}$$

$$-0.0000000000299618094618168=0.0885884561409400978011712 \quad . \tag{71}$$

The second Shapiro delay term contains ten zeroes after the decimal dot, so the overall result of the calculation of both terms in (70) - (71) is changed by the Shapiro term at and after the picosecond level (1 $p\sec = 10^{-9}$ sec). This result, based on the simple application of the celestial mechanics approach in the calculation of the propagation time is interesting, if compared with the calculations, based on the formulae (32) $T=\frac{R_{AB}}{c}+2\frac{G_{\oplus}M_E}{c^3}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)$. From the literature it can be seen that relativistic effects on light propagation from Satellite Laser Ranging (SLR) data are measured with an accuracy of 1 μm (1 micrometer), which corresponds to 0.01 $p\sec$ (see also paper [47]). The relativistic observables, defined for the $GRAIL$ mission in [32] are also accurate to 1 μm.

Let us present one final proof for the correctness of the applied in the papers [22], [23], [24] and [25], comparing the calculated propagation time of the signal with the celestial time t_{cel}, which can be calculated from $M=n(t-t_P)$ (M is the mean anomaly with the given numerical data), where t_P is the initial time of perigee passage.

$$\frac{\text{propagation time for electromagnetic signals}}{\text{celestial time from Kepler equation}}=\frac{0.0281179694658269766628639[sec]}{37.508256148[sec]} \tag{72}$$

$$=0.000749647473722373770335759582 \quad . \tag{73}$$

In other words, the propagation time is 10^4 times greater than the celestial time, which will mean that for this celestial time 37.5082561[sec] the satellite moves at a distance 145.125[km] (the velocity of the satellite is taken to be $v_S=3.874$ [$km/\sec$]) and the light signal will move at a distance 8435.3908[km].

9. New results for the propagation time in terms of elliptic functions-signal, emitted by a satellite on a space-distributed elliptical orbit

9.1. Parametrization of the space coordinates

Now we shall investigate the more complicated case of the propagation time of a signal, emitted by a satellite on a space distributed orbit [24]. The parametrization (with some minor differences) can be found in nearly all books on celestial mechanics, but a particularly clear derivation is given in the monograph [11]. Below we shall use the parametrization in the well-known monograph [8]

$$x=\frac{a(1-e^2)}{1+e\cos f}\left[\cos\Omega\cos(\omega+f)-\sin\Omega\sin(\omega+f)\cos I\right] \quad , \tag{74}$$

$$y=\frac{a(1-e^2)}{1+e\cos f}\left[\sin\Omega\cos(\omega+f)+\cos\Omega\sin(\omega+f)\cos I\right] \quad , \tag{75}$$

$$z=\frac{a(1-e^2)}{1+e\cos f}\sin(\omega+f)\sin I \quad , \tag{76}$$

where $r=\frac{a(1-e^2)}{1+e\cos f}$ is the radius-vector in the orbital plane, the angle Ω of the longitude of the right ascension of the ascending node is the angle between the line of nodes and the direction to the vernal equinox, the argument of perigee (periapsis) ω is the angle within the orbital plane from the ascending node to perigee in the direction of the satellite motion ($0 \leq \omega \leq 360^0$). The angle I is the inclination of the orbit with respect to the equatorial plane and the true anomaly f geometrically represents the angle between the line of nodes and the position vector $\overrightarrow{r}$ on the orbital plane. Since the angle f is related to the motion of the satellite and all the other parameters of the orbit do not change during the motion of the satellite, it can easily be found that

$$\sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{(v_f^x)^2 + (v_f^y)^2 + (v_f^z)^2} df = v_f df \quad , \tag{77}$$

where the velocity v_f , associated to the true anomaly angle f is given by

$$v_f = v = \frac{na}{\sqrt{1-e^2}} \sqrt{1 + e^2 + 2e\cos f} \quad . \tag{78}$$

Making use of the null cone equation (40) and also of the approximation $\beta = \frac{2V}{c^2} = \frac{2G_{\oplus}M_{\oplus}}{c^2 a} \ll 1$, one can obtain the general formulae for the propagation time in the form

$$T = \int \frac{v}{c}(1 - \frac{2V}{c^2})dt = \widetilde{T}_1 + \widetilde{T}_2 = \frac{1}{c}\int v dt - \frac{2}{c^3}\int vV dt \quad . \tag{79}$$

As it will be shown, the corrections $\widetilde{T}_1$ has two parts, but the whole correction $\widetilde{T}_2$ does not have two parts. The first part $\widetilde{T}_1$ can be written as

$$\widetilde{T}_1 = \frac{1}{c}\int v_f df = \frac{na}{c\sqrt{1-e^2}} \int \sqrt{1 + e^2 + 2e\cos f} df = \frac{na}{c\sqrt{1-e^2}} (\overline{T}_1^{(1)} + \overline{T}_1^{(2)}) \quad . \tag{80}$$

It is interesting that $\widetilde{T}_1$can be analytically calculated without the use of elliptic integrals, but also with the application of elliptic integrals.

9.2. Calculation of $\overline{T}_1^{(1)} + \overline{T}_1^{(2)}$ without elliptic integrals

For the first case without the use of elliptic integrals, the integral $\widetilde{T}_1^{(2)}$ is calculated in [24] as

$$\overline{T}_1^{(2)} = -\frac{(1+e)\sqrt{2}}{\sqrt{q}\sqrt{3e^2+2e+3}} \ln\left[\left(\frac{\sqrt{2}m_1(f_b;r_b) + \frac{1}{2}m_2(f_b;r_b)}{\sqrt{2}m_1(f_a;r_a) + \frac{1}{2}m_2(f_a;r_a)}\right)\left(\frac{m_3(f_a;r_a)}{m3(f_b;r_b)}\right)\right] \tag{81}$$

and $m_1(f;r)$, $m_2(f;r)$, $m_3(f;r)$ are complicated expressions, written in terms either of the initial and final true anomaly angles f_a and f_b or, of the initial distance r_a (at which the emission of the signal takes place) and the final point r_b of reception on the same orbit, corresponding to the propagation time $\overline{T}_1^{(2)}$. The first expression for $\overline{T}_1^{(1)}$ can be written in the same manner, since $\overline{T}_1^{(1)}$ and $\overline{T}_1^{(2)}$ are calculated from the derived integral [24]

$$\overline{T}_1^{(1)} + \overline{T}_1^{(2)} = -\frac{i}{\sqrt{q}}\left[\int \frac{d\overline{m}}{\left(\overline{m} - \frac{q^2}{2}\right)\sqrt{\overline{m} - \frac{q^2}{2}}} + \int \frac{d\overline{m}}{\left(\overline{m} + \frac{q^2}{2}\right)\sqrt{\overline{m} - \frac{q^2}{2}}}\right] \quad , \tag{82}$$

where the expression for $\overline{T}_1^{(2)}$ is obtained after integration of the second integral in (82) and the expression for $\overline{T}_1^{(1)}$- from the first integral in (82). The integration of both integrals is performed by using a formulae from the monograph [36] and given also in the paper [24].

9.3. Calculation of $\widetilde{T}_1$ by means of elliptic integrals of the second order

The integral (80) for $\widetilde{T}_1 = \frac{na}{c\sqrt{1-e^2}} \int \sqrt{1+e^2+2e\cos f}df$ can be calculated by making use of the substitution [24]

$$y = \sqrt{\frac{1}{q}\frac{(1+e\cos E)}{(1-e\cos E)}} \quad , \ q = \frac{1-e}{1+e} \tag{83}$$

and also of the well-known relation (11) $\tan\frac{f}{2} = \sqrt{\frac{1-\cos f}{1+\cos f}} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$ from celestial mechanics between the eccentric anomaly angle E and the true anomaly angle f [8]. Note that from the integral (80) it is not clear that it will be an elliptic integral, this becomes clear after the application of the convenient transformation (83).

Then we can write the integral $\widetilde{T}_1$ in the form of an elliptic integral of the second order (because of the term y^2 in the nominator), which is also of the first kind in the Legendre form

$$\widetilde{T}_1 = -2i\frac{na}{c}q^{\frac{3}{2}}\int\frac{y^2dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = -2i\frac{na}{c}q^{\frac{3}{2}}\widetilde{J}_2^{(4)}(y;q) \quad . \tag{84}$$

Note that for this calculation with elliptic integrals, there is no need to represent $\widetilde{T}_1$ in the form of any sum $\widetilde{T}_1 = \widetilde{T}_1^{(1)} + \widetilde{T}_1^{(2)}$. If we compare the last formulae with the formulae (80) $\widetilde{T}_1 = \frac{1}{c}\int v_f df = \frac{na}{c\sqrt{1-e^2}}(\overline{T}_1^{(1)} + \overline{T}_1^{(2)})$ (expressed by functions, which are not elliptic), two important conclusions can be made:

A. The elliptic integral $\widetilde{J}_2^{(4)}(y;q)$ can be expressed analytically. The paper [25] is dedicated to this problem (see the references there), and further some basic facts about a new approach will be mentioned briefly.

B. The elliptic integral $\widetilde{J}_2^{(4)}(y;q)$ is imaginary, so that the expression for the propagation time $\widetilde{T}_1$ is real-valued, as this was proved for expressions (55) and (56). Further this fact for $\widetilde{T}_1$ shall also be proved.

The integral $\widetilde{J}_2^{(4)}(y;q)$ can also be represented as

$$\widetilde{J}_2^{(4)}(y;q) = \int\frac{y^2dy}{\sqrt{(1-y^2)(1-q^2y^2)}} = -\frac{1}{q^2}\int\sqrt{1-q^2\sin^2\varphi}d\varphi + \frac{1}{q^2}\int\frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}} \quad , \tag{85}$$

which can be proved very easily [24]. The first integral in (85) is an elliptic integral of the second kind (denoted usually by $E(\varphi) = \int\sqrt{1-q^2\sin^2\varphi}d\varphi$, where $y = \sin\varphi$). So we obtain a relation between the second-order elliptic integral $\widetilde{J}_2^{(4)}(y;q)$ of the first kind in the Legendre form, the zero-order elliptic integral of the first kind in the Legendre form $\widetilde{J}_0^{(4)}(y;q)$ (the second integral in (85)) and the elliptic integral $E(\varphi)$. Elliptic integrals of higher order are investigated in the monographs [35], [44] and [45], the other cited above monographs [39], [41] and [42] do not deal with elliptic integrals of higher order, but only with mathematical problems of elliptic integrals and curves. A short summary of the different kinds of higher-order elliptic integrals is given in the paper [24]. For the moment, it is important to mention that the upper subscript in the notation for $\widetilde{J}_2^{(4)}(y;q)$ means that the polynomial in the denominator is of the fourth degree, the lower subscript 2 means that the integral is of the second order. Further analogous notations will also appear, when calculating the $O(\frac{1}{c^3})$ part of the integral (79).

9.4. Calculation of the second part of the integral (79) (the $O(\frac{1}{c^3})$ time correction) by means of elliptic integrals of the fourth order

Let us calculate the second part

$$T_2 = -\frac{2}{c^3}\int vV dt = -\frac{2G_{\oplus}M_{\oplus}}{c^3}\int \frac{v_f}{r}df \tag{86}$$

in the integral (79). The integral now acquires the form

$$T_2 = -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{na}{a\sqrt{1-e^2}}\int(1+e\cos f)\sqrt{1+2e\cos f+e^2}df = T_2^{(1)} + T_2^{(2)} \quad , \tag{87}$$

which allows to represent it as a sum of two parts $T_2^{(1)} + T_2^{(2)}$. The first integral is

$$T_2^{(1)} = -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{n}{(1-e^2)^{\frac{3}{2}}}\int\sqrt{1+2e\cos f+e^2}df = -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{n}{(1-e^2)^{\frac{3}{2}}}\widetilde{T}_1 \tag{88}$$

and is therefore equal to the previously calculated in (84) integral $\widetilde{T}_1$, multiplied by the coefficient $-\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{n}{(1-e^2)^{\frac{3}{2}}}$. The second term $T_2^{(2)}$ in the second correction T_2 is

$$T_2^{(2)} = -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{ne}{(1-e^2)^{\frac{3}{2}}}\int\cos f\sqrt{1+2e\cos f+e^2}df \;= -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{ne}{(1-e^2)^{\frac{3}{2}}}\widetilde{T}_2^{(2)} \quad , \tag{89}$$

where $\widetilde{T}_2^{(2)}$ is the notation for the more complicated integral

$$\widetilde{T}_2^{(2)} = \int\cos f\sqrt{1+2e\cos f+e^2}df \quad . \tag{90}$$

The whole correction $T_2^{(2)}$ is calculated in [24] to be

$$T_2^{(2)} = -inq^{\frac{5}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}(1+e^2)\widetilde{J}_2^{(4)}(\widetilde{y},q) + inq^{\frac{3}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}\frac{(1+e^2)}{(1-e^2)}\;\widetilde{J}_4^{(4)}(\widetilde{y},q) \quad , \tag{91}$$

where $\widetilde{J}_2^{(4)}(\widetilde{y},q)$ is the previous integral (85) and $\widetilde{J}_4^{(4)}(\widetilde{y},q)$ is the new, fourth-order elliptic integral

$$\widetilde{J}_4^{(4)}(\widetilde{y},q) = \frac{q^5}{i}\int\frac{\widehat{y}^4 d\widehat{y}}{\sqrt{(\widehat{y}^2-1)(1-q^2\widehat{y}^2)}} \quad , \tag{92}$$

written in terms of the variable $\widehat{y} = \frac{\widetilde{y}}{q} = \frac{\sqrt{1+2e\cos f+e^2}}{q}$.

9.5. A mathematical proof for the real-valuedness of all the $O(\frac{1}{c})$ and $O(\frac{1}{c^3})$ time corrections for the case of space-distributed orbits. The recurrent system of equations and finding the second and fourth order elliptic integrals

Proving the real-valuedness of all the components of the propagation time is an important step in proving the correctness of choosing the parametrization (74) - (76) of the space coordinates x, y, z in terms of the six Keplerian elements $(M, a, e, \Omega, I, \omega)$. For the previous case of plane orbits the proof was performed also in [24], but for this case of space-distributed orbits the proof will be much more complicated.

9.5.1. Real-valuedness of the expression for $\widetilde{T}_1$ Let us begin first with the proof of the real-valuedness of the first $O(\frac{1}{c})$ correction (80) $\widetilde{T}_1 = \frac{na}{c\sqrt{1-e^2}} \int \sqrt{1+e^2+2e\cos f} df$ [24]. From the representation of the integral (84) $\widetilde{T}_1 = -2i\frac{na}{c}q^{\frac{3}{2}}\widetilde{J}_2^{(4)}(y;q)$ it follows that $\widetilde{T}_1$ will be a real-valued expression if the second order elliptic integral $\widetilde{J}_2^{(4)}(y;q)$ will turn out to be imaginary.

For the purpose, let us perform the variable change

$$\widetilde{y} = \frac{\sqrt{1+2e\cos f+e^2}}{1+e} = \frac{y}{1+e} \qquad , \tag{93}$$

after which the integral acquires the form

$$\widetilde{T}_1 = i\frac{2na(1+e)}{cq\sqrt{1-e^2}} \int \frac{\widetilde{y}^2 d\widetilde{y}}{\sqrt{(1-\widetilde{y}^2)\left(1-\frac{\widetilde{y}^2}{q^2}\right)}} = \frac{2na(1+e)}{cq\sqrt{1-e^2}} \int \frac{\widetilde{y}^2 d\widetilde{y}}{\sqrt{(1-\widetilde{y}^2)\left(\frac{\widetilde{y}^2}{q^2}-1\right)}} \quad . \tag{94}$$

Now it is evident that if $\widetilde{T}_1$ is real, then the following identification should be made

$$\widetilde{J}_2^{(4)}(\widetilde{y},\frac{1}{q}) = \frac{1}{i}\int \frac{\widetilde{y}^2 d\widetilde{y}}{\sqrt{(1-\widetilde{y}^2)\left(\frac{\widetilde{y}^2}{q^2}-1\right)}} = \int \frac{\widetilde{y}^2 d\widetilde{y}}{\sqrt{(1-\widetilde{y}^2)\left(1-\frac{\widetilde{y}^2}{q^2}\right)}} \quad . \tag{95}$$

So let us prove that the expression $\left(1-\widetilde{y}^2\right)\left(1-\frac{\widetilde{y}^2}{q^2}\right)$ inside the square in the denominator in the second integral can take negative values. Respectively, then $\left(1-\widetilde{y}^2\right)\left(\frac{\widetilde{y}^2}{q^2}-1\right)$ in the first integral will be positive and thus in both cases the elliptic integral $\widetilde{J}_2^{(4)}(\widetilde{y},\frac{1}{q})$ will not be real-valued. Remembering the definition for the variable change (93) and also that $\cos f \leq 1$, the following series of inequalities can easily be proved

$$\widetilde{y} \leq 1 \ \ , \ \triangleright\, 1-\widetilde{y}^2 \geq 0 \ \ , \quad 1-\frac{\widetilde{y}^2}{q^2} \geq -\frac{(1-q^2)}{q^2} = 1-\frac{1}{q^2} \ = -\frac{4e}{(1-e)^2} \ . \tag{96}$$

Since $1-\frac{\widetilde{y}^2}{q^2}$ can take negative values from $-\frac{(1-q^2)}{q^2}$ to 0 and keeping in mind the definition $q^2 = \left(\frac{1-e}{1+e}\right)^2 < 1$, it becomes clear that $\widetilde{T}_1$ in (94) is a real-valued expression. Let us clarify also the statement "the expression inside the square in the denominator can take negative values".

It does not mean that only for values from $-\frac{(1-q^2)}{q^2}$ to 0 the expression for $\widetilde{J}_2^{(4)}(\widetilde{y},\frac{1}{q})$ is imaginary!

Let us present a more elegant and simple proof that the inequalities 96 are fulfilled. Inverting the sign of the last inequality in (96), we obtain the inequality

$$\frac{\widetilde{y}^2}{q^2} - 1 \leq \frac{4e}{(1-e)^2} \quad , \tag{97}$$

should be proved. Keeping in mind the definition for the variable $\widetilde{y}^2$ (93) and also for q^2, the inequality (97) is transformed to

$$\frac{(1+2e\cos f+e^2)}{(1+e)^2} - \frac{(1-e)^2}{(1+e)^2} \leq \frac{4e}{(1-e)^2}q^2 = \frac{4e}{(1+e)^2} \quad . \tag{98}$$

From here it follows

$$1+2e\cos f+e^2 \leq 4e+(1-e)^2 \qquad \Rightarrow 2e\cos f \leq 2e \quad , \tag{99}$$

which is fulfilled for all eccentricities e, because of the simple trigonometric inequality $\cos f \leq 1$ for the function cos and also because $0 < e < 1$.

Returning to the defining expression (84) $\widetilde{T}_1 = -2i\frac{na}{c}q^{\frac{3}{2}}\widetilde{J}_2^{(4)}(y;q)$ and after having proved that $\widetilde{T}_1$ is real-valued, it follows that $\widetilde{J}_2^{(4)}(y;q)$ (and also the integral (95) for $\widetilde{J}_2^{(4)}(\widetilde{y},\frac{1}{q})$) are also imaginary expressions.

9.5.2. Real-valuedness of the expression for $T_2^{(2)}$- proof by means of the recurrent system of equations for the elliptic functions According to formulae (91), $T_2^{(2)}$ was represented as a sum of two imaginary expressions $-inq^{\frac{5}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}(1+e^2)\widetilde{J}_2^{(4)}(\widetilde{y},q)$ and also $+inq^{\frac{3}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}\frac{(1+e^2)}{(1-e^2)}\widetilde{J}_4^{(4)}(\widetilde{y},q)$. Since we proved that $\widetilde{J}_2^{(4)}(\widetilde{y},q)$ is an imaginary expression, it remains to prove that $\widetilde{J}_4^{(4)}(\widetilde{y},q)$ is also an imaginary expression.

The proof now will be made in another way, by using the recurrent system of equations for the elliptic functions (see again the monographs [35], [44] and [45]), obtained after calculating the derivatives

$$\frac{d}{d\widetilde{y}}\left(\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}\right) \quad \text{and} \quad \frac{d}{d\widetilde{y}}\left(\widetilde{y}\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}\right) \tag{100}$$

and integrating the resulting equations from $\widetilde{y}_0$ to $\widetilde{y}_1$. Combining all the three equations, the following two equations can be derived for $\widetilde{J}_4^{(4)}(\widetilde{y},q)$ and $\widetilde{J}_3^{(4)}(\widetilde{y},q)$

$$\widetilde{J}_4^{(4)}(\widetilde{y},q) = \frac{1}{3q^2}\left[\widetilde{y}\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}\right]|_{\widetilde{y}=\widetilde{y}_0}^{\widetilde{y}=\widetilde{y}_1} + \frac{2(1+q^2)}{3q^2}\widetilde{J}_2^{(4)}(\widetilde{y},q) - \frac{1}{3q^2}\widetilde{J}_0^{(4)}(\widetilde{y},q) \quad , \tag{101}$$

$$\int\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}d\widetilde{y} = \frac{2}{3}\widetilde{J}_0^{(4)}(\widetilde{y},q) - \frac{1}{3}(1+q^2)\widetilde{J}_2^{(4)}(\widetilde{y},q) + \frac{1}{3}\left[\widetilde{y}\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}\right]|_{\widetilde{y}=\widetilde{y}_0}^{\widetilde{y}=\widetilde{y}_1} . \tag{102}$$

In the preceding section it was proved that the integral $\widetilde{J}_2^{(4)}(\widetilde{y},q)$ is imaginary, consequently since it enters expression (101) for $\widetilde{J}_4^{(4)}(\widetilde{y},q)$, this integral is also an imaginary one. Also, the integral $\int\sqrt{(1-\widetilde{y}^2)(1-q^2\widetilde{y}^2)}d\widetilde{y}$ will also be an imaginary one, because the integral $\widetilde{J}_2^{(4)}(\widetilde{y},q)$ enters the right-hand side of expression (102). Of course, in order to claim that the imaginary contribution $\frac{2(1+q^2)}{3q^2}\widetilde{J}_2^{(4)}(\widetilde{y},q)$ in (101) does not cancel with any of the other two contributions, one should investigate the first and the third terms in (101). Such a cancellation is not likely to happen, but the more rigorous mathematical approach would require to investigate whether the elliptic integral of zero-order in the Legendre form $\widetilde{J}_0^{(4)}(\widetilde{y},q)$ will be real-valued or imaginary. In monographs on elliptic functions, which are more related to the theory of analytical functions [48], [49], [44] such an analysis has been performed.

9.6. Real-valuedness and complex valuedness of elliptic integrals of the zero-order in the Legendre form-basic knowledge about the Christoffel-Schwartz integral

Zero-order elliptic integrals in the Legendre form such as $J_0^{(4)}(y,q) = \int\frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}}$ are analyzed by means of the well-known Christoffel - Schwartz integral (see especially [48], [49], [44], [50], but also many other monographs on analytical functions)

$$w = f(z) = C\int_0^z(s-a_1)^{\alpha_1-1}(s-a_2)^{\alpha_2-1}.......(s-a_n)^{\alpha_n-1}ds + C_1 \quad , \tag{103}$$

which represents a conformal mapping of the upper complex half-plane onto the inner part of an $n-$polygon. In the last formulae a_1,a_2,......a_n are points on the real axis and α_1, α_2....α_n denote the inner angles, represented by real numbers. Each one of these numbers is not greater than 2 and for them the following equality

$$\alpha_1 + \alpha_2 + ... + \alpha_n = n - 2 \tag{104}$$

is fulfilled. For the four-dimensional case of the zero-order integral $J_0^{(4)}(y,q)$, formulae (103) represents a mapping of the upper complex half-plane onto the rectangle (i.e. the $4-$polygon, which for the case turns out to be the rectangle). So the integral $J(y,q)$ is a partial case of the integral (103) for the following values of the parameters α_n, a_n and the constants C_1 and C:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2} \quad , \quad a_1 = -\frac{1}{q} \quad , \quad a_2 = -1 \ , \tag{105}$$

$$a_3 = 1 \quad , \quad a_4 = \frac{1}{k} \ , \quad C_1 = 0 \quad , \quad C = \frac{1}{q} \quad . \tag{106}$$

Therefore, the integral $J_0^{(4)}(y,q)$ can be represented in the form of the following Christoffel-Schwartz integral

$$J_0^{(4)}(y,q) = f(z) = \frac{1}{q}\int\limits_0^z \left(y + \frac{1}{q}\right)^{\frac{1}{2}-1} (y+1)^{\frac{1}{2}-1} (y-1)^{\frac{1}{2}-1} \left(y - \frac{1}{q}\right)^{\frac{1}{2}-1} dy \quad . \tag{107}$$

At the points a_1,a_2,a_3,a_4 the under-integral function in the integral $J_0^{(4)}(y,q)$ (similarly - also in the integral (103) is a branching multi-valued function, which maps the points on the real axis onto the points on the segments of the rectangle, situated on the complex plane. Each expression $(s-a_k)^{\alpha_k - 1}$ in (103) should be interpreted as the branch of the multi-valued function which on the real axis (when $z = s > a_k$) takes real positive values. Indeed, if s is in the segment $[0,1]$, then the function $f(z)$ in [48] takes values from zero to the value of the elliptic integral of the first kind:

$$K = \int\limits_0^1 \frac{ds}{\sqrt{(1-s^2)(1-q^2s^2)}} \quad . \tag{108}$$

In fact, this is an equivalent formulation of the period of the elliptic integral $K = F(\frac{\pi}{2}, e) = \int\limits_0^{\frac{\pi}{2}} \frac{d\overline{E}}{\sqrt{1-e^2\sin^2\overline{E}}}$. Since the function $\left(1-s^2\right)\left(1-q^2s^2\right)$ is a two-valued function at the branching points $s = \pm 1$ and $s = \pm\frac{1}{q}$, one has to choose the under-integral function so that to ensure the non-interruptness of this function, when $1 < s < \frac{1}{q}$. This means that for this case it should be written as [48]

$$\frac{1}{\pm i\sqrt{(s^2-1)(1-q^2s^2)}} \quad . \tag{109}$$

Consequently, for $s \subset [1, \frac{1}{q}]$ the function $w = f(z)$ maps the interval $[1, \frac{1}{q}]$ into the interval $[K, K + i\widetilde{K}^{,}]$ on the complex plane, where

$$\widetilde{K}^{,} = \int\limits_1^{\frac{1}{q}} \frac{ds}{\sqrt{(s^2-1)(1-q^2s^2)}} \quad . \tag{110}$$

For $s \subset [1, \frac{1}{q}]$ the function $w = f(z)$ (107) maps the interval $[1, \frac{1}{q}]$ into the interval $[K, K + i\widetilde{K}^{,}]$ on the complex plane, where

$$\widetilde{K}^{,} = \int_{1}^{\frac{1}{q}} \frac{ds}{\sqrt{(s^2-1)(1-q^2s^2)}} \quad . \tag{111}$$

Then the integral in (107) can be written as:

$$\int_{0}^{y} \frac{ds}{\sqrt{(1-s^2)(1-q^2s^2)}} = K + i\widetilde{K}^{,} = \tag{112}$$

$$= \int_{0}^{1} \frac{ds}{\sqrt{(1-s^2)(1-q^2s^2)}} + i\int_{1}^{y} \frac{ds}{\sqrt{(s^2-1)(1-q^2s^2)}} \quad . \tag{113}$$

In the same way,similarly the analysis can be performed for the other intervals on the real axis. Thus the following theorem about the Christoffel-Schwartz integral can be proved:

Theorem. The intervals $[0, 1]$, $[1, \frac{1}{q}]$, $[\frac{1}{q}, +\infty)$ on the real axis are mapped by the Christoffel - Schwartz integral (107) onto the rectangle with endpoints correspondingly $(0, 0)$, $(K, 0)$, $(K, K + i\widetilde{K}^{,})$, $(0, i\widetilde{K}^{,})$ on the complex plane, where $\widetilde{K}^{,}$ is represented by the integral (111).

The first significance of this theorem is that since the mentioned points are the points of a rectangle, it gives the opportunity to predict when the integral (107) is real-valued or imaginary. Such an analysis will be performed in another publication.

Secondly, it may be noted that the Christoffel-Schwartz theorem is not formulated for elliptic integrals of higher than zero order. So taking into account the results of the theorem for the zero-order integral $\widetilde{J}_0^{(4)}(\widetilde{y}, q)$ and combining the higher-order integrals (101) and (102) for respectively the integrals $\widetilde{J}_4^{(4)}(\widetilde{y}, q)$ and $\widetilde{J}_2^{(4)}(\widetilde{y}, q)$ from the recurrent system of equations, it may be established in another way whether these integrals are real-valued or complex-valued and how the interval $[0, 1]$, $[1, \frac{1}{q}]$, $[\frac{1}{q}, +\infty)$ is mapped by higher-order integrals. This is a problem, not solved for the moment in the theory of elliptic functions.

10. Brief summary of a new analytical algorithm for integrating elliptic integrals of the zeroth-order in the Legendre form

10.1. Transforming an integral in the Legendre form into an integral in the Weierstrass form

A new analytical algorithm for treatment of elliptic integrals in the Legendre form $\widetilde{J}_0^{(4)}(y, q) = \int \frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}}$ has been proposed in the paper [25] and the main motivation comes from the necessity to treat elliptic integrals of higher order

$$J_n^{(4)}(y) = \int \frac{y^n dy}{\sqrt{a_0y^4 + 4a_1y^3 + 6a_2y^2 + 4a_3y + a_4}} \ , \ J_n^{(3)}(x) = \int \frac{x^n dx}{\sqrt{ax^3 + bx^2 + cx + d}} \ , \tag{114}$$

widely applied in theoretical problems in various areas of physics such as relativistic treatment of perihelion advance (see the monograph [28], but also [51]), light deflection in Schwarzschild geometry and light trajectories around Black Holes, apparent supernova distances in cosmology, soliton equations and nonlinear motion of the pendulum and etc. Both integrals in (114) are of $n-$th order, because of the terms y^n and x^n in the nominators of these integrals. Below we shall only briefly outline the new algorithm, proposed in [25].

The algorithm is based on transforming the integral $\widetilde{J}_0^{(4)}(y,q)$ in the Legendre form into the integral $\widetilde{J}_0^{(3)}(x,g_2,g_3)$ in the s.c.Weierstrass form, where the integrals are defined as

$$\widetilde{J}_0^{(4)}(y,q)=\int\frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}}\quad,\quad \widetilde{J}_0^{(3)}(x,g_2,g_3)=-\sqrt{a}\int\frac{dx}{\sqrt{4x^3-g_2x-g_3}} \tag{115}$$

and the following transformation is applied

$$x=\frac{a}{y^2}+b\quad. \tag{116}$$

After applying the above transformation, it is proved that if the parameters a and b are determined as

$$a=\frac{9g_3}{g_2}K(q)\quad, \tag{117}$$

$$b=-\frac{a}{3}(1+q^2)=-\frac{3g_3}{g_2}K(q)\quad, \tag{118}$$

then the integrals $\widetilde{J}_0^{(4)}(y,q)$ and $\widetilde{J}_0^{(3)}(x,g_2,g_3)$ are equal. In (117) and (118) $K(q)$ is a rational function, depending on the modulus parameter q of the elliptic integral

$$K(q)\equiv\frac{q^4-q^2+1}{2q^4-5q^2+2}\quad. \tag{119}$$

Note that in the transformation (116) and the subsequently determined parameters (117) and (118), no any restrictions have been imposed on the modulus of the elliptic integral q to be a small quantity, so the analytical method is applicable to highly elliptic orbits, which are used in modern satellite technologies. This will be mentioned also further.

10.2. Another representation of the integral in the Weierstrass form without the conformal coefficient

The second representation is based on the s.c. theorem for "four-dimensional uniformization" [52], according to which an elliptic integral of the zero-order in its general four-dimensional form and in the Legendre representation

$$J_0^{(4)}(y)=\int\frac{dy}{\sqrt{a_0y^4+4a_1y^3+6a_2y^2+4a_3y+a_4}} \tag{120}$$

after a series of transformations (explicitly given in the paper [25], following [52]) can be brought to the elliptic integral in the Weierstrass representation

$$I_0^{(3)}(x)=\int_s^{\infty}\frac{d\sigma}{\sqrt{4\sigma^3-\overline{g}_2\sigma-\overline{g}_3}}\quad, \tag{121}$$

where for the polynomial of the fourth degree in the Legendre form

$$f(y)\equiv(1-y^2)(1-q^2y^2)=1-(1+q^2)y^2+y^4 \tag{122}$$

the new Weierstrass invariants $\overline{g}_2$ and $\overline{g}_3$ can be exactly calculated [25]. From the two representations of the integral (120)

$$\widetilde{J}_0^{(4)}(y,q)=\int\frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}}=-\sqrt{a}\int\frac{dx}{\sqrt{4x^3-g_2x-g_3}}=\int\frac{d\sigma}{\sqrt{4\sigma^3-\overline{g}_2\sigma-\overline{g}_3}} \tag{123}$$

the Weierstrass invariants g_2 and g_3 can be calculated to be

$$g_2 = \overline{g}_2 a^{\frac{4}{3}} = (3K(q))^4 \frac{\overline{g}_3^4}{\overline{g}_2^3} \quad , \quad g_3 = -a^2 \overline{g}_3 = -27^2 K(q)^6 \frac{\overline{g}_3^7}{\overline{g}_2^6} \quad , \tag{124}$$

where $\overline{g}_2$ and $\overline{g}_3$ are found to be complicated polynomial functions of the modulus q of the elliptic integral in the Legendre form [25] and $K = K(q)$ is the rational function in terms of q, given by the defining expression (119). We shall give the calculated expressions for $\overline{g}_2$ and $\overline{g}_3$, calculated for the polynomial (122), making use of the formulaes in the monograph [52]

$$\overline{g}_2 = q^2 + \frac{1}{12}(1+q^2)^2 \ > 0 \ \ , \ \ \overline{g}_3 = -\frac{1}{6}q^2(1+q^2) + \frac{(1+q^2)^3}{6^3} \quad . \tag{125}$$

Thus the Weierstrass invariants g_2 and g_3 (also the coefficient function $a(q)$) in the second integral in (123) are known functions. In the Conclusion part of this paper and on the baseof the theorem for "four-dimensional uniformization", it will be shown that an arbitrary elliptic integral (120) of the fourth-degree polynomial $a_0y^4 + 4a_1y^3 + 6a_2y^2 + 4a_3y + a_4$ can thus be parametrized with a complicated function, depending on the Weierstrass elliptic function $\rho(z)$ and its derivative.

10.3. Application of the Weierstrass integral and of the Weierstrass elliptic curve in the parametrizable form

Finding a solution $z - z_0$ of the second integral in (123) in the Weierstrass form $\int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}$ is equivalent to finding the solution of the cubic equation $y^2 = 4x^3 - g_2x - g_3$. This "equivalency" can be written as [50]

$$z - z_0 = \int\limits_{x_0}^{x} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \Leftrightarrow \ y^2 = 4x^3 - g_2x - g_3 \quad . \tag{126}$$

The cubic curve $y^2 = 4x^3 - g_2x - g_3$ is called also the "uniformization curve", because the functions of the complex variable z

$$x = \rho(z) \quad , \quad y = \rho^{'}(z) \equiv \frac{d\rho}{dz} \tag{127}$$

satisfy the cubic curve (126). In fact, the equality $x = \rho(z)$ is a solution also of the elliptic integral in (126) and this is the known problem of "inversion" for elliptic integrals. In (127) $\rho(z)$ is the Weierstrass function [50]

$$\rho(z) \equiv \frac{1}{z^2} + \sum \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + ... \tag{128}$$

From the second representation for the Weierstrass function $\rho(z)$ it becomes clear why in the literature it is denoted by $\rho(z; g_2, g_3)$.Taking into account the standard definition for the Weierstrass invariants g_2 and g_3 and also the defining equalities (124), the invariants can be represented as

$$g_2 = 60G_4 = 60 \sum \frac{'1}{\omega^4} \ = (3K(q))^4 \frac{\overline{g}_3^4}{\overline{g}_2^3} \quad , \tag{129}$$

$$g_3 = 140G_6 = 140 \sum \frac{'1}{\omega^6} = -27^2 \left(K(q) \right)^6 \frac{\overline{g}_3^7}{\overline{g}_2^6} \quad . \tag{130}$$

The invariants are defined as infinite convergent sums over the period ω of the two-periodic lattice and the prime "$'$" above the two sums means that the period $\omega = 0$ is excluded from the summation, so that the expressions would not tend to infinity.

There is also a unique mapping between the pair of Weierstrass invariants (g_2, g_3) and the points $z = m\omega_1 + n\omega_2$ of the two-dimensional lattice Λ on the complex plane

$$\Lambda \equiv \left\{ m\omega_1 + n\omega_2 \; ; \quad 0 \leq m \leq 1 \; , \; 0 \leq n \leq 1 \; , \; Imag\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\} \quad . \tag{131}$$

The following theorem, proved in Ch. 6 of the monograph by Knapp [53] allows us to understand how the result of integration for the elliptic integral (123) $\widetilde{J}_0^{(q)}(y) = \int \frac{dy}{\sqrt{(1-y^2)(1-q^2y^2)}}$ can be represented in an analytical form by means of the Weierstrass representation. This formulae will be explicitly written in another publication.

10.4. A theorem about the unique correspondence between the points on the two-dimensional complex lattice and the Weierstrass invariants (g_2, g_3) and also between elliptic curves and elliptic integrals in the Weierstrass form

Theorem [53] There exists an unique correspondence $(g_2(\Lambda), g_3(\Lambda)) \Leftrightarrow \Lambda$ between the lattices Λ on the complex plane $\mathbb{C}$ and the pair (g_2, g_3) of the complex numbers (the Weierstrass invariants) such that the discriminant of the cubic polynomial $4x^3 - g_2x - g_3$ is different from zero. If a, b, c are the roots of the above polynomial, i. e.

$$4x^3 - g_2x - g_3 = 4(z-a)(z-b)(z-c) \quad , \tag{132}$$

then the periods (ω_1, ω_2) on the complex lattice Λ (131) can be found from the following integrals

$$\omega_1 = \int_{\Gamma_1} \frac{dz}{2\sqrt{(z-a)(z-b)(z-c)}} \quad , \quad \omega_2 = \int_{\Gamma 2} \frac{dz}{2\sqrt{(z-a)(z-b)(z-c)}} \quad . \tag{133}$$

In the first integral the unique branch Γ_1 of the square root is chosen with cuts from a to b and from c to ∞ and in the second integral the branch Γ_2 is with cuts from b to c and from a to ∞.

Moreover, for every lattice Λ on the complex plane, defined according to (131), a biholomorphic mapping $\varphi : \mathbb{C}/\Lambda \mapsto E(\mathbb{C})$, where $E(\mathbb{C})$ denotes the elliptic curve $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. The mapping φ is defined by means of the Weierstrass elliptic function $\rho(z)$ and the inverse mapping - by the corresponding elliptic integral.

A detailed study whether the periods (ω_1, ω_2) are real and imaginary depending on the roots of the polynomial $4x^3 - g_2x - g_3$ is given in Ch. 6.5 the known monograph [48].

11. New physical and mathematical theory of the space-time interval on intersecting null cones for the case of plane elliptical orbits

11.1. Preliminary statement of the problem and the basic system of algebraic equations

This new approach, developed first in the paper [22] and later on summarized in [23] is a logical continuation of the approach of one null cone. In the preceding sections serious arguments were presented in favour of the fact that one can choose a reference system of space coordinates, parametrizing the null cone equation. In particular, numerical estimates were given which prove the consistency of the theoretical approach with some typical experimental data, related to satellite laser ranging.

The theoretical approach up to now was related to the problem about calculation of the propagation time of a signal, emitted by a moving along an elliptical orbit (plane orbit or also a space-oriented orbit) satellite. However, such an approach does not give an answer to the question: what is the propagation time if the signal is intercepted by a second satellite, which is moving during the time of propagation of the signal.

Evidently, there should be some "consistency" between the events of emission and perception of the signal. Since these two events are treated in the framework of General Relativity Theory, it will be quite natural to construct a new theory, based on two null cones (12) $ds_1^2 = -c^2\left(1+\frac{2V_1}{c^2}\right)(dT_1)^2 + \left(1-\frac{2V_1}{c^2}\right)\left((dx_1)^2+(dy_1)^2+(dz_1)^2\right) = 0$ and (13) $ds_2^2 = -c^2\left(1+\frac{2V_2}{c^2}\right)(dT_2)^2 + \left(1-\frac{2V_2}{c^2}\right)\left((dx_2)^2+(dy_2)^2+(dz_2)^2\right) = 0$ with origins at the signal-emitting and signal-receiving satellite.

For the moment, we shall not be able to give an answer to the question how to calculate the propagation time for a realistic model, when the signal-emitting- and signal-receiving satellites move on different space-distributed orbits, experiencing also various disturbing force and influences on the part not only of the Earth, but also the Moon, the Sun and other planets. However, in equations (20) - (22) a preliminary generalized model was presented, based on the dependence of the propagation times T_1 and T_2 on several celestial- mechanical parameters. Evidently, such a complicated approach will be related to serious mathematical and technical difficulties, without being sure that there will be any new physics or at least any new numerical estimate, being related to the process of emission and reception of the signal.

That is why, the more "idealized" approach which was used in the papers [22] and [23] is based only on the dependence of the two propagation times only on the two eccentric anomaly angles E_1 and E_2, characterizing the (independent) plane elliptical motion of the two satellites. Naturally, the two four-dimensional null cone equations (12) and (13) in terms of the variables $dT_1, dx_1, dy_1, dz_1, dT_2, dx_2, dy_2, dz_2$ are independent. They will become dependent if they are "intersected" by the hyper-plane equation, obtained after taking the differential of the Euclidean distance $R_{AB}^2 = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$. In terms of the two-dimensional parametrization (5) $x = a(\cos E - e)$, $y = a\sqrt{1-e^2}\sin E$ (now there will be two parametrizations for (x_1, y_1) and (x_2, y_2)) this equation for dR_{AB}^2 is written as

$$dR_{AB}^2 = 2(x_1-x_2)d(x_1-x_2) + 2(y_1-y_2)d(y_1-y_2) \quad . \tag{134}$$

In fact, the above equation of the four-dimensional hyperplane in terms of the variables dx_1, dx_2, dy_1, dy_2 is a simplified version of the more general hyper-plane equation (22), written in terms of 14 variables.

Our further purpose will be to solve the system of equations (12), (13) and (134) in terms of the parametrization (5), to find a resulting algebraic equation with respect to dR_{AB}^2 and to solve it as a differential equation in full derivatives with respect to the two set of parameters (e_1, E_1) and (e_2, E_2). The solution for R_{AB}^2 will allow us to make important physical conclusions.

11.2. Finding the solution of the system of algebraic equations

In the paper [22] it has been proved that combining the equations (12), (13) and (134), one can obtain the following differential equation with respect to dR_{AB}^2

$$dR_{AB}^2 = F_1(E_1, E_2)dE_1 + F_2(E_1, E_2)dE_2 \quad , \tag{135}$$

where $F_1(E_1, E_2)$ and $F_2(E_1, E_2)$ are the expressions

$$F_1(E_1, E_2) := 2e_1a_1^2(1-e_1\cos E_1)\sin E_1 - S_1(E_1, E_2) \quad , \tag{136}$$

$$F_2(E_1, E_2) = 2e_2a_2^2(1-e_2\cos E_2)\sin E_1 - S_2(E_1, E_2) \tag{137}$$

and $S_1(E_1, E_2)$ is the expression

$$S_1(E_1, E_2) := -2[a_1a_2\sqrt{(1-e_1^2)(1-e_2^2)}\sin E_2\cos E_1 +$$

$$+a_1 a_2 \sin E_1 \cos E_2 - e_2 a_1 a_2 \sin E_1] \quad . \tag{138}$$

The expression for $S_2(E_1, E_2)$ is the same as $S_1(E_1, E_2)$, but with interchanged $E_1 \Longleftrightarrow E_2$, i.e. $S_2(E_1, E_2) = S_1(E_2, E_1)$. Since equation (135) is a differential equation in full derivatives, the following differential equations follow

$$F_1(E_1, E_2) = \frac{\partial R_{AB}^2}{\partial E_1} \quad , \quad F_2(E_1, E_2) = \frac{\partial R_{AB}^2}{\partial E_2} \quad . \tag{139}$$

If the first equation is integrated then

$$R_{AB}^2 = \int F_1(E_1, E_2) dE_1 + \varphi(E_2) = \tag{140}$$

$$= -2e_1 a_1^2 \cos E_1 + \frac{1}{2} e_1^2 a_1^2 \cos(2E_1) +$$

$$+2a_1 a_2 \sqrt{\left(1 - e_1^2\right)\left(1 - e_2^2\right)} \sin E_1 \sin E_2 -$$

$$-2a_1 a_2 \cos E_1 \cos E_2 + 2e_2 a_1 a_2 \cos E_1 + \varphi(E_2) \quad , \tag{141}$$

where $\varphi(E_2)$ is a function, which has to be determined from the second equation $F_2(E_1, E_2) = \frac{\partial R_{AB}^2}{\partial E_2}$ in (139). If from (141) the derivative $\frac{\partial R_{AB}^2}{\partial E_2}$ is calculated and then is set up equal to $F_2(E_1, E_2)$ given by expression (137), the following simple differential equation for $\varphi(E_2)$ can be obtained

$$\frac{\partial \varphi(E_2)}{\partial E_2} = \left(2e_2 a_2^2 - 2e_1 a_1 a_2\right) \sin E_2 - e_2^2 a_2^2 \sin(2E_2) \quad . \tag{142}$$

If the equation is integrated and the result is substituted into (141), then the final expression for R_{AB}^2 is obtained

$$\widehat{R}_{AB}^2 = \left(-2e_1 a_1^2 \cos E_1 - 2e_2 a_2^2 \cos E_2\right) +$$

$$+ \left(2e_2 a_1 a_2 \cos E_1 + 2e_1 a_1 a_2 \cos E_2\right) +$$

$$+\frac{1}{2}\left(e_1^2 a_1^2 \cos(2E_1) + e_2^2 a_2^2 \cos(2E_2)\right) - 2a_1 a_2 \cos E_1 \cos E_2 +$$

$$+2a_1 a_2 \sqrt{\left(1 - e_1^2\right)\left(1 - e_2^2\right)} \sin E_1 \sin E_2 \quad . \tag{143}$$

Note that this expression is symmetrical and does not change under interchange of the indices 1 and 2, as it should be. In the next section it will be explained why the result of integration is denoted in (143) as $\widehat{R}_{AB}^2$ instead of keeping the original notation R_{AB}^2.

12. New physical consequences from the other expression for the Euclidean distance:space-time interval, condition for inter-satellite communications and geodesic distance

12.1. Euclidean distance and space-time interval-comparison between the corresponding formulaes and physical meaning of the space-time interval (distance)

The first important observation which can be made is that the expression (143) for R_{AB}^2 is that it is different from the expression for the Euclidean distance $R_{AB}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$R_{AB}^2 = \left[(a_1 \cos E_1 - a_2 \cos E_2) + (a_2 e_2 - a_1 e_1)\right]^2 +$$

$$+ \left[a_1 \sqrt{1 - e_1^2} \sin E_1 - a_2 \sqrt{1 - e_2^2} \sin E_2\right]^2 \quad . \tag{144}$$

written in terms of the two-dimensional elliptic coordinates (5). It is easily seen that when $E_1 = E_2$, $a_1 = a_2$ and $e_1 = e_2$ (the case of coinciding points $x_1 = x_2$ and $y_1 = y_2$), the Euclidean distance R_{AB} (144) is equal to zero, but the distance $\widehat{R}^2_{AB}$ (143) is not equal to zero. In other words, for $e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$ the expression (143) for $\widehat{R_{AB}}^2$ becomes equal to

$$\widehat{R_{AB}}^2 = 4a^2 \sin^2 E.(1 - e^2) + a^2(e^2 - 2) \quad . \tag{145}$$

Now it is interesting to note that $\widehat{R_{AB}}^2$ is positive for

$$\sin^2 E \geq \frac{2 - e^2}{4(1 - e^2)} \quad , \tag{146}$$

but for

$$\sin^2 E \leq \frac{2 - e^2}{4(1 - e^2)} \tag{147}$$

the expression (145) for $\widehat{R_{AB}}^2$ is negative. It can be also equal to zero for

$$\sin^2 E = \frac{2 - e^2}{4(1 - e^2)} \quad . \tag{148}$$

Consequently, (145) for $\widehat{R_{AB}}^2$ possesses the characteristics of the space-time interval, which is a typical notion for Special and General Relativity. That is the reason why in (143) the notation $\widehat{R}^2_{AB}$ was introduced, in order to distinguish the space-time distance $\widehat{R}^2_{AB}$ from the Euclidean distance R^2_{AB}.

Thus, there is no any mistake in obtaining the formulae (143) for $\widehat{R}^2_{AB}$, since it was derived from the intersection of the two four-dimensional null cones (12), (13) with the hyperplane equation (134). Thus, from a theoretical point of view a new result has been obtained - the intersection of the two null-cones (the null-cone equations are a partial case of the space-time interval) with the hyper-plane equation gives a space-time interval, which can be positive, negative or equal to zero.

12.2. The compatibility condition for inters-satellite communications - definition from a physical and mathematical point of view

From a physical point of view, it is clear that the space-time distance cannot be related to the physical process of signal propagation, because any signal (light, radio) propagates along a large-scale distance, which is of course only positive.

On the other hand, relation (143) for $\widehat{R}^2_{AB}$ was obtained from a set of equations, which included also the defining equation for the Euclidean distance $R^2_{AB} = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Therefore, it seems natural to ask whether the propagation of a signal takes place when the large-scale Euclidean distance is compatible with the space-time interval (143). The equality of the two expressions (143) and (144) is possible when the following equation is fulfilled

$$4a_1a_2\sqrt{\left(1 - e_1^2\right)\left(1 - e_2^2\right)} \sin E_1 \sin E_2 =$$

$$= a_1^2 + a_2^2 + (a_2e_2 - a_1e_1)^2 - \frac{1}{2}\left(e_1^2a_1^2 + e_2^2a_2^2\right) \quad . \tag{149}$$

The above equality shall be conditionally called "the compatibility condition for inter-satellite communications". The earlier established equality (148) $\sin^2 E = \frac{2 - e^2}{4(1 - e^2)}$ for the partial case

$e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$ is a consequence of the above equality. The limiting value $E_{\lim}$, for which the equality (148) is fulfilled is

$$E_{\lim} = \arcsin\left[\frac{1}{2}\sqrt{\frac{2-e^2}{1-e^2}}\right] =$$

$$= 45.002510943228 \text{ [deg]} \ = 0.7785441987624 \ [rad] \quad , \tag{150}$$

which is calculated for the eccentricity of the GPS orbit $e = 0.01323881349526$. Note also that $E_{\lim}$ does not depend on the large semi-major axis a of the orbit.

12.3. Consistency of the positivity of the space-time interval with the value for $E_{\lim}$ from the compatibility condition

The space-time interval (143) for equal eccentricities and semi-major axis, but for different eccentric anomaly angles $E_1 \neq E_2$ can be written as

$$\widehat{R}^2_{AB} = e^2a^2 - e^2a^2(\sin E_1 + \sin E_2)^2 -$$

$$-2a^2\cos(E_1 + E_2) \quad . \tag{151}$$

The space- time interval will be positive (i.e. $\widehat{R}^2_{AB} > 0$), if the following inequality is satisfied

$$e^2 - e^2(\sin^2 E_1 + \sin^2 E_2) +$$

$$+2(1-e^2)\sin E_1 \sin E_2 > 2\cos E_1 \cos E_2 \quad . \tag{152}$$

If we take into account the standard inequalities for the cos-function

$$\cos E_1 \leq 1 \quad , \ \cos E_2 \leq 1 \tag{153}$$

and also the simple trigonometric relations $\sin^2 E_1 = 1 - \cos^2 E_1$, $\sin^2 E_2 = 1 - \cos^2 E_2$, then the first two terms on the first line of the above inequality can be written as

$$e^2 - e^2(\sin^2 E_1 + \sin^2 E_2) =$$

$$= e^2\cos^2 E_1 - e^2 + e^2\cos^2 E_2 \leq e^2 - e^2 + e^2 = e^2 \quad . \tag{154}$$

Substituting into inequality (152), it can be derived

$$2\cos E_1 \cos E_2 < e^2 + 2\sin E_1 \sin E_2 \quad , \tag{155}$$

which can be represented also as

$$\cos(E_1 + E_2) < \frac{e^2}{2} \quad \Longrightarrow \quad E_1 + E_2 > \arccos(\frac{e^2}{2}) \quad . \tag{156}$$

For the typical value of the eccentricity $e = 0.01323881349526$ of the GPS orbit, it can be obtained

$$E_1 + E_2 > 89.994978993712 \text{ [deg]} \ . \tag{157}$$

Note that the sign is greater because cos is a decreasing function with the increase of the angle (in the first quadrant). For the third and the fourth quadrant cos is an increasing function and the sign should be the reverse one.

If one sets up $E_1 = E_2 = E$ in (156), then

$$\sin^2 E > \frac{1}{2}(1 - \frac{e^2}{2}) \implies E > \overline{E} = \arcsin \frac{\sqrt{2-e^2}}{2} \quad , \tag{158}$$

where the numerical result for $\overline{E}$ is twice as smaller than the value 89.994978993712 [deg] in (157)

$$\overline{E} = 44.997489496856 \ \ [\text{deg}] \quad . \tag{159}$$

It should be clarified that this numerical value is a little lower that the limiting value $E_{\text{lim}} =$ 45.002510943228 [deg] (150). So it may seem that in the interval $\overline{E} < E < E_{\text{lim}}$ the space-time interval (143) is negative, but yet the space-time interval (154) in terms of the two eccentric anomaly angles E_1 and E_2will be positive, consequently $E > \overline{E} = 44.997489496856$ [deg] (158) and (159) will be fulfilled. This would have been a contradiction, but this is not the case. The compatibility limiting value E_{lim} is an exact result, while trigonometric approximations are used for derivation of the inequalities (157) and (158).

12.4. Restriction on the ellipticity of the orbit. Consequences for the RadioAstron space project and for satellites on large elliptical orbits

Since $\sin E \leq 1$, one should have also the inequality

$$\sin E = \frac{1}{2}\sqrt{\frac{(2-e^2)}{(1-e^2)}} \leq 1 \quad , \tag{160}$$

which is fulfilled for

$$e^2 \leq \frac{2}{3} \quad \text{or} \quad e \leq 0.816496580927726 \quad . \tag{161}$$

Surprisingly, highly eccentric orbits (i.e. with the ratio $e = \frac{\sqrt{a^2-b^2}}{a}$ tending to one, where a and b are the great and small axis of the ellipse) are not favourable for inter-satellite communications. For GPS satellites which have very low eccentricity orbits (of the order 0.01) and for communication satellites on circular orbits ($e = 0$), inter-satellite communications between moving satellites can be practically achieved. For the first time the above calculation shows that such communications depend on the eccentricity of the orbit, which should be experimentally checked.

Another example can be given with the RadioAstron space mission (a system of an Earth based radio-telescope and a moving near-Earth space antenna) with a large semi-major axis of $a \approx 2 \times 10^8$ m with a variable orbital eccentricity ranging from $e = 0.59$ to the large value $e = 0.966$, which is higher than the value 0.816496580927726. So from the theory developed in this paper and also in the papers [22] and [23] it will follow that for eccentricity in the interval $0.59 < e < 0.816$, inter-satellite communications of RadioAstron with another satellites on the same orbit will be possible, but this will not be possible for eccentricities in the interval $0.816 < e < 0.966$. We should stress that this result is not applicable for the radio communications between the Earth-based radio telescope and the satellite on orbit, but only for the satellites on highly elliptic plane orbits.

In view of the restriction on the ellipticity of the orbit, it is perhaps interesting to mention that the traditional methods for autonomous navigation inter-satellite orientation (and ranging), which are occasionally suited for nearly circular orbits (and with small eccentricity), are inapplicable and limited when spacecrafts on large elliptical orbits are considered [54]. The reason is that large elliptical orbit satellites (such as SBIRS of the USA and the Russian Molniya series of communication satellites) operate slowly in apogee and operate quickly when passing the perigee, so navigational methods typical for GPS cannot be used. Although the analysis in the monograph of Liu [54] is purely classical, it would be interesting to include the General Relativity methods. It should be reminded again that GPS satellites (also the $GLONASS$ satellites) are very-low eccentricity orbits.

12.5. New results:the notion of the geodesic distance from the compatibility condition and the space-time interval

If the compatibility condition (149) is substituted into the expression (143) for the space-time interval, the following expression is obtained

$$\widetilde{R}_{AB}^2 = \frac{1}{2}(a_1^2 + a_2^2) + \frac{1}{2}(a_2e_2 - a_1e_1)^2 + \frac{1}{4}\left(a_1^2e_1^2 + a_2^2e_2^2\right) -$$
$$-\left(2e_1a_1^2\cos E_1 + 2e_2a_2^2\cos E_2\right) -$$
$$-\left(e_1^2a_1^2\sin^2 E_1 + e_2^2a_2^2\sin^2 E_2\right) - 2a_1a_2\cos E_1\cos E_2 +$$
$$+2a_1a_2\left(e_2\cos E_1 + e_1\cos E_2\right) \quad . \tag{162}$$

The distance $\widetilde{R}_{AB}^2$ is called the "geodesic distance" and it is different from the space-time distance (interval) (143) and from the Euclidean distance (144). In order to understand why $\widetilde{R}_{AB}^2$ is called a geodesic distance, let us find the difference between the square of the geodesic distance and the Euclidean distance

$$R_{AB}^2 - \widetilde{R}_{AB}^2 = \frac{1}{2}(a_1^2 + a_2^2) - e_1e_2a_1a_2 +$$
$$+\frac{1}{4}(a_1^2e_1^2 + a_2^2e_2^2) - 2a_1a_2\sqrt{\left(1 - e_1^2\right)\left(1 - e_2^2\right)} \quad . \tag{163}$$

12.6. Equality to zero of the geodesic distance when the compatibility condition is fulfilled-the case of equal eccentricities, eccentric anomaly angles and semi-major axis

For the case $e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$ the geodesic distance (162) acquires the form

$$\widetilde{R}_{AB}^2 = -a^2 + \frac{1}{2}a^2e^2 + 2a^2(1 - e^2)\sin^2 E \quad . \tag{164}$$

This expression becomes equal to zero for this value of $E_{\lim}$, for which the compatibility condition (149) is fulfilled, i.e. $\sin^2 E = \sin^2 E_{\lim} = \frac{1}{4}\frac{(2-e^2)}{(1-e^2}$. It is easy to check that for this value of $E_{\lim}$ the space-time interval (145) $\widehat{R}_{AB}^2 = 4a^2\sin^2 E.(1 - e^2) + a^2(e^2 - 2)$ is also equal to zero. Consequently, for this simplest case the geodesic distance is compatible with the space-time distance.

12.7. Geodesic distance greater than the (non-zero) Euclidean distance - the case of equal eccentricities and semi-major axis, but different eccentric anomaly angles

It is instructive to compare the non-zero Euclidean distance (given by 144 for $e_1 = e_2 = e$ and $a_1 = a_2 = a$, but for $E_1 \neq E_2$) and to compare it with the geodesic distance (162). Now it can be noted however that the difference $R_{AB}^2 - \widetilde{R}_{AB}^2$ in (163) does not depend on the eccentric anomaly angles. So for $e_1 = e_2 = e$ and for $a_1 = a_2 = a$ one can represent (163) as

$$\widetilde{R}_{AB} = \sqrt{R_{AB}^2 + a^2(1 - \frac{3}{2}e^2)} \quad . \tag{165}$$

Taking into account the restriction (161) $e^2 \leq \frac{2}{3}$ on the value of the ellipticity of the orbit, the second term under the square root in (165) is positive. Due to this

$$\widetilde{R}_{AB} \geq R_{AB} \quad , \tag{166}$$

which means that the geodesic distance, travelled by the signal is greater than the Euclidean distance. Importantly, it may be concluded also that we established the same physical meaning of the geodesic distance, but following a different "logical" sequence of definitions, which have nothing to do with any null geodesic distance. The geodesic distance $\widetilde{R}_{AB}$ is in fact a distance, defined on the intersection of two algebraic varieties (two four-dimensional null cones) with a hyper-plane equation. Let us remind also that first the space-time interval $\widehat{R}^2_{AB}$ (143) was defined as a solution of a system of three algebraic equations, then it was required the space-time interval to be compatible with the Euclidean distance R_{AB} (144)- as a result the compatibility condition (149) was obtained and finally-the geodesic distance $\widetilde{R}_{AB}$ (162) was derived.

Since the important property of the geodesic distance $\widetilde{R}_{AB}$ to be greater than the Euclidean distance is proved for some partial cases, it remains to prove it for the general case of different eccentricities, semi-major axis and eccentric anomaly angles. If this happens to be the case, the consistency of the formalism will be proved.

12.8. Geodesic distance greater than the (non-zero) Euclidean distance - the general case of different eccentricities, semi-major axis and eccentric anomaly angles

In order to perform the proof, let us write the condition for inter-satellite communications (149) (also called "compatibility condition") in the form

$$\sin E_1 \sin E_2 = p = \frac{P_1(e_1, a_1; e_2, a_2)}{Q_1(e_1, a_1; e_2, a_2)} \quad , \tag{167}$$

where $P_1(e_1, a_1; e_2, a_2)$ and $Q_1(e_1, a_1; e_2, a_2)$ for given values of the two eccentricities e_1, e_2 and the semi-major axis a_1, a_2 are the numerical parameters

$$P_1(e_1, a_1; e_2, a_2) := a_1^2 + a_2^2 + (a_2 e_2 - a_1 e_1)^2 - \frac{1}{2}(e_1^2 a_1^2 + e_2^2 a_2^2) \ , \tag{168}$$

$$Q_1(e_1, a_1; e_2, a_2) := 4 a_1 a_2 \sqrt{(1 - e_1^2)(1 - e_2^2)} \quad . \tag{169}$$

Since

$$\sin E_1 \sin E_2 \leq 1 \quad , \tag{170}$$

from the preceding relations (167) - (170) it can be obtained

$$\begin{aligned} &-\frac{1}{2}(a_1^2 + a_2^2) + a_1 a_2 e_1 e_2 \geq \\ &\geq -\frac{1}{4}(e_1^2 a_1^2 + e_2^2 a_2^2) + \frac{1}{2}(e_1^2 a_1^2 + e_2^2 a_2^2) \\ &-2 a_1 a_2 \sqrt{(1 - e_1^2)(1 - e_2^2)} \quad . \end{aligned} \tag{171}$$

Substituting the terms in the left-hand side of the above inequality in the expression (163) for $\widetilde{R}^2_{AB} - R^2_{AB}$ (we change the sign in both sides of the inequality), it can be obtained

$$\widetilde{R}^2_{AB} - R^2_{AB} \geq -\frac{1}{4}(e_1^2 a_1^2 + e_2^2 a_2^2) + \frac{1}{2}(e_1^2 a_1^2 + e_2^2 a_2^2)$$

$$-2a_1a_2\sqrt{(1-e_1^2)(1-e_2^2)} - \frac{1}{4}(e_1^2a_1^2 + e_2^2a_2^2)$$
$$+2a_1a_2\sqrt{(1-e_1^2)(1-e_2^2)} \quad . \tag{172}$$

All the terms in the right-hand side of the above inequality cancel, so one obtains for this case again the inequality (166) $\widetilde{R}_{AB} \geq R_{AB}$. So again, in analogy with the previous partial case (165), when $\widetilde{R}_{AB} = \sqrt{R_{AB}^2 + a^2(1-\frac{3}{2}e^2)}$ was fulfilled and in the framework of the two null cones formalism, the geodesic distance retains its property of being greater than the Euclidean distance. Similar was the case with the Shapiro delay formulae (32) $\Delta t = \frac{R_{AB}}{c} + 2\frac{G_{\oplus}M_E}{c^3}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)$.

13. Algebraic treatment of the space-time interval and the geodesic distance

13.1. Zero space-time interval from non-zero Euclidean distance - analysis of fourth-order algebraic equations by means of higher algebra theorems

Now we shall explore the problem when the space-time interval can be zero for the case of non-zero Euclidean distance. For the purpose, expression (151) for the case of equal to zero space-time interval $\widehat{R}_{AB}^2 = 0$ can be written as

$$2\sqrt{(1-\sin^2 E_1)(1-\sin^2 E_2)} =$$
$$= e^2 - e^2(\sin^2 E_1 + \sin^2 E_2) + 2(1-e^2)\sin E_1 \sin E_2 \quad . \tag{173}$$

After some transformations and introducing the notation $\sin^2 E_1 = y$, the above expression can be presented in the form of a quartic (fourth-degree) algebraic equation

$$y^4 + a_1y^3 + a_2y^2 + a_3y + a_4 = 0 \quad . \tag{174}$$

The coefficient functions of this equation are given in Appendix C of the paper [22]. Consequently, the problem about finding those values of the eccentric anomaly angle E_1 for which the space-time interval (151) is zero is equivalent to the algebraic problem of finding all the roots of the above fourth-order algebraic equation, which are within the circle $\mid y \mid = \mid \sin^2 E_1 \mid < 1$ (we exclude the boundary points $y = \sin^2 E_1 = 1$). It is well-known that an algebraic equation of fourth degree will always possess roots. The problem is that these roots should be within the circle $\mid y \mid < 1$. This is a well-known problem in algebra theory and is treated by the well-known Schur theorem. It is interesting to mention that the theorem was published yet in the 1918 year [56] and is mentioned also in most textbooks on higher algebra.

13.2. The Schur theorem from algebra theory for the roots of an $n-th$ degree polynomial within the circle $\mid y \mid < 1$

Below we present the formulation of the Schur theorem, which can be found in the monograph by Nikola Obreshkoff [55] and also in the paper [22]. In Appendix B of this paper the proof of the theorem is given, also taken from the cited monograph. The formulation of the theory is presented also in the paper [23].

Theorem (Schur) The necessary and sufficient conditions for the polynomial of $n-$th degree

$$f(y) = a_0y^n + a_1y^{n-1} + ... + a_{n-2}y^2 + a_{n-1}y + a_n \tag{175}$$

to have roots only in the circle $\mid y \mid < 1$ are the following ones:

1. The fulfillment of the inequality

$$\mid a_0 \mid > \mid a_n \mid . \tag{176}$$

2. The roots of the polynomial of the $(n-1)-$th degree

$$f_1(y) = \frac{1}{y}\left[a_0 f(y) - a_n f^*(y)\right] \tag{177}$$

should be contained in the circle $\mid y \mid < 1$, where $f^*(y)$ is the s.c. "inverse polynomial", defined as

$$f^*(y) = y^n f(\frac{1}{y}) = a_n y^n + a_{n-1}y^{n-1} + ... + a_2 y^2 + a_1 y + a_0 \quad . \tag{178}$$

In case of fulfillment of the inverse inequality

$$\mid a_0 \mid < \mid a_n \mid \tag{179}$$

the $(n-1)$ degree polynomial $f_1(y)$ (again with the requirement the roots to remain within the circle $\mid y \mid < 1$) is given by the expression

$$f_1(y) = a_n f(y) - a_0 f^*(y) \quad . \tag{180}$$

Concerning the necessary conditions, the Schur theorem has one another advantage - if the condition (177) (or (180)) about the roots of the polynomial $f_1(y)$ is not fulfilled, then the polynomial $f(y)$ will not have any roots within the circle. This allows to apply the theorem not only with respect to the space-time interval algebraic equation (which is proved to have roots in Appendix C of the paper [22]), but also with respect to the geodesic equation, which should not have any roots within the circle $\mid y \mid < 1$ (this is proved in Appendix E of the paper [22]). The last fact may be confirmed by independent calculations, since it has been already proved that the geodesic distance is greater than the Euclidean distance, so it cannot become equal to zero. This is fully consistent from a physical point of view, since the light or signal propagation is related to the geodesic distance and not to the space-time interval, which can be equal to zero or even become negative. In this aspect, it is really amazing how the physical interpretation is consistent with the mathematical results about these two algebraic equations.

It is important to mention that these conclusions are valid in view of the fact that the eccentricity e is very small (in celestial mechanics, it is of the order of 0.01), and on the base of this it is possible to compare terms with inverse powers in e in the corresponding inequalities - the higher inverse powers in e will lead to a larger number. For example, a term of the order of $\frac{1}{e^2}$ will give a number of the order of 10000, but as it was shown, there will be terms proportional to $\frac{1}{e^{10}}$, $\frac{1}{e^{12}}$ and even $\frac{1}{e^{14}}$, which are extremely large numbers. It is important that terms which differ by two orders in inverse powers of e will have greatly different numerical values.

13.3. The general strategy for using the Schur theorem - the "chain" of lower-degree polynomials

The basic fact which is a consequence of the Schur theorem and which shall be used further in the proof is that the polynomial of the $(n-1)$ degree (177) or (180) is a sufficient condition for the existence of roots within the unit circle of the initial polynomial of $n-$th degree. But then, if a new polynomial of $(n-2)$ degree is constructed according to formulaes (177) or (180), then this polynomial can become a sufficient condition for the roots of the $(n-1)$ degree polynomial. In such a way, a chain of lower-degree polynomials is constructed - each polynomial represents a necessary and at the same time a sufficient condition for the construction of a lower degree polynomial. The last constructed polynomial will be of first order, and from it the condition for the roots to be contained in the unit circle can easily be found. Note the important role of the necessary and sufficient condition - if from the linear polynomial the condition for the roots is found, then it will be a sufficient condition for the second-order polynomial, further this polynomial will be a necessary and sufficient condition for the third-order polynomial and etc. In such a way, the first-order polynomial will turn out to be a sufficient condition for the roots to remain within the unit circle with respect to the initial n-th degree polynomial, provided also that for each polynomial the corresponding inequalities between the coefficient functions are fulfilled. It can be claimed that this "chain" of lower-degree polynomials, together with the corresponding inequalities between the coefficient functions, represents a modified version of the Schur theorem. So from the point of view of pure mathematics, such a modified version of the Schur theorem without any doubt is interesting, the peculiar moment is that the physical information (availability of roots with respect to the space - time equation and absence of any roots with respect to the geodesic equation) is very important for the proof of such a modified version of the theorem. Of course, the proof is limited for the investigated case of polynomials of fourth degree.

13.4. Some consistent numerical checks, confirming the approach

Now let us discuss some numerical relations, following from the equation for the zero space-time interval (173), which in Appendix C of the paper [22] on the base of the Schur theorem and the "chain" of algebraic equations is proved to have roots within the interval

$$15.64[\text{deg}] < E_1 < 56.88[\text{deg}] \quad , \quad 15.64[\text{deg}] < E_2 < 56.88[\text{deg}] \tag{181}$$

If the two inequalities are summed up, then one can obtain

$$31.28[\text{deg}] < E_1 + E_2 < 113.76[\text{deg}] \quad . \tag{182}$$

Importantly, the earlier found restriction (157) $E_1+E_2 > 89.994978993712$ [deg] falls within the range of the inequality (182). This is a confirmation of the correctness of the formalism, because (157) was obtained on the base of trigonometric estimates, having nothing to do with searching the roots of the cubic equation. The value (150) for $E_{\text{lim}} = \arcsin\left[\frac{1}{2}\sqrt{\frac{2-e^2}{1-e^2}}\right] = 45.002510943228$ [deg] from the condition (149) for inter-satellite communications is also compatible with the inequality (182), which is a second confirmation about the correctness of the approach and the performed calculations.

There is also a third interesting numerical confirmation. From the inequality (170) $\sin E_1 \sin E_2 \leq 1$, represented now in the form $\sin E_2 = \frac{p}{\sin E_1} < 1$ and representing the condition for inter-satellite communications (see equations (171) - (172)) we can write $E_1 > \arcsin p = 30.002899$ [deg], which turns out to be higher than the lower bound in 15.64 [deg] $< E_1 < 56.88$ [deg] (181). Consequently, there is an interval

$$15.64\ [\text{deg}] < E_1 < 30.002899\ [\text{deg}] \quad , \tag{183}$$

where the space-time interval can exist (and can have zeroes) but the condition for inter-satellite communications and the resulting from it geodesic distance equation (162) cannot be defined. This confirms the conclusion that the space-time distance is a more broader notion and has a more general meaning in comparison with the Euclidean distance and the geodesic distance. This also means that the notion of space-time distance can be defined independently from the geodesic distance.

In this review paper we shall not treat the other case of the geodesic algebraic equation, which is also a fourth - degree algebraic equation and which is treated in details in the paper [22]. The important property of the geodesic distance $\widetilde{R}^2_{AB}$ to be greater than the Euclidean distance, if "translated" into the algebraic terminology, means that the algebraic equation does not have any roots within the circle $\mid y \mid < 1$. This turns out to be the case.

14. Conclusion

In this paper a new generalized model of the Shapiro delay formulae is presented, which is based on the null cone equations formalism in General Relativity Theory. The paper summarizes the new approaches, proposed in several other previous papers [22], [23], [24] and [25], but at some places new mathematical proofs are added, also some supplementary theorems, which prove the correctness of the applied theoretical approach - this is for example the theorem about the compatibility of the null cone equation with the null geodesic equations, proved in the known General Relativity monograph [28]. Chronologically, the first developed approach was the s.c. "two null cones intersecting formalism" [22], [23] and afterwards the formalism of calculating the propagation time of a signal, emitted by a satellite moving along a space-distributed orbit was proposed in [24]. So the future development of this theory will be 1. to put together all these separate elements in the theory in [24] in the generalizing formalism of the "two null intersecting four-dimensional cones", developed in [22], [23]. 2. Since some consistency arguments have been added (the theorem from [28]) and also some numerical estimates that the approach of parametrizing the space coordinates in the null cone equations in terms of the coordinates of the elliptic orbit of the satellite is consistent, at the beginning of this paper some basic facts have been added, concerning the disturbed motion in celestial mechanics. Also, for the first time the two null cone equations were presented for the most general case of the six Keplerian parameters, parametrizing the orbit of the satellite. This is the formalism, which shall be developed in subsequent papers.

Let us formulate briefly the general purpose of the formalism, which is difficult to develop to a full extent, because knowledge from several areas of physics and mathematics needs to be combined: celestial mechanics, General Relativity theory, theory of elliptic curves and elliptic integrals (including elliptic integrals of higher order), algebraic geometry and especially the theory of intersection of algebraic varieties, theory of analytical functions (Christoffel-Schwartz theorem and any possibilities for practical applications to higher-order elliptic integrals, not just for the zero-order elliptic integrals, for which it is valid). So the general purpose of the formalism is: to describe analytically and numerically the processes of signal exchange (of course, with account of GR effects) between moving satellites on one orbit or on different space-distributed orbits, possibly also taking into account disturbing effects in the motion. In fact, such a research programm requires the incorporation into the formalism of the basic approaches: 1. Finding the propagation times for the signal for different kinds of orbit of the satellites - plane elliptic, space-distributed and orbits, on which act the s.c. "disturbing forces". 2. Calculating the propagation times T_1 and T_2, which are correspondingly the "signal-emitting" propagation time T_1 and the "signal-receiving" propagation time T_2 in the two null-cone equations (12) $ds_1^2 = -c^2\left(1+\frac{2V_1}{c^2}\right)(dT_1)^2 + \left(1-\frac{2V_1}{c^2}\right)\left((dx_1)^2+(dy_1)^2+(dz_1)^2\right) = 0$ and (13) $ds_2^2 = -c^2\left(1+\frac{2V_2}{c^2}\right)(dT_2)^2 + \left(1-\frac{2V_2}{c^2}\right)\left((dx_2)^2+(dy_2)^2+(dz_2)^2\right) = 0$. Let us note that orbits of the two satellites may have different characteristics and parametrizations - for example, the second null-cone equation might be replaced by the null cone equation, derived from a metric in a rotating frame around the Earth [5]

$$0 = -ds^2 = -(1-\frac{\omega_E^2 r^2}{c^2})(cdT)^2 + 2\omega_E^2 r^2 d\phi dT + \left[(dr)^2+(rd\phi)^2+(dz)^2\right] \quad , \tag{184}$$

where ω_E is the uniform rotating angular velocity of the Earth. Such a case is not investigated in the cited publications [22], [23], [24] and [25], since this is not related to the problem about inter-satellite communications. But in principle, such a problem can be investigated.

2. Further, applying the formalism of "two intersecting null cones" [22], [23], the equations for the space-time interval $\widehat{R}_{AB}$ (143), for the geodesic distance $\widetilde{R}_{AB}$ (162) and the so called "compatibility condition for inter-satellite communications" (149) can be defined. One of the important conclusions in the papers [22], [23] is related to the clarification of the properties of

these three equations and also the fact that these important physical notions appear in a strictly determined sequential order. Namely, after solving the two null cone equations, intersected by the hyper-plane equation $dR_{AB}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ (for the two-dimensional case of a plane orbit, solved in the cited papers), the formulae for the space-time interval $\widehat{R}_{AB}$ (143) was obtained in terms of the variables $a_1, a_2, e_1, e_2, E_1, E_2$ and after comparing it with the Euclidean distance R_{AB}^2 (144), it became clear that $\widehat{R}_{AB}$ can be positive, negative or equal to zero. Note that the variables E_1 and E_2 are related to the motion of the satellite- for some other parametrizations of the space coordinates they will be another. Further, it was required that the space-time interval is comparable to the Euclidean distance and thus, the compatibility condition (149) was obtained. Then, if the compatible condition is substituted back into the formulae for the space-time interval $\widehat{R}_{AB}$ (143), the geodesic distance (162) $\widetilde{R}_{AB}^2$ was obtained and it was proved that it is strictly positive for all the partial and also for the general case. This positivity for all the cases is one of the strongest arguments in favour of the consistency of the formalism.

All these facts, established in the papers [22], [23] can serve as a starting point for applying the approach of "two intersecting null cones" in another setting, meaning for more complicated and realistic situations. In other words, it is important to prove all these properties for the space-time interval and the geodesic distance for other cases -for example, for the case of the space-distributed orbits, when the more general parametrization of the space coordinates (74) - (76) [8] has been used. This more general case may be subdivided into two cases: case A. The variable, related to the motion of the satellite is the true anomaly angle f- respectively, in the framework of the two null cone formalism, instead of E_1, E_2 the changing variables will be f_1 and f_2. case B. The whole entity of parameters $(M, a, e, \Omega, I, \omega)$ is changing. This corresponds to the case when a signal is being send by a satellite on a space-distributed orbit with parameters $(M_1, a_1, e_1, \Omega_1, I_1, \omega_1)$ and the orbit is "deformed" to an orbit $(M_2, a_2, e_2, \Omega_2, I_2, \omega_2)$, where the signal-receiving satellite is situated and during the propagation time of travel of the signal, the second satellite is also moving. So for this case, the two null cones formalism displays a "consistency" in the events of emission and reception, but on the level of General Relativity (since two null cone equations are used) and also on the classical "celestial mechanics level" (because of the parametrizations in terms of the space coordinates of the satellites on the two elliptic orbits). We remind that here the mean anomaly $M = n(t - t_{per})$ (where $n = \sqrt{\frac{G_{\oplus} M_{\oplus}}{a^3}}$ is the mean motion) is related to the true anomaly angle f and to the eccentric anomaly angle E (see formulae (11)).

14.1. Proposals for further development of the approach, based on elliptic integrals of higher order, the recurrent system of equations for higher order integrals and some new analytical algorithms for calculation of elliptic integrals of zero-order in the Legendre form

As a first step in this direction, one may use the calculated propagation times in terms of the higher order elliptic integrals for the case of the parametrization (74)-(76) of the space coordinates of the space-distributed orbit. The propagation times for this cased were proved to be expressed by higher-order (second and fourth) elliptic integrals, which by means of the recurrent system for elliptic equations (101) and (102) (see [35]) were expressed through the zero-order elliptic integral $\widetilde{J}_0^{(4)}(\widetilde{y}, q)$ in the Legendre form.

Here an interesting opportunity arizes also to apply the analytical approach, developed in the paper [25] and in the section "Application of the Weierstrass integral and of the Weierstrass elliptic curve in the parametrizable form" of this paper. For the purpose, one may use the representation $\widetilde{J}_0^{(4)}(\widetilde{y}, q)$ (123) in the form of two equivalent, but different integrals in the Weierstrass form. From the comparison of the two equivalent integrals, the Weierstrass invariants g_2 and g_3 were determined as complicated rational functions (124) and moreover, the obtained formulaes did not presume any smallness of the modulus q of the elliptic integral. The knowledge of the functions g_2 and g_3 from the formulae (124) allows to define the Weierstrass function (128) $\rho(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \ldots$. Now, since the Weierstrass function is known, but the value of the integral $\widetilde{J}_0^{(4)}(y,q)$ should be expressed in terms of the $\widetilde{y}-$coordinates in the integral $\widetilde{J}_0^{(4)}(y,q)$, two more expressions are needed from the s.c."theorem for four-dimensional uniformization", taken from the monograph [52] and presented also in the paper [25]

$$y = y_0 + \frac{1}{4}\frac{f^{\prime}(y_0)}{\left[\rho(z;g_2,g_3) - \frac{1}{24}f^{\prime}(y_0)\right]} \quad , \tag{185}$$

$$Y = -\frac{1}{4}\frac{f^{\prime}(y_0)\rho^{\prime}(z;g_2,g_3)}{\left[\rho(z;g_2,g_3) - \frac{1}{24}f^{\prime}(y_0)\right]^2} \quad , \tag{186}$$

where $Y = f(y)$ is the fourth-order algebraic polynomial

$$Y \equiv f(y) \equiv a_0y^4 + 4a_1y^3 + 6a_2y^2 + 4a_3y + a_4 \Leftrightarrow \tag{187}$$

$$\Leftrightarrow J_0^{(4)}(y) = \int \frac{dy}{\sqrt{a_0y^4 + 4a_1y^3 + 6a_2y^2 + 4a_3y + a_4}} \quad , \quad f(y_0) = 0 \quad . \tag{188}$$

In other words, similar to the uniformization (126) $z - z_0 = \int\limits_{x_0}^{x} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \Leftrightarrow y^2 = 4x^3 - g_2x - g_3$ of the cubic curve with the uniformization functions $x \equiv \rho(z)$ and $y \equiv \rho^{'}(z)$ and its equivalence with the elliptic integral, the equalities (187) - (188) mean that the fourth degree polynomial (187) is equivalent to the elliptic integral (188) (in its general form with a fourth-degree polynomial in the denominator) and the corresponding uniformization functions y (185) and Y (186) "uniformize" the fourth-degree algebraic polynomial (187). Note that in the uniformization functions (185) and (186) the Weierstrass invariants g_2 and g_3 (124) are the ones in the second integral $-\sqrt{a}\int \frac{dx}{\sqrt{4x^3-g_2x-g_3}}$ in (123), which means that additionally the "conformal" function

$$-\sqrt{a} = -\sqrt{\frac{9g_3}{g_2}K(q)} = -\sqrt{\frac{9g_3}{g_2}\frac{(q^4 - q^2 + 1)}{(2q^4 - 5q^2 + 2)}} \tag{189}$$

has to be calculated with g_2 and g_3 defined as $g_2 = (3K(q))^4\frac{\overline{g}_3^4}{\overline{g}_2^3} \quad , \quad g_3 = -27^2K(q)^6\frac{\overline{g}_3^7}{\overline{g}_2^6}$ (124). But of course, the uniformization may be performed also with respect to the invariants $\overline{g}_2$ and $\overline{g}_3$ in the third integral $\int \frac{d\sigma}{\sqrt{4\sigma^3 - \overline{g}_2\sigma - \overline{g}_3}}$ in (123), so it is interesting to compare the results from both approaches.

In such a way and if the two null cones formalism is applied, this will enable us to express the propagation times in a complicated way. Further, it remains to solve the corresponding differential equations for dR_{AB}^2, to find the expressions for the space-time interval $\widehat{R}_{AB}^2$ and for the geodesic distance $\widetilde{R}_{AB}^2$ and thus to see whether the important properties of the space-time interval to be positive, negative and zero and for the geodesic distance to be only positive will be valid also for this case.

14.2. Most important experimental consequence from the developed theory of two null intersecting cones and the plane elliptical orbits of the satellites

Now it may be summarized that with respect to the approach of the two null intersecting cones, applied to plane elliptical orbits, there are two important consequences from the theory:

1. The found limiting value for the eccentric anomaly angle (150) $E_{\lim} = \arcsin\left[\frac{1}{2}\sqrt{\frac{2-e^2}{1-e^2}}\right] = 45.002510943228$ [deg], above which the space-time interval $\widehat{R}^2_{AB}$ is positive and the compatibility condition (149) for inter-satellite communications is fulfilled. Evidently, the corresponding restriction in the above proposed ""two null cones" approach, applied to the two space-distributed orbits, will be with respect to the true anomaly angle $f_{\lim}$. Although $E_{\lim}$ is found from a formalism, which is idealistic from a physical and experimental point of view, it should be taken into account, when inter-satellite communications between satellites on one orbit take place, since it corresponds to satellite configuration of 8 satellites per orbit and at angular distance 45 deg, which is typical for the *GLONASS* satellite configuration (8 satellites per orbit). Note however that the value $E_{\lim}$ differs slightly from the value 45 deg, which means that there will be a slight difference of 0.002510943228 [deg] from the "exact" angular distance 45 deg. This will result in a slight increase of the distance between the satellites, which can be calculated and perhaps should be taken into account. It is also interesting that for the limiting value $E_{\lim}$ in (150) the true anomaly angle f can be calculated from the known formulae in celestial mechanics $\cos f = \frac{\cos E - e}{1 - e\cos E}$ to be $f = 45.541436900412$ [deg] for the typical *GPS* orbit eccentricity $e = 0.01323881349526$. We note that this value for f is also near to the value $E_{\lim} = 45.002510943228$ [deg].

2. Quite an interesting consequence from the theory of two intersecting null cones is the restriction on the eccentricity of the orbit (161) $e^2 \leq \frac{2}{3}$ or $e \leq 0.816496580927726$, which follows from the simple inequality (160) $\sin E = \frac{1}{2}\sqrt{\frac{(2-e^2)}{(1-e^2)}} \leq 1$ for the $\sin -$function. It is really amazing how this simple inequality leads to the fact that the geodesic distance $\widetilde{R}_{AB}$ is greater than the Euclidean distance $\widetilde{R}_{AB} \geq R_{AB}$ (166), because of the simple formulae (165) $\widetilde{R}_{AB} = \sqrt{R^2_{AB} + a^2(1 - \frac{3}{2}e^2)}$. That is why in the forthcoming developments of the theory of two intersecting null cones, for example on the base of the space-distributed orbits, it is extremely important to check whether such a restriction on the ellipticity will also hold. Quite probably, it will depend also on the other parameters $(a, e, \Omega, I, \omega)$ of the space-distributed orbit. The restriction on the ellipticity is particularly important in reference to the future developments of satellite constellations on highly elliptical orbits [54]. The communications between such satellites on such orbits, taking into account the General Relativity effects is still an unexplored theoretical topic.

3. An important consequence from the theory about the propagation time of a signal, emitted by a satellite, moving along a plane orbit or a space-distributed orbit is the real-valuedness of the expression for the propagation time. Although for the second case the proof from a mathematical point of view is more complicated, this is a proof also of the correctness of the formalism and also opens the possibility for extending the formalism in the sense, explained in the previous section. It is important to note that the proposed mathematical proofs for the real-valuedness of the integrals $\frac{a}{c}\int\sqrt{1 - e^2\cos^2 E}dE$ and $-\frac{2G_{\oplus}M_{\oplus}}{c^3}\int\sqrt{\frac{1+e\cos E}{1-e\cos E}}dE$ in the expression (51) for the propagation time T and further, in proving the real-valuedness of the expression (84) $\widetilde{T}_1 = -2i\frac{na}{c}q^{\frac{3}{2}}\widetilde{J}^{(4)}_2(y;q)$, expression (88) for $T^{(1)}_2 = -\frac{2G_{\oplus}M_{\oplus}}{c^3}.\frac{n}{(1-e^2)^{\frac{3}{2}}}\widetilde{T}_1$ and the two integrals $-inq^{\frac{5}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}(1 + e^2)\widetilde{J}^{(4)}_2(\widetilde{y}, q)$ and $inq^{\frac{3}{2}}\frac{2G_{\oplus}M_{\oplus}}{c^3}\frac{(1+e^2)}{(1-e^2)}\ \widetilde{J}^{(4)}_4(\widetilde{y}, q)$ in the expression (91) for $T^{(2)}_2$ did not imply that any restrictions on the modulus of the elliptic integrals q or on the ellipticity of the orbits are imposed. Concerning the two null cone formalism in this paper, the only restriction on the ellipticity was $e \leq 0.816496580927726$, but there was no requirement for the ellipticity to be small of the order of 0.01. But such an assumption turned out to be useful later on, when the properties of the space-time interval and the geodesic distance were confirmed by the approach of fourth-order algebraic equations.Only the algebraic equation for

the space-time distance was considered in this paper, the full investigation on the two algebraic equations can be found in [22]. The problem however remains, also with reference of the two null cones approach and the use of space-distributed orbits: can the properties of the space-time distance and the geodesic distance from an algebraic point of view be proved without assuming necessarily the smallness of the ellipticity of the orbit?

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