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## Measurement uncertainties in regression analysis with scarcity of data

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**Abstract.** The evaluation of measurement uncertainty, in certain fields of science, faces the problem of scarcity of data. This is certainly the case in the testing of geological soils in civil engineering, where tests can take several days or weeks and where the same sample is not available for further testing, being destroyed during the experiment. In this particular study attention will be paid to triaxial compression tests used to typify particular soils. The purpose of the testing is to determine two parameters that characterize the soil, namely, cohesion and friction angle. These parameters are defined in terms of the intercept and slope of a straight line fitted to a small number of points (usually three) derived from experimental data. The use of ordinary least squares to obtain uncertainties associated with estimates of the two parameters would be unreliable if there were only three points (and no replicates) and hence only one degrees of freedom.

### 1. Introduction

The triaxial compression test involves having the soil specimen loaded to failure, by compression, when submitted to a specified confined stress. Through a series of tests, usually three, with different confinement stresses, the required experimental data to draw the failure line (stress coordinates representing failure, see Figure 2) are obtained, from which it is possible to determine its intercept and the slope. This information is then used to determine two parameters that characterize the soil, namely, the cohesion and friction angle. The coordinate axes represent the effective normal stress and the shear stress resulting from different consolidation stresses applied to the soil specimen. The triaxial compression test is performed according to the standard CEN ISO/TEC 17892-9 [1].

The study will focus on problems linked to the construction of a straight line given data from these experiments. The use of ordinary least squares to obtain uncertainties associated with estimates of the two parameters would be unreliable if there were only three points (and no replicates) and hence one degrees of freedom. These uncertainties would be more reliably obtained in terms of the uncertainties associated with estimates of quantities representing the coordinates of each point provided by measurement models for those quantities. Sometimes, however, there may be measured values available from a small number (two or possibly three) of experiments at each consolidation stress

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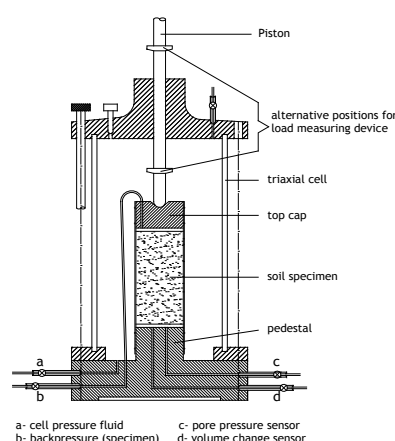
point. It will be discussed how such data should be used in conjunction with the latter uncertainties to provide combined uncertainties at each point.

The application of the GUM uncertainty framework [2] to the evaluation of measurement uncertainty is accepted in most areas of metrology, including testing. However, if the conditions supporting the underlying theory relating to model linearity, the number of input quantities in the measurement model and their probability distributions are not met, the outcome will represent an approximation that is difficult to assess. Experimental data from a set of triaxial compression tests will be used to highlight the importance and usefulness of an alternative method such as the Monte Carlo method – MCM [3], which has been successfully applied in areas of metrology where more conventional methodologies have proved inadequate [4].

In summary, this work aims at a rigorous evaluation of the measurement uncertainties for tests in the civil engineering field, using experimental data from triaxial compression tests in soils, through the application of a set of statistical tools not commonly applied in metrology outside national measurement institutes. This objective arises also from the growing need to evaluate measurement uncertainties associated with testing, according to the international standard NP EN ISO 17025:2005, and may represent a first step to the generalization of this methodology to other areas.

## 2. Test procedure for triaxial compression in soils

One of the most used tests for the characterization, in laboratories, of the shear resistance of soils is the triaxial test. The test specimen is cylindrical and protected by a confining elastic membrane that prevents fluid from penetrating the specimen. The specimen is placed on a pedestal, in a triaxial chamber, so that the axes of the specimen and the chamber are aligned, as illustrated in figure 1. Within the chamber there is a fluid, normally de-aired water, whose pressure is controlled through a pressure connection system (connection “a” in Figure 1). On the other hand, the top of the specimen is loaded by a load frame, with the force exerted through a load cell. During the test there is also the need to apply another pressure, called back pressure (connection “b” in Figure 1), within the specimen, and finally, to measure the porous pressure in the specimen, called the pore pressure (connection “c” in Figure 1).



**Figure 1.** Scheme of the triaxial chamber.

For the present study isotropic consolidated and drained triaxial compression tests were performed on specimens that had been previously saturated, according to the CEN ISO/TEC 17892-9 [1] standard. The specimens were obtained by compaction, for the same moisture content and compaction energy, so that the same bulk density was obtained. The specimens were compacted with a diameter

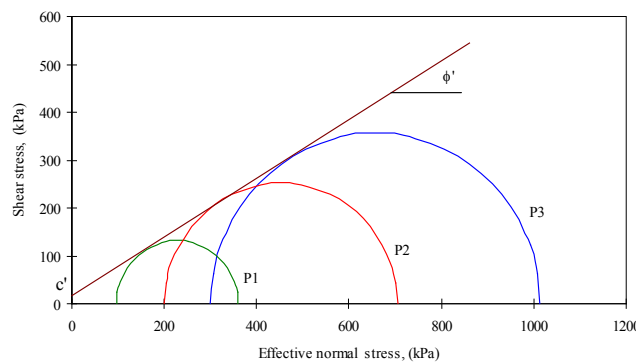
close to 70 mm, and a height/diameter relation close to 2 (the test standard recommends this relation to be in the interval [1.85, 2.25]).

## 2.1. Triaxial compression tests

In the context of this study nine specimens in total were tested, with consolidation stress values of 100 kPa, 200 kPa and 300 kPa, to allow a comparison of the common situation of having three specimens, one at each consolidation stress value (baseline condition), with a situation where there is a double repetition at each of those testing points. In this work only the results of the shear phase will be presented, in which, among others, the quantities measured include: horizontal (radial) stress ( $\sigma_3$ ), pore pressure ( $u$ ), axial force ( $P$ ), height variation ( $\Delta H$ ) and volume variation ( $\Delta V$ ).

The results presented in the following figures correspond to a set of three of those tests. For some of the consolidation stresses tests were repeated to obtain experimental reproducibility, as will be referred to in more detail later during uncertainty evaluation.

In Figure 2 the failure stress condition is illustrated, using Mohr circles and the corresponding failure line.



**Figure 2.** Evaluation of the resistance parameters using Mohr circles at failure.

In practice, it is common to represent the failure stress condition just by a single point, in a  $(p, q)$  or  $(p', q')$  coordinate system, where

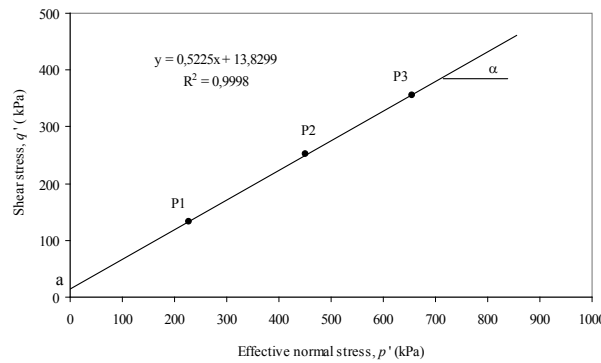
$$p = (\sigma_1 + \sigma_3)/2, \quad p' = (\sigma_1' + \sigma_3')/2, \quad q = q' = (\sigma_1 - \sigma_3)/2 = (\sigma_1' - \sigma_3')/2. \quad (1)$$

In Table I we display the consolidation stresses and also the failure values of  $p'$  and  $q'$  for the same tests. Figure 3 displays the more common graphical representation of a straight-line regression, obtained by ordinary least squares, whose parameters (slope and intercept) are the basis for the evaluation of the friction angle  $\phi'$  and the cohesion  $c'$  [1], obtained using an Excel spreadsheet.

TABLE I  
Results of tests for different consolidation stresses

Specimen	P1	P2	P3
Consolidation stress/kPa	100	200	300
$q'$ at failure/kPa	132	252	355
$p'$ at failure/kPa	228	452	655

The usual approach is to determine the resistance parameter values, from such a straight-line regression, with no uncertainty associated with them. In this case, the data displayed in Figure 3 yields,  $\phi' = \arcsin(\tan \alpha) = 31.5$  and  $c' = a/\cos(\phi') = 16.2$ .



**Figure 3.** Straight-line regression, by ordinary least squares, for the evaluation of the friction angle  $\phi'$  and the cohesion  $c'$ .

### 3. Uncertainty evaluation

The evaluation of uncertainty related to the triaxial compression test of soils, described in the preceding section, will be carried out using the two indicated approaches, the GUM uncertainty framework and MCM.

#### 3.1. The GUM uncertainty framework

In its simplest expression, the application of the GUM uncertainty framework assumes independent quantities, or *at most* correlations associated with measured values are negligible, and that the expansion of the Taylor formula on which this methodology is based uses just first order derivatives. The linearity of the measurement model is, in this way, implicitly assumed in this approach. Even so, the calculation involved in forming first order derivatives may entail considerable algebraic effort that is not required by MCM.

The combined standard uncertainty is the square root of the combined variance, and is given by

$$u^2(y) = \sum_{i=1}^N \sum_{j=1}^N \frac{\partial f}{\partial X_i} \bigg|_{X=x} u(x_i, x_j), \frac{\partial f}{\partial X_j} \bigg|_{X=x} \quad (2)$$

where the partial derivatives  $\partial f / \partial X_i$ , referred to as sensitivity coefficients, are evaluated at the estimates  $x_i$  of  $X_i$ . Another way of writing (1), more appropriate for numerical implementations, is  $u^2(y) = c^T V_x c$  with sensitivity coefficients  $c$  and covariances  $V_x$  [6].

In the case of the triaxial compression test, the final tests results – friction angle and cohesion – are based, as mentioned earlier, on the slope of the straight-line regression function and its intercept with the ordinate axis. This function is in general established based on just three points, as is normal practice in this type of test. The ordinate and abscissa axes represent the shear stress and the effective mean normal stress, respectively. These quantities, on the other hand, are calculated using expression (1) and the following expressions:

$$\sigma'_1 = \sigma_1 - u, \quad \sigma'_3 = \sigma_3 - u, \quad (3)$$

$$\sigma_1 = \sigma_3 + \frac{P}{A}. \quad (4)$$

The pore pressure  $u$  in the specimen, indicated by expression (3), relates the total stress and the effective stress, not only for the axial stress ( $\sigma_1$ ) but also for the horizontal stress ( $\sigma_3$ ). Substituting in (1) the terms of the axial and horizontal stresses by their definitions (3) and (4), the following expressions (5) and (6) are obtained, with which the uncertainty evaluation will be performed:

$$q' = \frac{P}{2A}, \quad (5)$$

$$p' = \frac{2\sigma_3 - 2u + (P/A)}{2}, \quad (6)$$

where the area  $A$  is calculated using expression (7):

$$A = \frac{V_i - \Delta V_c - \Delta V}{H_i - \Delta H_c - \Delta H}. \quad (7)$$

It should be noted that the evaluation of the area  $A$  of the specimen involves quadratic terms ([1, p18]), which brings a degree of complexity to the evaluation of the sensitivity coefficients needed for applying the GUM uncertainty framework. The assumption of linearity, in this case, is no longer valid.

### 3.2. The MCM alternative method

The Monte Carlo method (MCM), as referred to above, has some advantages over the GUM uncertainty framework, namely not requiring the assumption of linearity and therefore being affected by the corresponding approximation entailed by it, or, for coverage intervals, the assumption of normality. In many problems, MCM has a relatively simple implementation, because problems with the GUM uncertainty framework do not arise only in complicated cases, but also, for example, in problems with few input quantities in non-linear models or in models where the dominant uncertainty source is associated with a non-Gaussian distribution.

For  $q = 1, \dots, M$ , let  $\xi_q = (\xi_{1,q}, \dots, \xi_{n,q})^T$ , where  $\xi_{j,q}$  is a random draw made from the PDF for the  $j$ th input quantity  $X_j$ . Thus,  $y_q = f(\xi_q)$  represents a random draw from the distribution that characterizes  $Y = f(X)$ . A best estimates, associated standard uncertainty and coverage intervals can be readily determined from the set of values  $\{y_q, q = 1, \dots, M\}$ . In summary, the implementation of MCM requires the existence of a mathematical model, which can be as simple as an additive model of two or three quantities, requires assigning a distribution to each input quantity, and requires also the ability to draw at random from those PDFs.

## 4. Discussion of results

The GUM uncertainty framework requires the use of expression (2) for the determination of measurement uncertainties, and the evaluation of the sensitivity coefficients using partial derivatives, whereas for MCM, expressions (5) and (6) are used directly, each quantity being represented by the PDF assigned to it. Each of these expressions is evaluated  $M$  times ( $10^6$  in our case) and from the resulting set of values of the output quantity the required statistical parameters are calculated. A graphical representation of the set of values is also possible to illustrate the shape of the output distribution.

After the evaluation of  $q'$  and  $p'$ , and the associated uncertainties, one is in a position to determine the parameters of a straight-line regression function – the slope and intercept – based on which it is then possible to determine the friction angle and the cohesion characterizing a particular soil, which are the required output quantities. In situations where there are uncertainties associated with the pair values of the linear regression, the correct statistical procedure implies to incorporate those uncertainties in the process of evaluation the regression parameters, in such a way that the output quantities may also be adequately quantified in terms of measurement uncertainties.

With these input data, using both methods, the stress values of the failure line and their standard uncertainties, required for the straight-line regression, were determined. The results obtained are shown in Tables II and III. The difference between uncertainties given by both approaches are quite noticeable regarding quantity  $q'$ , which corroborates the previous statement of a non-linear expression associated with the evaluation of area  $A$  and the approximations that the GUM uncertainty framework could entail.

TABLE II  
Results from GUM uncertainty framework

Specimen	$q'/\text{kPa}$	$u_{q'}/\text{kPa}$	$p'/\text{kPa}$	$u_{p'}/\text{kPa}$
1	132	2.60	228	0.47
3	252	2.60	452	0.47
5	355	2.60	655	0.46

TABLE III  
Results from MCM

Specimen	$q'/\text{kPa}$	$u_{q'}/\text{kPa}$	$p'/\text{kPa}$	$u_{p'}/\text{kPa}$
1	132	1.63	228	0.47
2	252	1.63	452	0.47
3	355	1.62	655	0.47

It is apparent how the uncertainty associated with the area is affected in the case of  $p'$  whereas for the variable  $q'$  this effect is negligible. It was also established that the PDF assigned to each input variable is in this case very important, and in fact if Gaussian PDFs were assumed to all variables the difference in the standard uncertainties found for  $p'$  would also be negligible, which reinforces the message of caution in the application of the GUM uncertainty framework.

The point pairs  $(p', q')$  and the respective uncertainties of Table III are the input data for the application of the least squares method to a straight-line regression for the 3 points considered. The procedure adopted for the determination of this straight-line regression was iterative Generalised Distance Regression – GDR [6]. The main results obtained in the evaluation of the straight-line function may be represented by the vector  $\mathbf{a}$  of dimension  $2 \times 1$  of parameter estimates and the associated covariance matrix  $\mathbf{U}_a$  of dimension  $2 \times 2$ :

$$\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{U}_a = \begin{bmatrix} u^2(a) & u(a,b) \\ u(a,b) & u^2(b) \end{bmatrix}.$$

In this approach for the determination of estimates of the straight-line regression parameters it is assumed that the straight-line model is valid and that the uncertainties associated with the experimental data give a valid measure of the expected deviations of the data from the straight-line due to random effects influencing the data. After the parameters have been estimated, the observed deviations of the data from the best fit straight-line may be compared with the expected deviations. This comparison is generally made on the basis of an aggregated measure of the deviations, expressed by the sum  $R$  of the squares of residual values obtained. If  $R$  is much larger or smaller than expected, there is cause to question the validity of the model and/or the uncertainties considered. The comparison is carried out assuming that the input distributions are Gaussian (normal), and with this assumption,  $R$  is associated with a chi-squared distribution with  $m-2$  degrees of freedom, where  $m$  is the number of data points, and the percentiles of this distribution may be used to associate a probability with the observed value  $R$  [6].

The application of this methodology to the data of Table III, that is to the pairs of points ( $p', q'$ ) and respective uncertainties, produced the values of  $a$ ,  $b$ ,  $\phi'$  (friction angle) and  $c'$  (cohesion) shown in Table IV and in Figure 5.

TABLE IV  
Results of straight-line regression for specimens 1 to 3

	$\phi' / ^\circ$	$u_{\phi'} / ^\circ$	$c' / \text{kPa}$	$u_{c'} / \text{kPa}$
No validation	31.6	----	16.7	----
After validation	31.6	0.63	16.7	5.13

The validation referred to in the previous Table is established by comparing the observed  $R$  value with the chi-squared distribution with  $\nu = m - 2$  degrees of freedom and probability 95 %. As the observed  $R$  value was outside the corresponding interval, a value  $\hat{\sigma}^2 = R/(m-2)$  was determined to adjust the matrix  $U_a$  proportionally to this factor. The uncertainty values  $u(a)$  and  $u(b)$  were inflated in this way whereas the values of  $a$  and  $b$  remained unchanged.

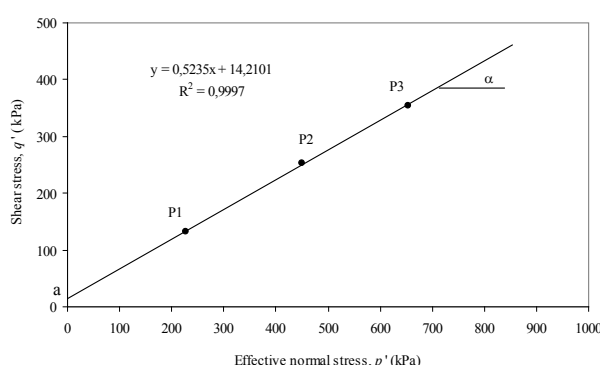


Figure 5-Straight-line regression with uncertainties evaluated using MCM

Having available the experimental dispersion values related to the consolidation stresses of 100 kPa, 200 kPa and 300 kPa, for a total of 3 tests at each of these stresses, it was decided to use these values in two ways. First, the linear regression was repeated with these experimental uncertainties, much larger than the previous uncertainties, and the results in Table V and Figure 6 were obtained. Then, the friction angles and cohesion for each of these 3 tests were calculated and an estimative for the experimental dispersion of  $\phi'$  e de  $c'$  was obtained.

TABLE V  
Results of straight-line regression for specimens 1 to 5 with experimental dispersion

	$\phi' / ^\circ$	$u_{\phi'} / ^\circ$	$c' / \text{kPa}$	$u_{c'} / \text{kPa}$
Validation accepted	30.6	0.36	22.12	2.55

Note how the introduction of larger uncertainties in the model of linear regression led to the acceptance of the model validation, with no need for an adjustment, and how the uncertainties of  $\phi'$  and  $c'$ , without the adjustment factor, resulted in lower values. The values of direct experimental standard deviation for  $\phi'$  and  $c'$  were of 0.92 and 5.9, respectively.



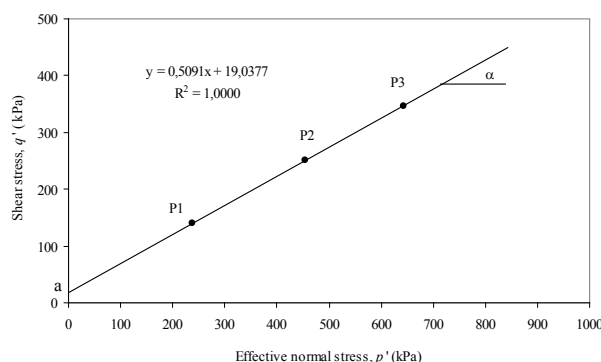


Figure 6- Straight-line regression with uncertainties by experimental dispersion

## 5. Conclusions

The GUM uncertainty framework overestimates uncertainties when there are non linear terms in the mathematical models, particularly in situations where there are rectangular distributions involved in the uncertainty budget, so that the assumption of a Gaussian output is inadequate;

Using either the GUM or the MCM the resulting uncertainties proved to be smaller than expected, with respect to the linear regression model used, which means that probably not all uncertainty sources and/or existent correlations were taken into account, and that in complex systems having experimental data is a very important tool for validation, and mathematical formulation can be insufficient;

The linear regression with uncertainties based on experimental dispersion results in a valid linear regression model, but ideally more replicates should be used. The uncertainties associated with the friction angle and the cohesion, when obtained solely by the experimental data are smaller than those obtained by generalised distance regression, with inflated (scaled) covariance matrix;

For the friction angle the obtained uncertainty values permit to conclude that for this type of soil, with specimens casted in laboratory under specified conditions, a value not larger than  $0.65^\circ$  is probably a good indication of the expected dispersion of values. With respect to the value of cohesion, the variation of uncertainty values is larger, and the indication of a typical uncertainty value is more difficult, in the range of values tested.

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