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Reconstructing Atiyah-Hitchin manifold in the generalized Legendre transform

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Abstract. We revisit the Atiyah-Hitchin manifold using the generalized Legendre transform approach. Originally it is examined by Ivanov and Roček, and it has been further explored by Ionaş, with a particular focus on calculating the explicit forms of the Kähler potential and the Kähler metric. Notably, there exists a distinction between the former study and the latter. In the framework of the generalized Legendre transform approach, a Kähler potential is formulated through the contour integration of a specific function with holomorphic coordinates. It's essential to note that the choice of the contour in the latter differs from that in the former. This discrepancy in contour selection may result in variations in both the Kähler potential and, consequently, the Kähler metric. Our findings demonstrate that the former exclusively yields the real Kähler potential, aligning with its defined properties. In contrast, the latter produces a complex Kähler potential. We present the derivation of the Kähler potential and metric for the Atiyah-Hitchin manifold in terms of holomorphic coordinates, considering the contour specified by Ivanov and Roček.

1. Introduction

A pioneering method for constructing hyperkähler metrics involves the generalized Legendre transform approach, as outlined in previous works [1, 2, 3, 4]. This approach establishes a connection between the Kähler potentials of specific hyperkähler manifolds and a linear space. This is also related to the theory of twistor spaces of hyperkähler manifolds [2]. In this methodology, a Kähler potential is derived through the contour integration of a function with holomorphic coordinates (refer to (3) and (4)). This integration, termed the F-function, has been applied to various hyperkähler metrics, such as the Eguchi-Hanson family of self-dual instantons [5, 6], the Taub-NUT family of self-dual instantons [6], and the metric proposed by Calabi [7].

Following these studies, the Atiyah-Hitchin manifold [8], representing the metric on the centered moduli space of two Bogomol'nyi-Prasad-Sommerfield $SU(2)$ monopoles, has been considered within the context of the generalized Legendre transform approach, as presented in [9]. In [9], the F-function generating the Atiyah-Hitchin manifold is proposed, and the corresponding Kähler 2-form is derived, precisely matching one of the Atiyah-Hitchin manifold.

However, the explicit forms of the Kähler potential and metric are not obtained therein due to the intricate and messy nature of the calculations. The first explicit calculation of the Kähler potential and metric for the Atiyah-Hitchin manifold within the generalized Legendre transform approach has been undertaken in [10, 11]. This calculation retains the original holomorphic coordinates in the F-function, enabling the derivation of the metric in terms of these coordinates. Deriving the metric while manifestly keeping the complex structure is crucial for investigating geometric properties. Nevertheless, it is imperative to emphasize that several aspects of [10, 11] require reconsideration. Primarily, the choice of the integration contour in the F-function in [10] differs from that in [9]. However, it can be demonstrated that this variance leads to a complex Kähler potential rather than a real one. In contrast, the choice made in [9] aligns with the definition of the Kähler potential, resulting in a real outcome. Thus, the calculations in [10] pertaining to the derivation of the Kähler potential and metric should be revisited in light of the integration contour specified in [9].

In this proceeding, we reassess the Atiyah-Hitchin manifold within the framework of the generalized Legendre transform approach. We adopt the integration contour from [9] in the F-function and derive the Kähler potential and metric with holomorphic coordinates based on that choice. Our analysis demonstrates that the contour selection in [9] yields a real Kähler potential. We provide a comprehensive presentation of all the necessary steps to derive the Kähler potential and metric, starting from the F-function defining the Atiyah-Hitchin manifold. This proceeding is grounded in the research presented in [12].

2. The generalized Legendre transform

We briefly explain the generalized Legendre transform construction of hyperkähler manifolds as presented in the work by Lindström and collaborators [4]. Our starting point is a polynomial given by

$$\eta^{(2j)} = \frac{\bar{z}}{\zeta^j} + \frac{\bar{v}}{\zeta^{j-1}} + \frac{\bar{t}}{\zeta^{j-2}} + \cdots + x + (-)^j(\cdots + t\zeta^{j-2} - v\zeta^{j-1} + z\zeta^j), \tag{1}$$

where $z, t, \cdots, x$ are holomorphic coordinates, and ζ is the coordinate of the Riemann sphere $\mathbb{C}P^1 = S^2$. This polynomial is referred to as an $\mathcal{O}(2j)$-multiplet. Equation (1) must satisfy the reality condition

$$\eta^{(2j)}\left(-\frac{1}{\bar{\zeta}}\right) = \overline{\eta^{(2j)}(\zeta)}. \tag{2}$$

The Kähler potential for a hyperkähler manifold is constructed using a function involving $\eta^{(2j)}$:

$$F = \oint_C \frac{d\zeta}{\zeta} G(\eta^{(2j)}), \tag{3}$$

where G is an arbitrary holomorphic (possibly single or multi-valued) function, and the contour C is chosen so that the result of the integration is real. Equation (3) is referred to as the F-function.

The Kähler potential can be derived from the F-function through a two-dimensional Legendre transform with respect to v and $\bar{v}$:

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, t, \bar{t}, \cdots, x) - uv - \bar{u}\bar{v}, \tag{4}$$

along with the extremizing conditions

$$\frac{\partial F}{\partial v} = u, \tag{5}$$

$$\frac{\partial F}{\partial t} = \cdots = \frac{\partial F}{\partial x} = 0. \tag{6}$$

The Kähler metric can then be obtained by differentiating the Kähler potential with respect to z, $\bar{z}$, u, and $\bar{u}$.

2.1. The function F for the Atiyah-Hitchin manifold

Let's explore a $\mathcal{O}(4)$-multiplet $\eta^{(4)} = \eta^{(4)}(\zeta)$ which is expressed in a Majorana normal form:

$$\eta^{(4)} = \frac{\bar{z}}{\zeta^2} + \frac{\bar{v}}{\zeta} + x - v\zeta + z\zeta^2. \tag{7}$$

To facilitate further analysis, it's convenient to represent this form in terms of its roots and a scale factor. The reality condition (refreal) ensures that the four roots of $\eta^{(4)}$ remain invariant under the antipodal map $\zeta \mapsto -1/\bar{\zeta}$. Therefore, (7) can be expressed as:

$$\eta^{(4)} = \frac{\rho}{\zeta^2} \frac{(\zeta - \alpha)(\bar{\alpha}\zeta + 1)}{(1 + |\alpha|^2)} \frac{(\zeta - \beta)(\bar{\beta}\zeta + 1)}{(1 + |\beta|^2)}. \tag{8}$$

Without loss of generality, we assume the scale factor ρ to be positive.

In accordance with [9] and [10], the F-function, $F = F(z, \bar{z}, v, \bar{v}, x)$, of the Atiyah-Hitchin manifold is given by:

$$F = F_2 + F_1 = -\frac{1}{2\pi i h} \oint_{\Gamma_0} \frac{d\zeta}{\zeta} \eta^{(4)} + \oint_{\Gamma} \frac{d\zeta}{\zeta} \sqrt{\eta^{(4)}}. \tag{9}$$

Here, h is a constant coupling scale. The contour Γ_0 encircles the origin of the ζ-plane counterclockwise, and the contour $\Gamma = \Gamma_m \cup \Gamma'_m$ winds around two branch cuts between α and $-1/\bar{\beta}$, and β and $-1/\bar{\alpha}$, referring to a double contour. The choice of the double contour is initially proposed in [9], while [10] only opts for Γ_m as Γ, referred to as a single contour. In the case of the single contour, it can be shown that the function F_1 is not real, leading to the Kähler potential not being real. However, the double contour ensures that both F_1 and the Kähler potential are real-valued.

In the following, we demonstrate the reality of F. We can rewrite F_1 as:

$$F_1 = 2z\frac{\partial F_1}{\partial z} + 2\bar{z}\frac{\partial F_1}{\partial \bar{z}} + 2v\frac{\partial F_1}{\partial v} + 2\bar{v}\frac{\partial F_1}{\partial \bar{v}} + 2x\frac{\partial F_1}{\partial x}. \tag{10}$$

If we define, for $n \in \mathbb{Z}$:

$$\mathcal{I}_n = \oint_{\Gamma} \zeta^n \frac{d\zeta}{2\zeta\sqrt{\eta^{(4)}}}, \tag{11}$$

then the partial derivatives in (10) can be expressed as:

$$\frac{\partial F_1}{\partial z} = \mathcal{I}_2, \quad \frac{\partial F_1}{\partial \bar{z}} = \mathcal{I}_{-2}, \quad \frac{\partial F_1}{\partial v} = -\mathcal{I}_1, \quad \frac{\partial F_1}{\partial \bar{v}} = \mathcal{I}_{-1}, \quad \frac{\partial F_1}{\partial x} = \mathcal{I}_0. \tag{12}$$

For (11), we can prove the relation:

$$\mathcal{I}_{-n} = (-1)^n \overline{\mathcal{I}_n}. \tag{13}$$

The proof is given in Appendix A in [12]. From (13), it follows that $\mathcal{I}_0$ is real-valued, and (10) can be rewritten as:

$$F_1 = 2x\mathcal{I}_0 - 2(v\mathcal{I}_1 + \overline{v\mathcal{I}_1}) + 2(z\mathcal{I}_2 + \overline{z\mathcal{I}_2}). \tag{14}$$

Therefore, we've established that F_1 is real-valued. As will be shown in (15), the Kähler potential is real from (4) because F_2 is also real-valued.

Now let's evaluate the Kähler potential from (9) by the generalized Legendre transformation. To do that, it is necessary to perform the integrals in (9). Explicitly, they are F_2 and $\mathcal{I}_n$ $(n = 0, 1, 2)$ in F_1. F_2 can be evaluated by a straightforward application of Cauchy's integral formula. Then, we get:

$$F_2 = -\frac{x}{h}. \tag{15}$$

For the evaluation of $\mathcal{I}_n$ in F_1, we require several steps. First, let us rewrite $\mathcal{I}_n$ only by using the single contour Γ_m. When deforming Γ'_m to Γ_m, we need to pick up the residues of the integrand of $\mathcal{I}_n$. This integrand has two simple poles - one at $\zeta = 0$ and the other at $\zeta = \infty$. Therefore, we have:

$$\mathcal{I}_n = 2\oint_{\Gamma_m} \zeta^n \frac{d\zeta}{2\zeta\sqrt{\eta^{(4)}}} + 2\pi i R(0,n) + R(\infty,n), \tag{16}$$

where $R(\zeta, n)$ denotes the residue for the integrand of $\mathcal{I}_n$ at $\zeta \in 0, \infty$. Evaluating $R(\zeta, n)$, we find that F_1 becomes:

$$F_1 = 4\left\{x\mathcal{I}_0(\Gamma_m) - \left(v\mathcal{I}_1(\Gamma_m) - z\mathcal{I}_2(\Gamma_m) - \frac{\pi i}{4}\cdot\frac{v}{\sqrt{z}} + \text{c.c.}\right)\right\}, \tag{17}$$

Here, we define:

$$\mathcal{I}_n(\Gamma_m) = \oint_{\Gamma_m} \zeta^n \frac{d\zeta}{2\zeta\sqrt{\eta^{(4)}}} \quad (n = 0, 1, 2). \tag{18}$$

In the following subsections, the integral $\mathcal{I}_n(\Gamma_m)$ will be evaluated.

2.2. *Calculation of* $\mathcal{I}_n(\Gamma_m)$

To compute $\mathcal{I}_n(\Gamma_m)$ for $n = 0, 1, 2$, we extensively apply the theory of Weierstrass elliptic functions. We aim to express $\mathcal{I}_n(\Gamma_m)$ using the Weierstrass normal form. To achieve this, we introduce a single transformation:

$$\frac{(\zeta-\alpha)(1+\bar{\alpha}\beta)}{(\zeta-\beta)(1+|\alpha|^2)} = \frac{X-e_3}{e_1-e_3}, \tag{19}$$

where

$$e_1 = -\frac{\rho}{3}(k^2-2), \quad e_2 = \frac{\rho}{3}(2k^2-1), \quad e_3 = -\frac{\rho}{3}(k^2+1), \quad k = \frac{|1+\bar{\alpha}\beta|}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}}. \tag{20}$$

Let X_ζ be the image of ζ through this birational map. Using this notation, we have

$$X_0 = e_3 + (e_1 - e_3)\frac{\alpha}{\beta}\frac{1+\bar{\alpha}\beta}{1+|\alpha|^2} = e_3 + \rho\cdot\frac{\alpha}{\beta}\frac{1+\bar{\alpha}\beta}{1+|\alpha|^2}, \tag{21}$$

$$X_\infty = e_3 + \rho\cdot\frac{1+\bar{\alpha}\beta}{1+|\alpha|^2}. \tag{22}$$

Then, Eq. (19) turns out to be

$$\zeta = \beta\frac{X-X_0}{X-X_\infty}. \tag{23}$$

The contour Γ_m on the ζ-plane is mapped to one on the X-plane via Eq. (23), denoted by the same symbols, namely, Γ_m. Therefore, $\mathcal{I}_n(\Gamma_m)$ is expressed as

$$\mathcal{I}_n(\Gamma_m) = \oint_{\Gamma_m}\left(\beta\frac{X-X_0}{X-X_\infty}\right)^n \frac{dX}{Y}, \tag{24}$$

where

$$Y^2 = 4(X-e_1)(X-e_2)(X-e_3) = 4X^3 - g_2 X - g_3, \tag{25}$$

with

$$g_2 = \frac{4}{3}\rho^2(1-k^2+k^4), \quad g_3 = \frac{4}{27}\rho^3(k^2-2)(2k^2-1)(k^2+1). \tag{26}$$

For the case $n=0$, it is straightforward to see that

$$\mathcal{I}_0(\Gamma_m) = \oint_{\Gamma_m} \frac{dX}{Y} = \frac{2}{\sqrt{\rho}}K(k) \equiv 2\omega_1. \tag{27}$$

To evaluate Eq. (24) with $n=1,2$, we need to use the Weierstrass $\wp$-function, ζ-function, and σ-function. We define $u_\zeta \in \mathbb{C}/\Lambda$ for $\zeta \in \mathbb{C}\cup\infty$ by the equation

$$X_\zeta = \wp(u_\zeta), \tag{28}$$

and set $Y_\zeta = \wp'(u_\zeta)$. We divide u_ζ into the real part and the imaginary part with respect to the antiholomorphic involution $\zeta \mapsto -1/\bar{\zeta}$ on $\mathbb{C}\cup\infty$, i.e.,

$$u^\pm_\zeta = u_\zeta \pm u_{-1/\bar{\zeta}}. \tag{29}$$

Taking ζ to be infinity, we obtain

$$u^\pm_\infty = u_\infty \pm u_0. \tag{30}$$

We denote $(x_\pm, y_\pm)$ as the (X,Y)-coordinates of the point corresponding to $u^\pm_\infty$ via the Abel map. They can be obtained as

$$x_\pm = \frac{x \pm 6|z|}{3}, \quad y_+ = iv_+(x_+ - x_-), \quad y_- = v_-(x_- - x_+)\,. \tag{31}$$

Here we define

$$v_+ = \mathrm{Im}\frac{v}{\sqrt{z}}, \quad v_- = \mathrm{Re}\frac{v}{\sqrt{z}}. \tag{32}$$

In the following, we shall calculate $\mathcal{I}_1(\Gamma_m)$. First of all, we find

$$\mathcal{I}_1(\Gamma_m) = \beta\left[\oint_{\Gamma_m}\frac{dX}{Y} + \frac{X_\infty - X_0}{Y_\infty}\oint_{\Gamma_m}\frac{Y_\infty}{X - X_\infty}\frac{dX}{Y}\right]. \tag{33}$$

The first term in (33) can be readily evaluated with the use of (27). To calculate the second term, it is necessary to calculate the integral

$$\pi(X_\zeta) \equiv -\oint_{\Gamma_m}\frac{Y_\zeta}{X - X_\zeta}\frac{dX}{Y}\,, \quad \zeta \in \mathbb{C}\cup\{\infty\}\,, \tag{34}$$

with $\zeta = \infty$. We can calculate this for arbitrary ζ by using the Abel map and the formula

$$\frac{\wp'(v)}{\wp(u) - \wp(v)} = -\zeta(u+v) + \zeta(u-v) + 2\zeta(v)\,. \tag{35}$$

Then, we have

$$\pi(X_\zeta) = 4\det\begin{pmatrix} u_\zeta & \omega_1 \\ \zeta(u_\zeta) & \zeta(\omega_1)\end{pmatrix} \pmod{2\pi i\mathbb{Z}}\,. \tag{36}$$

In the case for $\zeta = \infty$, after several steps, we find

$$\pi(X_\infty) = \frac{1}{2}\left\{\pi(x_+) + \pi(x_-)\right\} + 2\omega_1 \frac{Y_\infty}{X_\infty - X_0} + a\pi i\,, \quad a \in \mathbb{Z}. \tag{37}$$

We substitute (27) and (37) into (33) and find

$$\mathcal{I}_1(\Gamma_m) = \frac{1}{4\sqrt{z}}\left\{\pi(x_+) + \pi(x_-) + 2a\pi i\right\}\,. \tag{38}$$

Next, we evaluate $\mathcal{I}_2(\Gamma_m)$. We first observe

$$\left(\frac{X - X_0}{X - X_\infty}\right)^2 = 1 + \frac{2(X_\infty - X_0)}{Y_\infty}\frac{Y_\infty}{X - X_\infty} + \left(\frac{X_\infty - X_0}{Y_\infty}\right)^2\left(\frac{Y_\infty}{X - X_\infty}\right)^2. \tag{39}$$

The integrals of the first term and the second term can be computed using Eq. (27) and Eq. (37), respectively. The integral of the second term is evaluated as

$$\oint_{\Gamma_m} \frac{2(X_\infty - X_0)}{Y_\infty}\frac{Y_\infty}{X - X_\infty}\frac{dX}{Y} = \frac{1}{2\beta\sqrt{z}}\left[\pi(x_+) + \pi(x_-) + 2a\pi i\right] - 4\omega_1. \tag{40}$$

To perform the integration of the third term in Eq. (39), we find

$$\left(\frac{Y_\infty}{X - X_\infty}\right)^2 = 2(X - X_\infty) - \frac{12X_\infty^2 - g_2}{2Y_\infty}\frac{Y_\infty}{X - X_\infty} - Y\frac{d}{dX}\left(\frac{Y}{X - X_\infty}\right). \tag{41}$$

By using this, we obtain

$$\begin{aligned}&\oint_{\Gamma_m}\left(\frac{Y_\infty}{X - X_\infty}\right)^2\frac{dX}{Y}\\ &= -4\eta_1 - 4\omega_1\left(\frac{x}{3} - 2\beta^2 z\right) + \frac{1}{2}\left(\frac{v}{\sqrt{z}} - 4\beta\sqrt{z}\right)\left[\pi(x_+) + \pi(x_-) + 2a\pi i\right].\end{aligned} \tag{42}$$

Therefore, with the use of Eq. (40) and Eq. (42), we have

$$\mathcal{I}_2(\Gamma_m) = -\frac{1}{z}\left[\eta_1 + \omega_1\cdot\frac{x}{3} - \frac{1}{8}\frac{v}{\sqrt{z}}\left(\pi(x_+) + \pi(x_-) + 2a\pi i\right)\right], \tag{43}$$

where η_1 is the quasi-period of ζ-function.

2.3. The function F in terms of elliptic integrals

We revisit the computation of F_1, aiming to evaluate the second term in Eq. (17) using Eq. (38) and Eq. (43). Consequently, we obtain

$$v\mathcal{I}_1(\Gamma_m) - z\mathcal{I}_2(\Gamma_m) - \frac{\pi i}{4}\frac{v}{\sqrt{z}} = \eta_1 + \omega_1\cdot\frac{x}{3} + \frac{v}{8\sqrt{z}}\left(\pi(x_+) + \pi(x_-)\right) + \frac{\pi i}{4}(a-1)\cdot\frac{v}{\sqrt{z}}. \tag{44}$$

Utilizing Eq. (31), we observe that the integrands of $\pi(x_\pm)$ are purely imaginary and real, respectively, leading to $\pi(x_+) \in i\mathbb{R}$ and $\pi(x_-) \in \mathbb{R}$. This results in

$$\frac{v}{\sqrt{z}}\left(\pi(x_+) + \pi(x_-)\right) + \text{c.c.} = 2\left(iv_+\pi(x_+) + v_-\pi(x_-)\right), \tag{45}$$

where $v_\pm$ are defined in Eq. (32). Using Eq. (31), we express F_1 from Eq. (17) as

$$F_1 = -8\eta_1 + 8(x_+ + x_-)\omega_1 - \left(iv_+\pi(x_+) + v_-\pi(x_-)\right) + 2\pi(a-1)v_+. \tag{46}$$

Here, we've employed $x = 6(x_+ + x_-)$. Consequently, the final expression for $F = F_2 + F_1$ is

$$F = -8\eta_1 + \left(8\omega_1 - \frac{3}{2h}\right)(x_+ + x_-) - \left(iv_+\pi(x_+) + v_-\pi(x_-)\right) + 2\pi(a-1)v_+. \tag{47}$$

2.4. Deriving the Kähler potential and the Kähler metric

The Kähler potential $K = K(z, \bar{z}, u, \bar{u})$ describing the Atiyah-Hitchin manifold is expressed through the generalized Legendre transformation as follows:

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, x) - (uv + \bar{u}\bar{v}), \tag{48}$$

subject to the conditions given by Eq. (5) and Eq. (6). The Kähler potential K satisfies the hyperkähler Monge-Ampère equation:

$$\det \begin{pmatrix} K_{z\bar{z}} & K_{z\bar{u}} \\ K_{u\bar{z}} & K_{u\bar{u}} \end{pmatrix} = 1. \tag{49}$$

Upon solving the conditions in Eq. (5) and Eq. (6) and eliminating v and $\bar{v}$, we obtain:

$$K = -8\eta_1 + 2(x_+ + x_-)\omega_1. \tag{50}$$

It is important to highlight that this Kähler potential is real-valued, consistent with the definition of the Kähler potential. This result stems from the choice of the double contour.

Next, we proceed to derive the Kähler metric. We introduce holomorphic coordinates Z and U defined by:

$$Z = 2\sqrt{z}, \quad U = u\sqrt{z}. \tag{51}$$

By this coordinate change, the hyperkähler Monge-Ampère equation is preserved:

$$\det \begin{pmatrix} K_{Z\bar{Z}} & K_{Z\bar{U}} \\ K_{U\bar{Z}} & K_{U\bar{U}} \end{pmatrix} = 1. \tag{52}$$

In the following, we calculate the components $K_{Z\bar{Z}}$, $K_{U\bar{Z}}$, $K_{Z\bar{U}}$, and $K_{U\bar{U}}$ of the metric with respect to the holomorphic coordinates (Z, U).

We initiate by evaluating $d\eta_1$ and $dx_\pm$ using dZ, $d\bar{Z}$, dU, and $d\bar{U}$. From Eq. (5), we find that U and $\bar{U}$ can be expressed as:

$$U = -\frac{1}{2}(\pi(x_+) + \pi(x_-)) - \pi i(a-1), \quad \bar{U} = -\frac{1}{2}(-\pi(x_+) + \pi(x_-)) + \pi i(a-1), \tag{53}$$

where a is the integer given by Eq. (37). This leads to:

$$dU = -\frac{1}{2}(d\pi(x_+) + d\pi(x_-)), \quad d\bar{U} = -\frac{1}{2}(-d\pi(x_+) + d\pi(x_-)). \tag{54}$$

Here, $d\pi(x_\pm)$ can be written in terms of $dx_\pm$, dg_2, and dg_3. In addition, dg_2 and dg_3 can be converted to $d\omega_1$ and $d\eta_1$. The expression of $d\pi(x_\pm)$ is:

$$d\pi(x_\pm) = \frac{4(x_\pm\omega_1 + \eta_1)}{y_\pm}dx_\pm + \frac{8(x_\pm^2 - V\eta_1)}{y_\pm}d\omega_1 - \frac{8(x_\pm + V\omega_1)}{y_\pm}d\eta_1, \tag{55}$$

where:

$$V = \frac{-3g_3\omega_1 + 2g_2\eta_1}{12\eta_1^2 - g_2\omega_1^2}. \tag{56}$$

Using Eq. (55) and Eq. (54) with $d\omega_1 = 0$ (obtained from Eq. (6)), we get:

$$\begin{aligned} d\eta_1 &= \frac{A_-(dU - d\bar{U}) - A_+(dU + d\bar{U}) + 4A_+A_-(\bar{Z}dZ + Zd\bar{Z})}{8(A_-B_+ - A_+B_-)}, && (57) \\ dx_\pm &= \frac{(-B_+ + B_-)dU - (B_+ + B_-)d\bar{U} + 4A_\pm B_\pm(\bar{Z}dZ + Zd\bar{Z})}{4(A_-B_+ - A_+B_-)}. && (58) \end{aligned}$$

By using Eq. (57) and Eq. (58), we can derive $K_{Z\bar{Z}}$, $K_{U\bar{Z}}$, $K_{Z\bar{U}}$, and $K_{U\bar{U}}$. Putting

$$\mathcal{Q} = (\eta_1 + e_1\omega_1)(\eta_1 + e_2\omega_1)(\eta_1 + e_3\omega_1) = \eta_1^3 - \frac{g_2}{4}\omega_1^2\eta_1 + \frac{g_3}{4}\omega_1^3, \tag{59}$$

we have:

$$K_{Z\bar{Z}} = -\frac{2}{\mathcal{Q}|Z|^2}\mathcal{K}4, \tag{60}$$

$$K_{U\bar{Z}} = \frac{v_-\mathcal{K}3+ + iv+\mathcal{K}3-}{2\mathcal{Q}\bar{Z}}, \tag{61}$$

$$K_{Z\bar{U}} = \frac{v_-\mathcal{K}3+ - iv+\mathcal{K}3-}{2\mathcal{Q}Z}, \tag{62}$$

$$K_{U\bar{U}} = -\frac{1}{2\mathcal{Q}|Z|^2}\mathcal{K}_2, \tag{63}$$

where:

$$\begin{aligned}\mathcal{K}_2 &= \left(\frac{g2}{4} - 3x_+x_-\right)\eta_1^2 \\ &\quad - \left(\frac{3g_3}{2} + (x_+ + x_-)\frac{g_2}{2}\right)\omega_1\eta_1 + \left(\frac{g_2^2}{16} + 3(x_+ + x_-)\frac{g_3}{4} + x_+x_-\frac{g_2}{4}\right)\omega_1^2,\end{aligned} \tag{64}$$

$$\mathcal{K}_{3\pm} = \eta 1^3 + 3x_\pm\omega_1\eta_1^2 + \frac{g_2}{4}\omega_1^2\eta_1 - \left(\frac{g_3}{2} + x_\pm\frac{g_2}{4}\right)\omega_1^3, \tag{65}$$

$$\begin{aligned}\mathcal{K}_4 &= \eta 1^4 + 2(x_+ + x_-)\omega_1\eta_1^3 + \left(\frac{g_2}{4} + 3x_+x_-\right)\omega_1^2\eta_1^2 \\ &\quad - \frac{g_3}{2}\omega_1^3\eta_1 - (x_+ + x_-)\frac{g_3}{4}\omega_1^4.\end{aligned} \tag{66}$$

Finally, $K_{U\bar{U}}$ is derived by substituting Eq. (60)–Eq. (62) into the Monge-Ampère equation Eq. (52). We have provided all the steps of the calculation of the Kähler potential the Kähler metric.

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