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Determining the curve with the quickest descent under gravitational force alone

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Abstract. Various curvatures can be drawn between two different points. However, what curve will yield the fastest descent for an object to travel from the higher point to the lower point under the force of gravity alone? This essay utilizes theoretical methods such as calculus, optimization, and parametric equations and experimental approaches to show that a curve - shaped like a cycloid - will yield the fastest descent. This conclusion has practical applications for the design of roller coaster trajectories along with other entertainment facilities.

1. Introduction

The discovery of the brachistochrone curve originated as a problem that troubled the distinguished physicist Galileo. In 1696, Johann Bernoulli posed a challenge for other scientists to find the brachistochrone curve, which Jakob Bernoulli, Isaac Newton, and Gottfried Wilhelm Leibniz found to be the cycloid. According to Britannica, a cycloid is defined as "the curve generated by a point on the circumference of a circle that rolls along a straight line", shown in figure 1.

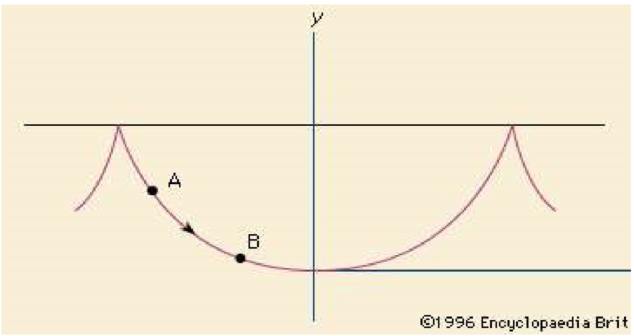

Figure 1 Diagram of a Cycloid

More recently, scholars have completed research into this field of study. For instance, Gladkov and Bogdanova (2020) proved that in the absence of friction forces, the minimization problem for the motion time for any motion along a curvilinear trough under the action of the gravity force can always be reduced to the brachistochrone problem. Pinky and Jyotindra (2019) used spline collocation to discuss the brachistochrone curve and demonstrate that this method is effective and accurate. Spline collocation method attained the best approximation for the non-linear problems without any assumption and linearization. However, scholars rarely used experimentational data to prove the brachistochrone problem. Therefore, this essay aims to use detailed experimentations along with simplistic mathematical proofs to explore the brachistochrone problem with the assumption of negligible friction forces.

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2. problem formulation

Based on this definition, a function between the vector from the origin to any endpoint on the cycloid and the included angle (theta) can be derived. The vector $\vec{r}$ can equate to $\vec{r} = x(\theta)\imath + y(\theta)\jmath$, which is the sum of the x and y components, $x(\theta)\imath$ and $y(\theta)\jmath$. As shown in Figure 2, the vector $\overrightarrow{AB}$ is the sum of $\overrightarrow{AO} + \overrightarrow{OB}$ according to the laws of adding vectors. Furthermore, $\overrightarrow{AO} = \overrightarrow{AC} + \overrightarrow{CO}$, where $\overrightarrow{CO}$ is the radius of the circle, r; and $\overrightarrow{OB} = -rsin(\theta) - rcos(\theta)$.

Let the included angle $A\hat{O}C = \theta$ (in rads).

$\overrightarrow{AC}$ equals the portion of the circumference that the circle has travelled, which is the arc length $= \theta r$.

$$\because\ \overrightarrow{AB} = x(\theta)\imath + y(\theta)\jmath = \overrightarrow{AO} + \overrightarrow{OB} \tag{1}$$
$$\overrightarrow{AO} = \overrightarrow{AC} + \overrightarrow{CO} = \theta r_x + r_y$$
$$\overrightarrow{OB} = -rsin(\theta)_x - rcos(\theta)_y$$

Then adding up the x and y components separately.

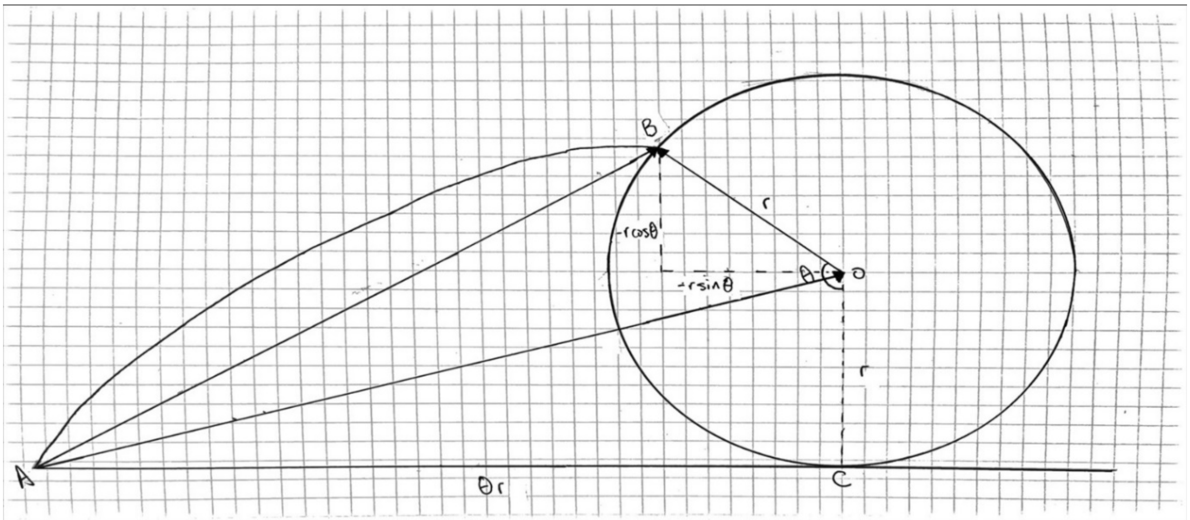

Figure 2 Tracing the Path of a Cycloid

$$\therefore \begin{cases} x(\theta)\imath = \theta r - rsin(\theta) = r(\theta - \sin(\theta)) \\ y(\theta)\jmath = r - rcos(\theta) = r(1 - \cos(\theta)) \end{cases}$$

Therefore, the parametric equation for a cycloid is $x = r(\theta - \sin(\theta))$ and $y = r(1 - \cos(\theta))$.

3. problem analysis

This lab is aimed to test the effects of different curves on the time required for a body to descend. To guarantee the significance of this lab – and not merely test an existing theory – the independent variable will be selected from some common curves that the roller coaster's track from to accelerate the ride, which are a straight line, a cycloid, a quatre-arc, and a semi-arc. The lower point will be at a 15° angle of depression in respect to the higher point, guaranteeing some initial acceleration. Furthermore, they will be ranked by their deviation from the shape of a cycloid. Since these curves are symmetrical shapes, the deviation will be measured by the distance between the mid-point of the designated curve and that of the cycloid.

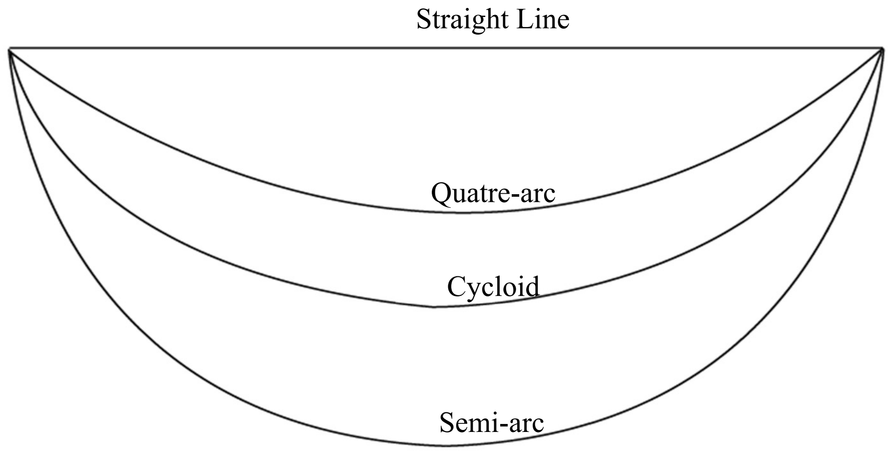

Figure 3 Variations of the Curve

Then, these curves in Figure 3 will be drawn using a mathematical software, then copied to an illustrator file, and cut out onto wooden boards using a laser cutter. The curve will be glued to a horizontal plane and the surface will be sanded out to minimize friction. On the contrary, the dependent variable will be measured by using a timer to record the time of descent for a rolling solid ball from the higher point to the lower point.

3.1 Hypothesis:
If the deviation between the shape of a certain curve and that of a cycloid increase, then the time of descent will increase because cycloid is the curve that yields the fastest descent from a higher point to a lower point, with negligible friction and air resistance, and under the force of gravity alone.

Suppose $O(0,0)$ and $A(p,q)$ are two points of different height, and the mass travels from point O to point A. Assume that the curve that yields the fastest descent is the function $y = y(x)$. According to conservation of energy where the potential energy at point O before the release of the mass equals the kinetic energy at point A.

$$mgy = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gy}$$

In this equation, v is the velocity of the mass, g is the gravitational acceleration, and y is the vertical distance that the mass travelled.

At the point (x,y) of the curve, the velocity of the mass is also equivalent to

$$v = \frac{ds}{dt} = \sqrt{1+y'^2}\frac{dx}{dt}$$

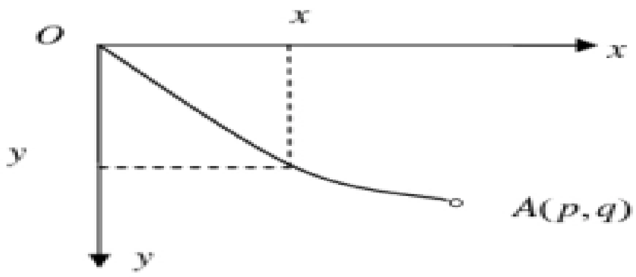

Figure 4 Graph of the Brachistochrone

See Appendix 1 for the proof of the identity $ds = \sqrt{1+y'^2}dx$. In this equation, ds is arc length of the curve and t is time. Expressing this rate of change in terms of dt

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$$dt = \frac{\sqrt{1+y'^2}}{v}dx = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Since the mass travels the horizontal distance from 0 to p, then integrating in terms of time where J is the functional of y

$$t = J(y) = \int_0^p \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Therefore, the time required for the mass to travel from O to A is a function of y, and the problem of fastest decent becomes finding a curve y that allows the functional to have the minimum value out of all functions that satisfy the boundary conditions of all the continuous function $y(x)$.

The boundary condition being

$$y(0) = 0, y(p) = q$$

The problem now becomes a variational problem where the goal is to find the extreme values of the functional. Solving the equation

$$J[y(x)] = \int_0^p \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Using the Euler-Lagrange equations (6), within the range of

$$y(0) = 0, y(p) = q$$

gives the equation

$$y = 2rsin^2x\frac{\theta}{2} = r(1-cos\theta)$$

And integrating in terms of θ

$$x = r(1-\sin\theta) + x_0$$

Because the curve passes through $(0,0)$ and (p, q) so $x_0 = 0$. Hence the parametric equations are

$$\begin{cases} x = r(\theta - \sin(\theta)) \\ y = r(1-\cos(\theta)) \end{cases}$$

proving the curve to be a cycloid.

3.2 Variables:

Table 1： variables and definition

Variable	Name and Unit of Measurement	How will I measure or control it?
Independent Variable	The shape of the curve: straight line, cycloid, quatre-arc, and semi-arc. Thus, the deviation between these curves and a cycloid will be measured by the distance in meters from the mid-point of the curve to the mid-point of a cycloid.	I will be manipulating this variable by changing the track that the ball will roll on which has different deviations when compared with a cycloid. The trajectories will be made using two pieces of wooden board which form the designated curve.
Dependent Variable	The time in seconds of descent for the mass to travel from the higher point to the lower point	I will be measuring this variable using a digital stopwatch and multiple trials will be completed to test the effect of an increasing deviation to the time of descent.
Controlled Variable	The position of the endpoints	I will control this variable by using the same template for the two coordinates on illustrator when I am drawing the curves.

		The wooden boards will also be placed on the same level to guarantee that the endpoints correspond.
Controlled Variable	The weight of the mass	I will control this variable by using the same ball which will be a marble with light weight and low elasticity.
Controlled Variable	The roughness of the surface of the wooden boards	I will control this variable by using the same type of wood with the same texture.
Controlled Variable	The initial velocity of the mass	I will control this variable by using a glass rod as a blockage for the golf ball and moving the rod upwards to release the ball without exerting any additional force.

3.3 Materials:2 pieces of laser-cut wooden board for each curve (12 in total);20 g Marble x1;Digital stopwatch x1;10 cm glass rod x1

3.4 Procedure:

- Tape the wooden boards of the same curve parallel to each other, leaving a 2 cm gap in between.
  - Place the marble at the higher endpoint of the trajectory and use the glass rod to block the marble from rolling down.
  - Release the marble by moving the glass rod upwards and simultaneously start the stopwatch.
  - End the stopwatch at the moment when the marble reaches the lower endpoint.
  - Record the time of descent on the data collection table for the corresponding curve.
  - Complete two more trials for this same trajectory.
  - Repeat steps 2~7 for the rest of the curves.

Safety Precautions: Place a box under the lower endpoint of the track with any type of cushioning to avoid the marble from rolling or dropping elsewhere

3.5 Data Collection:

Table 2: The effects of a curve's deviation from a cycloid on the time of descent

	Time of Descent (s)						
Shape of the curve in ascending deviation order (cm)	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Average	Uncertainties
Cycloid 0 cm	0.42	0.46	0.43	0.42	0.45	0.436	± 0.020
Quarter-arc 3.5 ± 0.5 cm	0.47	0.50	0.50	0.56	0.51	0.508	± 0.045
Semi-arc 5.3 ± 0.5 cm	0.59	0.56	0.55	0.61	0.65	0.592	± 0.045
Straight line 10 ± 0.5 cm	0.86	0.83	0.82	0.78	0.82	0.822	± 0.045

3.6 Graph:
By graphing via a scattered plot, the linear growth of the y values can be easily seen.

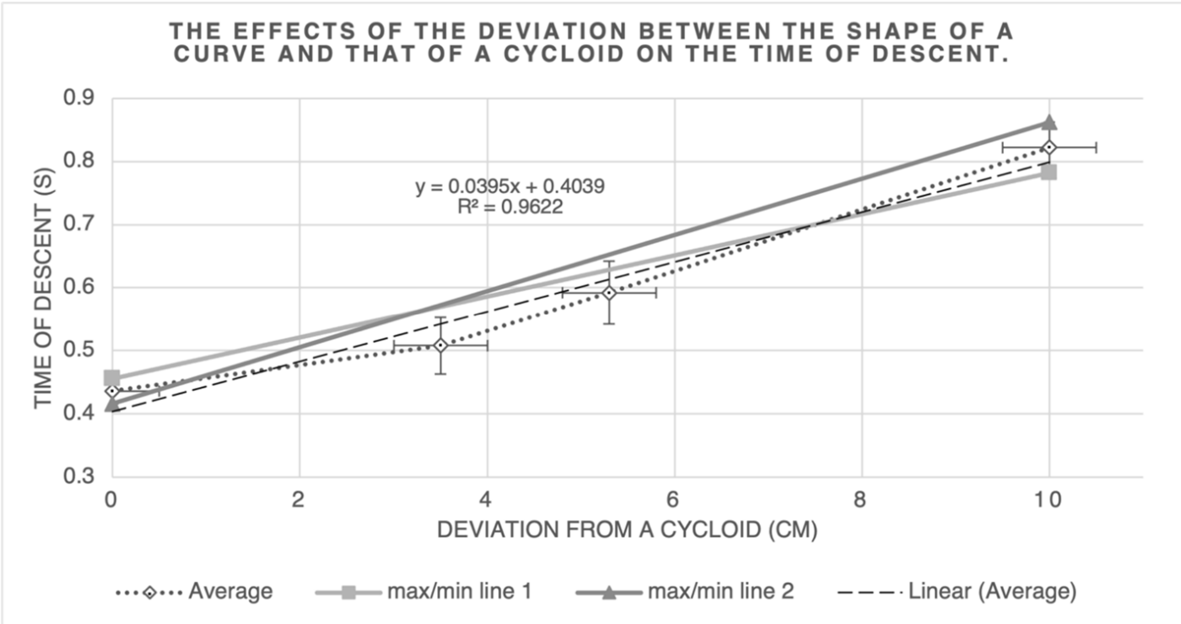

Figure 5 The Effects of the Deviation

4. Conclusion:

The data shows a linear correlation between a curve's deviation from the cycloid and the time of descent it yields for 96% of the data collected, given by the linear trendline $y = 0.0395x + 0.4039$. Interpreting this equation, within the domain of $x \in (0,10)$, when the degree of deviation increases by 1 cm, the mass travels an extra 0.0395 seconds. This trendline also suggests that the time of descent can be calculated for any given x within the domain the experiment. Nonetheless, the y-intercept indicates the time of descent when the mass travels on the curvature of a cycloid because that is when there is no deviation, giving 0.4 seconds for 96% of the data.

The data do support my hypothesis because this linear correlation shows that when a curve's deviation from a cycloid increase, the time of descent increases linearly. This distribution of data also shows that cycloid yields the fastest descent because when curves deviate from the cycloid, they travel for a longer time. Similarly, Yang Hans et al (2011) conducted an experiment at the University of British Columbia on the Experimental Analysis of the Brachistochrone using a Kinematical Approximation Method. Their experimental data has shown that "the times of descent increase for curves as they deviate from the shape of the cycloid", which supports my hypothesis.

Contextually, this research can be used in roller coaster design. The physics behind a roller coaster is simple—the transfer of potential energy into kinetic energy. A crawler track first brings the roller coaster to the first peak, where potential energy is maximized, and then, gravity acts as the only force to accelerate the ride down the track transferring the stored potential energy into kinetic energy. This corresponds to the purpose of this investigation. Roller coaster track designer can use my research to adjust the speed of the roller coaster by altering the track's deviation from the curvature of a cycloid. Specifically, designers can use this linear correlation to predict the extent of increment in speed in respect to its deviation, creating the track with the best user experience.

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