Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Stable solitary vortices in two-dimensional quasi-integrable systems

To cite this article: Atsushi Nakamula *et al* 2023 *J. Phys.: Conf. Ser.* **2667** 012010

Journal of Physics: Conference Series **2667** (2023) 012010 doi:10.1088/1742-6596/2667/1/012010

Stable solitary vortices in two-dimensional quasi-integrable systems

Atsushi Nakamula[1], Kiori Obuse[2], Nobuyuki Sawado[3*], Kohei Shimasaki[3], Kouichi Toda[4]

[1]Department of Physics, School of Science, Kitasato University, Sagamihara, Kanagawa 252-0373, Japan

[2]Graduate School of Environment, Life, Natural Science and Technology, Okayama University, Okayama 700-8530, Japan

[3]Department of Physics and Astronomy, Tokyo University of Science, Noda, Chiba 278-8510, Japan

[4]Department of Mathematical Physics, Toyama Prefectural University, Imizu, Toyama 939-0398, Japan

E-mail: *sawadoph@rs.tus.ac.jp

Abstract. Some solitary vortices to 2+1 quasi-integrable systems are discussed in the context of the planetary atmosphere. The Williams-Yamagata-Flierl (WYF) equation is one of the best candidates for the great red spot. We calculate the long-term simulation of the equation and find that the stable vortex is supported by a background zonal flow of a certain strength. The Zakharov-Kuznetsov (ZK) equation is a mimic of the WYF equation and considerably owes a great deal of its stability to the vortex. To learn more about the origin of longevity, we investigate the Painlevé test of the static ZK equation.

1. Introduction

The Great Red Spot (GRS) is a unique, extraordinary object in every respect. It is located at the south latitude $\phi = 22$ degrees and in the jet streams (the shear flows) with the depth at about 3000 km. It has an amazing longevity of over 400 years which suggests that there are some underlying nonlinear dynamics. In this paper, we discuss the properties of some nonlinear differential equations in two dimensions and how the quasi-integrable nature supports the existence of the vortices with certain longevity.

The candidates of such quasi-integrable equations inspired by shallow-water fluid dynamics are the Charney-Hasegawa-Mima equation and the Williams-Yamagata-Flierl (WYF) equation. Both of them have distinct stable vortex solutions, although it is not at all evident how they behave. In particular, the WYF equation has no conserved quantities thus we must say that it is completely different from the known integrable system. Nonetheless, there is still an effect of the underlying integrable nature through the Zakharov-Kuznetsov (ZK) equation, which is one of the mimics of the WYF equation. We investigate the Painlevé test of the static version of the ZK equation.

We summarize the basic parameters of the Jupiter[1]

$$\text{The mean radius: } R = 69,911 \text{ km}\,, \quad \text{the angular velocity: } \Omega = 1.76 \times 10^{-4} \text{ s}^{-1},$$

and also of the GRS [1]

$$\begin{aligned}&\text{The size: } L \sim 16,000 \text{ km}\,, \quad \text{the height: } H \sim 300 \text{ km } (200 \sim 500)\text{km}\,,\\ &\text{the speed: } V \sim 106 \text{ m s}^{-1}\,, \quad \text{the gravity acceleration } g \sim 24.0 \text{ m s}^{-2}\,.\end{aligned}$$

2. Two-dimensional governing equations of a perfect fluid

We construct the governing equation of the two-dimensional, non-divergent flow on the $\beta-$plane. We introduce the velocity vector

$$\vec{V}(x,y,t) = (u(x,y,t), v(x,y,t)), \quad (x,y) \in \mathbb{R}_2\,. \tag{1}$$

The equations of the motion on the β-plane at a certain point with latitude ϕ on a sphere

$$\frac{Du}{Dt} - (f+\beta y)v = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad \frac{Dv}{Dt} + (f+\beta y)u = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \qquad \frac{D}{Dt} := \frac{\partial}{\partial t} + \vec{V}\cdot\nabla\,, \tag{2}$$

where ρ is the constant density of the matter of fluid and $p(x,y,t)$ is the pressure. Equation (2) contain two parameters describing the Coriolis parameter $f = 2\Omega\sin\phi$ and $\beta = 2\Omega\cos\phi/R$. For the GRS of the Jupiter, we roughly estimate the values as

$$f = 1.32 \times 10^{-4}\text{s}^{-1}\,, \quad \beta = 4.66 \times 10^{-12}\text{m}^{-1}\text{s}^{-1}\,. \tag{3}$$

Since the flow is non-divergent $\nabla \cdot \vec{V} = 0$, the velocity can be expressed in terms of the stream function $\eta(x,y,t)$ as

$$u(x,y,t) = -\frac{\partial \eta(x,y,t)}{\partial y}\,, \quad v(x,y,t) = \frac{\partial \eta(x,y,t)}{\partial x}\,. \tag{4}$$

From (2), we obtain the equation for the vorticity

$$\frac{\partial Q}{\partial t} + J[\eta, Q] = 0\,, \quad Q = \Delta\eta + \beta y, \tag{5}$$

where the Jacobian $J[A,B] := \partial_x A \partial_y B - \partial_y A \partial_x B$ and the Laplacian $\Delta := \partial_x^2 + \partial_y^2$. The GRS is most likely a “shallow fluid dynamics” phenomenon. The shallow water equations on a β-plane on a planetary surface are

$$\begin{cases} \hat{\varepsilon}\dfrac{Du}{Dt} - (1+\hat{\beta}y)v = -\dfrac{\partial\eta}{\partial x}\,, \\ \hat{\varepsilon}\dfrac{Dv}{Dt} + (1+\hat{\beta}y)u = -\dfrac{\partial\eta}{\partial y}\,, \\ \dfrac{D\eta}{Dt} + \left(\eta + \dfrac{\hat{s}}{\hat{\varepsilon}}\right)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0\,, \end{cases} \tag{6}$$

where the time derivative is defined as $D/Dt := \partial/\partial t + \vec{V}\cdot\nabla = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$. The equations (6) are written in terms of the three parameters $\hat{\varepsilon}, \hat{\beta}, \hat{s}$, which are related with the

[1] https://nssdc.gsfc.nasa.gov/planetary/factsheet/jupiterfact.html

several scales of the phenomenon of our concern: the velocity V, the horizontal scale L, and the height scale H

$$\hat{\varepsilon} = \frac{V}{fL}\,, \quad \hat{\beta} = \frac{\beta L}{f}\,, \quad \hat{s} = \frac{gH}{f^2L^2}\,. \tag{7}$$

The system is written by a couple of nonlinear differential equations and is not the best one for finding the vortex solution. Rather, it is effective to solve a governing equation obtained estimates for Eqs.(6). We discuss this in detail in the next section.

3. The model of the Great Red Spot

As we saw in Eq.(7), the strength of the parameters $\hat{\varepsilon}, \hat{\beta}, \hat{s}$ reflect the environment of the phenomenon of our interest. As a result, once we make a choice about the order, the appropriate governing equation is found. In this manner, a geostrophic regime is established.

3.1. Quasi-geostrophic regime: the Charney-Hasegawa-Mima equation

For the quasi-geostrophic regime, the following order of strength of the parameters is assumed

$$\hat{\varepsilon} \sim \hat{\beta} \ll 1\,, \quad \hat{s} \sim O(1)\,. \tag{8}$$

We expand u, v, η by the scale parameter ϵ ($\sim \hat{\varepsilon}, \sim \hat{\beta}$)

$$\begin{aligned} u &= u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots\,, \\ v &= v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots\,, \\ \eta &= \eta^{(0)} + \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + \cdots\,. \end{aligned} \tag{9}$$

Plugging (9) into (6), we obtain the leading order contribution, *i.e.*, the geostrophic balance,

$$O(\epsilon^0): \qquad u^{(0)} = -\frac{\partial \eta^{(0)}}{\partial y}\,, \quad v^{(0)} = \frac{\partial \eta^{(0)}}{\partial x}\,, \quad \frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0, \tag{10}$$

which supports the relation of the velocity and the stream function (4). In the next order, we get the coupled equations

$$\begin{cases} \hat{\varepsilon}\dfrac{Du^{(0)}}{Dt} - \epsilon v^{(1)} - \hat{\beta} y v^{(0)} = -\epsilon \dfrac{\partial \eta^{(1)}}{\partial x}\,, \\ \hat{\varepsilon}\dfrac{Dv^{(0)}}{Dt} + \epsilon u^{(1)} + \hat{\beta} y u^{(0)} = -\dfrac{\partial \eta^{(1)}}{\partial y}\,, \\ \dfrac{\hat{\varepsilon}}{\hat{s}}\dfrac{D\eta^{(0)}}{Dt} + \epsilon\left(\dfrac{\partial u^{(1)}}{\partial x} + \dfrac{\partial v^{(1)}}{\partial y}\right) = 0\,. \end{cases} \tag{11}$$

In Eq.(11), we successfully eliminate $u^{(1)}, v^{(1)}$ and we finally obtain the equation

$$\frac{\partial}{\partial t}\left(\Delta\eta^{(0)} - \frac{1}{\hat{s}}\eta^{(0)}\right) + J\left[\eta^{(0)}, \Delta\eta^{(0)} - \frac{1}{\hat{s}}\eta^{(0)} + \frac{\hat{\beta}}{\hat{\varepsilon}}y\right] = 0, \tag{12}$$

which is called as the Charney-Hasegawa-Mima equation [2, 3]. The solution of the Eq.(12) contains the typical dipole solution [4], also the tripole or the quadrupole solutions with the inclusion of the background shear flow [5].

The plot in Fig.1, which depict the parameter dependence for L, show that the condition (8) is only satisfied for the very tiny size of the vortex L.

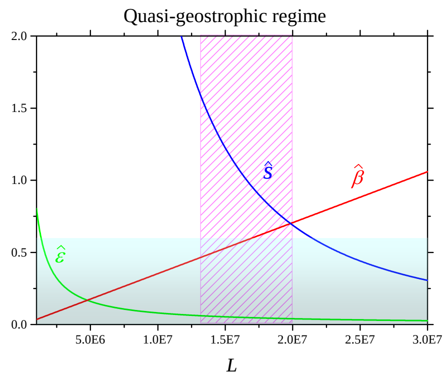

Figure 1. Parameters for the quasi-geostrophic regime estimated via (7). The red shaded area indicates a plausible size of the GRS and the blue shaded area corresponds to the condition (8).

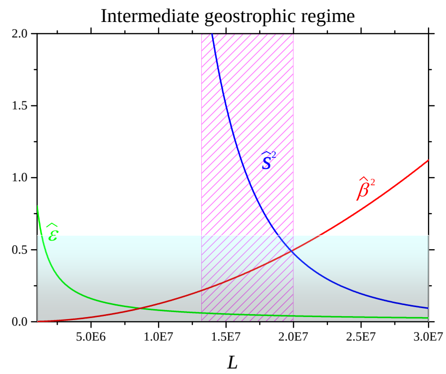

Figure 2. Same as Fig.1, but the parameters are of the intermediate geostrophic regime. The blue shaded area is now the condition (13). We have a window for this condition.

3.2. Intermediage geostrophic regime: Williams-Yamagata-Flierl equation

The intermediate-geostrophic regime [6, 7] is realized by order of parameters

$$\hat{\varepsilon} \sim \hat{\beta}^2\,, \quad \hat{s} \sim \hat{\beta}\,, \quad \hat{\beta} < 1\,. \tag{13}$$

Figure 2 shows there is a window for this condition. Now we expand u, v, η in terms of $\hat{\beta}$. The leading order contribution is

$$O(\hat{\beta}^0): \qquad u^{(0)} = -\frac{\partial \eta^{(0)}}{\partial y}\,, \quad v^{(0)} = \frac{\partial \eta^{(0)}}{\partial x}\,, \quad \frac{D^{(0)}\eta^{(0)}}{Dt} + \frac{\hat{s}\hat{\beta}}{\hat{\varepsilon}}(u_x^{(1)} + v_y^{(1)}) = 0\,, \tag{14}$$

where $D^{(0)}/Dt := \partial/\partial t + \vec{V}^{(0)} \cdot \nabla$, and $u_x^{(1)} := \partial u^{(1)}/\partial x$, $v_y^{(1)} := \partial v^{(1)}/\partial y$. The next order is

$$\begin{aligned} O(\hat{\beta}^1): \qquad & v^{(1)} + y v^{(0)} = \frac{\partial \eta^{(1)}}{\partial x}\,, \quad u^{(1)} + y u^{(0)} = -\frac{\partial \eta^{(1)}}{\partial y}\,, \\ & \hat{\beta}\frac{D^{(0)}\eta^{(1)}}{Dt} + \frac{D^{(1)}\eta^{(0)}}{Dt} + \frac{\hat{s}\hat{\beta}^2}{\hat{\varepsilon}}(u_x^{(2)} + v_y^{(2)}) + \eta^{(0)}\hat{\beta}(u_x^{(1)} + v_y^{(1)}) = 0\,, \end{aligned} \tag{15}$$

where $D^{(1)}/Dt := \partial/\partial t + \vec{V}^{(1)} \cdot \nabla$, and $u_x^{(2)} := \partial u^{(2)}/\partial x$, $v_y^{(2)} := \partial v^{(2)}/\partial y$. In terms of Eq.(15), the zeroth order continuity equation (14) is reduced to the equation of the Rossby wave with the phase velocity $V = \frac{\hat{s}\hat{\beta}}{\hat{\varepsilon}}$

$$\left(\frac{\partial}{\partial t} - \frac{\hat{s}\hat{\beta}}{\hat{\varepsilon}}\frac{\partial}{\partial x}\right)\eta^{(0)} = 0\,. \tag{16}$$

As in the discussion for the Rossby wave, we would like to consider the physics of the long-time scale. Therefore we introduce a variable describing the long-time scale $T := \hat{\beta} t$, $\hat{\beta} \ll 1$. We rewrite the continuity equation of the first order (15) such as

$$\frac{D^{(0)}\eta^{(1)}}{Dt} + \frac{D^{(1)}\eta^{(0)}}{DT} + \frac{\hat{s}\hat{\beta}}{\hat{\epsilon}}(u_x^{(2)} + v_y^{(2)}) + \eta^{(0)}(u_x^{(1)} + v_y^{(1)}) = 0\,. \tag{17}$$

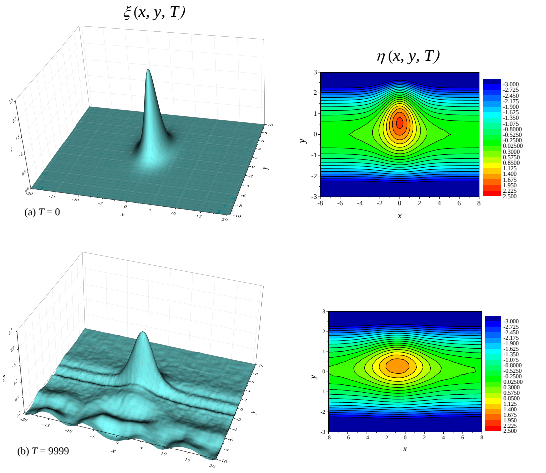

Figure 3. The profiles $\xi(x, y, t)$ and $\eta(x, y, t)$ in $(f_0, f_1) = (0.0, 1.2)$ for the iteration number (a) 0, (b) 9999×10^4 [8].

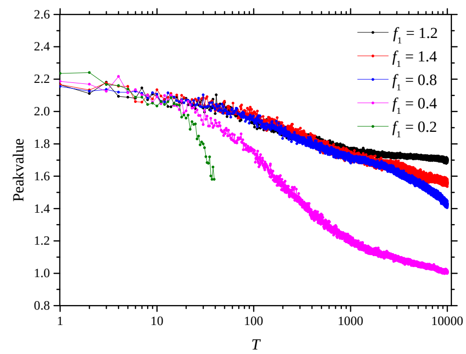

Figure 4. The long-term behavior of the vortex in terms of the peak amplitude for the case with $f_0 = 0.0$ with $f_1 = 0.2, 0.4, 0.8, 1.2, 1.4$. The solution with $f_1 = 1.2$ has a longevity [8].

The second-order equations are

$$O(\hat{\beta}^2): \quad \hat{\epsilon}\frac{D^{(0)}u^{(0)}}{Dt} - \hat{\beta}^2 yv^{(1)} - \hat{\beta}^2 v^{(2)} = -\hat{\beta}^2\eta_x^{(2)}\,, \quad \hat{\epsilon}\frac{D^{(0)}v^{(0)}}{Dt} + \hat{\beta}^2 yu^{(1)} + \hat{\beta}^2 u^{(2)} = -\hat{\beta}^2\eta_y^{(2)}\,, \tag{18}$$

where $\eta_x^{(2)} := \partial\eta^{(2)}/\partial x,\ \eta_y^{(2)} := \partial\eta^{(2)}/\partial y$. After some manipulations using (18), Eq.(17) becomes

$$\frac{\hat{\beta}}{\hat{s}}\frac{\partial\eta^{(1)}}{\partial t} - \frac{\hat{\beta}^2}{\hat{\epsilon}}\eta_x^{(1)} + \frac{\hat{\beta}}{\hat{s}}\frac{\partial\eta^{(0)}}{\partial T} - \frac{\partial}{\partial t}\nabla^2\eta^{(0)} - J[\eta^{(0)}, \nabla^2\eta^{(0)}] + \frac{2\hat{\beta}^2}{\hat{\epsilon}}y\eta_x^{(0)} - \frac{\hat{\beta}}{\hat{s}}\eta^{(0)}\eta_x^{(0)} = 0\,. \tag{19}$$

The first two terms concerning $\eta^{(1)}$ are the secular term for the equation of $\eta^{(0)}$, which is same form as the linear Rossby wave equation (16). In order to avoid the resonance, we replace the fourth term of Eq.(19) with Eq.(16). The governing equation for the long-timescale $T := \hat{\beta}t$ is

$$\frac{\hat{\beta}}{\hat{s}}\frac{\partial\eta^{(0)}}{\partial T} - \frac{\hat{\beta}}{\hat{s}}\eta^{(0)}\eta_x^{(0)} - \frac{\hat{s}\hat{\beta}}{\hat{\epsilon}}(\nabla^2\eta^{(0)})_x + \frac{2\hat{\beta}^2}{\hat{\epsilon}}y\eta_x^{(0)} - J[\eta^{(0)}, \nabla^2\eta^{(0)}] = 0\,. \tag{20}$$

This is the so-called Williams-Yamagata-Flierl equation of the intermediate geostrophic regime. The equation possesses *anti-cyclonic* $(\eta > 0)$ vortex solutions when the appropriate background zonal currents, $u^0(y) := f_0 + f_1 y,\quad f_0, f_1 \in \mathbb{R}$ are applied. We introduce a new variable

$$\xi(x, y, t) := \eta(x, y, t) + \int_0^y u^0(y')dy' \tag{21}$$

and, after some redefinition of the coordinates and the field, we get a form of the equation

$$\frac{\partial\xi}{\partial T} - 2\xi\frac{\partial\xi}{\partial x} - P(y)\frac{\partial}{\partial x}(\nabla^2\xi) + 2Q(y)\frac{\partial\xi}{\partial x} - 2J[\xi, \nabla^2\xi] = 0\,, \tag{22}$$

$$P(y) := 1 + 2u^0(y), \quad Q(y) := y + \frac{d^2u^0(y)}{dy^2} + \int_0^y u^0(y')dy'\,. \tag{23}$$

Figure 3 shows a vortex solution in the initial condition and the solution at a sufficiently large time. Figure 4 plots the peak value of the amplitude ξ for several strength of the zonal currents f_1.As f_1 increases, the decay of the peaks gradually decreases and the solution reaches the maximal stability at $f_1 = 1.2$.

4. 2+1 quasi-integrable systems: Zakharov-Kuznetsov equation

Unfortunately, the WYF equation possesses no conserved quantities [8], thus we must conclude that it is entirely distinct from any known integrable system. Nonetheless, for the longevity of the GRS, an integrable nature in the underlying nonlinear dynamics is of the dominant role. The ZK equation can be regarded as the mimics of the WYF equation and also has stable vortex solutions. Therefore, we think it is worthwhile to research the integrable property to comprehend the physics of the GRS. The ZK equation originally was the model of plasma with a uniform magnetic field [9] and have been studied as a two-dimensional extension of the well-known KdV dynamics [10, 11, 12]. The equation

$$\frac{\partial \phi}{\partial t} + 2\phi \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}(\nabla^2 \phi) = 0 \tag{24}$$

possesses meta-stable solitary vortices. The single soliton is stable in a dynamical sense. However, especially in the scattering of the two solitons with dissimilar height, the taller soliton becomes much taller while the shorter one turns out to be much shorter and radiates ripples [11]. The equation possesses the solutions propagating in a specific direction with uniform speeds. Here we set the direction in the positive x orientation with the velocity c, namely assuming $\phi = \Phi(x - ct, y)$. Plugging it into Eq.(24) we obtain

$$\nabla^2 \Phi = c\Phi - \Phi^2 \,, \tag{25}$$

where $\nabla^2 = \partial_{\tilde{x}}^2 + \partial_y^2$ and $\tilde{x} = x - ct$. The solutions to Eq.(24) keeping circular symmetry satisfy the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi_c}{dr}\right) = c\Phi_c - \Phi_c^2 \,, \qquad \Phi_c := \Phi_c(r)\,, \tag{26}$$

where $r := \sqrt{\tilde{x}^2 + y^2}$. With the boundary condition $\Phi_c \to 0$ as $r \to \infty$, the solutions are obtained in terms of a simple numerical study; they form one parameter family of c such as $\Phi_c(r) := cF(\sqrt{c}r)$.

The ZK equation is a known quasi-integrable system that has four conserved quantities. However, there is no rigorous discussion of the integrable property about the ZK equation. Now we would like to remind the Painlevé property. A n-th order nonlinear ODE

$$\mathcal{F}\left(z, w; w^{(1)}, ..., w^{(n)}\right) = 0, \qquad w^{(n)} := \frac{d^n w}{dz^n} \tag{27}$$

has the Painlevé property if and only if its general solution has only poles as movable singularities. And when Eq.(27) possesses the Painlevé property, it is *integrable*. It is clear that a necessary condition for the Painlevé property of Eq.(27) is that an ansatz for the general solution in the form of a formal Laurent series expansion

$$w(z) = \frac{1}{(z - z_0)^\alpha} \sum_{j=0}^{\infty} w_j \,(z - z_0)^j \tag{28}$$

is possible [13, 14, 15]. Here α is a positive integer, z_0 is arbitrary, and $n - 1$ of the coefficients w_j are arbitrary with $w_0 \neq 0$. The test to check if Eq.(27) possesses the Painlevé property is called *the Painlevé test.*

We will apply the Painlevé test in the sense of the Ablowitz-Ramani-Segur(ARS) method for Eq.(26) put forward in [13]. A leading term of the solutions to Eq.(26) has the form

$$\Phi_c \sim \Phi_0 \,(r - r_0)^{-\alpha} \tag{29}$$

as $r \to r_0$. Here r_0 is arbitrary. Now (29) gives

$$\frac{d^2\Phi_c}{dr^2} \sim \alpha(\alpha+1)\,\Phi_0\,(r-r_0)^{-(\alpha+2)}\,, \qquad \Phi_c^2 \sim \Phi_0^2\,(r-r_0)^{-2\alpha}\,. \tag{30}$$

Then we find $\alpha = 2$ and $\Phi_0 = -6$, by balancing the most singular terms of Eq.(26).

Next consider a solution $\phi = \phi(x,y,t)$ of Eq.(26) in the formal Laurent series expansion

$$\Phi_c = \sum_{j=0}^{\infty} \Phi_j \rho^{j-2}\,, \qquad \rho := r - r_0\,. \tag{31}$$

After substituting of the expansion (31) into Eq.(26), the recursion relations for the ϕ_j are presented as follows

$$\sum_{j=0}^{\infty}(j-2)\,(j-3)\,\rho^{j-4} + \frac{1}{r}\sum_{j=0}^{\infty}(j-2)\,\Phi_j\rho^{j-3} = c\sum_{j=0}^{\infty}\Phi_j\rho^{j-2} - \sum_{j=0}^{\infty}\sum_{k=0}^{j}\Phi_{j-k}\Phi_k\rho^{j-4}\,. \tag{32}$$

(32) reads

$$\left[r_0\sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}\Phi_{j-k}\Phi_k + (j-2)(j-3)\Phi_j\right\} + \sum_{j=1}^{\infty}\left\{\sum_{k=0}^{j-1}\Phi_{j-k-1}\Phi_k + (j-3)^2\Phi_{j-1}\right\} \right.$$
$$\left. -c\,r_0\sum_{j=2}^{\infty}\Phi_{j-2} - c\sum_{j=3}^{\infty}\Phi_{j-3}\right]\rho^{j-4} = 0\,. \tag{33}$$

After trivial algebraic calculations of (33), one can obtain $(j+1)(j-6)\Phi_j = F(\rho\,;\ \Phi_0,\ldots,\Phi_{j-1})$. One can find $F(\rho\,;\ \Phi_0,\ldots,\Phi_{j-1}) = 0$ only in the case of $j=-1,6$. Here $j=-1$ corresponds to the arbitrariness of r_0. And Φ_6 may be arbitrary, corresponding to $j=6$. To see whether Φ_6 is arbitrary, we are checking (32):

$$\begin{aligned}
&j=0: \quad \Phi_0 = -6\,; \qquad j=1: \quad \Phi_1 = \frac{6}{5r_0}\,; \qquad j=2: \quad \Phi_2 = -\frac{49}{50\,r_0^2} + \frac{c}{2}\,;\\
&j=3: \quad \Phi_3 = \frac{113}{125\,r_0^3}\,; \qquad j=4: \quad \Phi_4 = -\frac{4583}{5000\,r_0^4} - \frac{c^2}{400}\,;\\
&j=5: \quad \Phi_5 = \frac{-81139 + 1375\,c^2\,r_0^4}{75000\,r_0^5}\,; \qquad j=6: \quad 0\times\Phi_6 = \frac{6272}{3125\,r_0^5}\ \ (=:\beta)\,.
\end{aligned} \tag{34}$$

That is, Φ_6 seems not to be arbitrary because of $\beta \neq 0$. Therefore Eq.(26) does not pass the Painlevé test in the sense of the ARS method. That means Eq.(26) does not possess the Painlevé property since the general solution Φ_c does not admit the sufficient number of arbitrary coefficients in Eq.(31). However, for $1 \ll |r_0| < \infty$, β can be regarded as 0, since the power of r_0 is -5 in β. Consequently Eq.(26) can be considered as integrable-like system, or *quasi-integrable* system and its solution is the form like

$$\Phi_c \sim -\frac{6}{(r-r_0)^2} + \frac{c}{2} - \left\{\frac{c}{200}\,(r-r_0)\right\}^2 + \cdots\ , \tag{35}$$

in a certain range of parameter, *i.e.*, $1 \ll |r_0| < \infty$. Thus the ZK equation to not exactly integrable but can share some properties that integrable equations have.

It seems to be interesting and important to study the singular behaviors of solutions of the ZK equation in detail. For that, we will apply the Painlevé test in the sense of the Weiss-Tabor-Carnevale method for Eq.(24) put forward in [16]. It is one of the feature works in this research project.

5. Summary

We studied several equations of the two-dimensional, non-divergent flow. We obtained two-dimensional vortex solutions in the Williams-Yamagata-Flierl equation (23). The equation can be one of the best candidates for the great red spot. The model is not integrable in any sense: It has no conserved quantities and then the soliton-like stability of the solutions is not the consequence of such mathematical origin. The existence of the vortex is caused by the cooperation of several origins including external effects, *i.e.*, the scalar/vector nonlinearities, and the sheared flow. The scalar nonlinearity is equipped in the Zakharov-Kuznetsov equation, which has a relatively stable soliton solution, and the stability is realized by the underlying the KdV dynamics. For a deeper understanding of the stability of the vortex, we studied the Painlevé test of the ZK equation. We were successful in demonstrating that the ZK equation is quasi-integrable in the sense of the Painlevé property, and as a result, the solution ought to display a certain stability nature.

Acknowledgments N.S. and K.S. would like to thank all the conference organizers of QTS12 and Prof.Čestmir Burdík for the hospitality and also many kind considerations. K.S. was supported by Tokyo University of Science. A.N., N.S., and K.T. were supported in part by JSPS KAKENHI Grant Number JP20K03278, and also JP23K02794.

References

[1] Parisi M, Kaspi Y, Eli G, Durante D, Bolton S J, M L S, Buccino D R, Fletcher L N, M F W, Guillot T, Helled R, Luciano I, Li C, Oudrhiri K and Wong M H 2021 *Science* **374** 964–968
[2] Charney J G and Eliassen A 1949 *Tellus* **1** 38–54
[3] Hasegawa A and Mima K 1978 *The Physics of Fluids* **21** 87–92
[4] Larichev V and Reznik G 1976 *Doklady Akademii nauk SSSR* **231** 1077–1079
[5] Horihata S, Irie H and Sato M 1990 *Journal of the Physical Society of Japan* **59** 1242–1251
[6] Williams G and Yamagata T 1984 *Journal of The Atmospheric Sciences - J ATMOS SCI* **41** 453–478
[7] Charney J G and Flierl G R 1981 *Evolution of physical oceanography* **504** 548
[8] Koike Y, Nakamula A, Nishie A, Obuse K, Sawado N, Suda Y and Toda K 2022 *Chaos Solitons and Fractals: the interdisciplinary journal of Nonlinear Science and Nonequilibrium and Complex Phenomena* **165** 112782 (*Preprint* 2204.01985)
[9] Zakharov V and Kuznetsov E A 1974 *Soviet Physics JETP* **29** 594–597
[10] Petviashvili V I and Yan'kov V V 1982 *Dokl. Akad. Nauk SSSR* **267** 825–828
[11] Iwasaki H, Toh S and Kawahara T 1990 *Physica D: Nonlinear Phenomena* **43** 293–303 ISSN 0167-2789
[12] Klein C, Roudenko S and Stoilov N 2021 *Journal of Nonlinear Science* **31** 1–28
[13] Ablowitz M J R A and H S 1980 *Journal of Mathematical Physics* **21** 715–721
[14] Schmitz R 1997 *Applied Mathematics Letters* **10** 5–9
[15] Conte R 2000 *The Painlevé property one century later* (Springer-Verlag)
[16] Weiss J, Tabor M and Carnevale G 1983 *Journal of Mathematical Physics* **24** 522–526