Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

On resolving efficient domination number of path and comb product of special graph

To cite this article: I Kusumawardani *et al* 2022 *J. Phys.: Conf. Ser.* **2157** 012012

On resolving efficient domination number of path and comb product of special graph

I Kusumawardani[2], Dafik[1,2,*], E Y Kurniawati[1], I H Agustin[1,3], R Alfarisi[1,4]

[1]CGANT-University of Jember, Indonesia
[2]Department of Post. Mathematics Education, University of Jember, Indonesia
[3]Department of Mathematics, University of Jember, Indonesia
[4]Department of Primary School, University of Jember, Indonesia
[*]Corresponding author

E-mail: d.dafik@unej.ac.id

Abstract. We use finite, connected, and undirected graph denoted by G. Let $V(G)$ and $E(G)$ be a vertex set and edge set respectively. A subset D of $V(G)$ is an efficient dominating set of graph G if each vertex in G is either in D or adjoining to a vertex in D. A subset W of $V(G)$ is a resolving set of G if any vertex in G is differently distinguished by its representation respect of every vertex in an ordered set W. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be a subset of $V(G)$. The representation of vertex $v \in G$ in respect of an ordered set W is $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is called a resolving set of G if $r(u|W) \neq r(v|W)\ \forall\ u, v \in G$. A subset Z of $V(G)$ is called the resolving efficient dominating set of graph G if it is an efficient dominating set and $r(u|Z) \neq r(v|Z)\ \forall\ u, v \in G$. Suppose $\gamma_{re}(G)$ denotes the minimum cardinality of the resolving efficient dominating set. In other word we call a resolving efficient domination number of graphs. We obtained $\gamma_{re}G$ of some comb product graphs in this paper, namely $P_m \triangleright P_n$, $S_m \triangleright P_n$, and $K_m \triangleright P_n$.

1. Introduction

There are numerous kind of graphs. In this paper, we use graphs that have characteristics finite, connected and undirected denoted by G. Chartrand and Lesniak [13] have introduced that a graph G as a set of finite and nonempty objects named vertices along with a set of unordered pairs of distinct vertices of G named edges which is possibly empty. $V(G)$ and $E(G)$ repectively denotes the vertex set and the edge set of graf G. There are so many various kinds of studies in graph theory, some of them are dominating set and metric dimension of graph.

Boutin [7] has presented the stereotype of resolving set in a graph. A subset W of $V(G)$ is categorized as a resolving set of G if each vertex in G is differently distinguished by its distance representation in respect to every vertex in an ordered set W. Consider $W = \{w_1, w_2, w_3, \ldots, w_k\}$ is a subset of $V(G)$. The distance representation of vertex $x \in G$ in respect of an ordered set W is $r(x|W) = (d(x, w_1), d(x, w_2), ..., d(x, w_k))$. The set W is called a resolving set of G if $r(x|W) \neq r(y|W)\ \forall\ x, y \in G$. $dim(G)$ denotes the metric dimension of graph G which is referred to as the minimum cardinality of a resolving set. There are a few sorts of metric dimension. Some of them have been researched in [26, 4, 27]. We provide several previous results on the metric dimension in [22, 11, 30, 31, 32, 35, 14, 2].

Another study carried out in graph theory apart from resolving set is dominating set. Du and Wan [19] have explained the concept of dominating set. D is called a dominating set of graph G if D is a subset of $V(G)$ and each vertex in G which is not in D is adjacent to at least a vertex in D. $\gamma(G)$ denotes the domination number which is the minimum cardinality of a dominating set in G. A few sorts of dominating set have been researched in [24, 28]. Chartrand, et al. [15] stated that if a subset of $V(G)$ has no two vertices within the set are adjacent, so it is called an independent set.

Deng, et al. [18] have explained one type of dominating set called the efficient dominating set. If D is an independent subset of $V(G)$ which each vertex in $G-D$ is adjacent to precisely a vertex in D, it is categorized as the efficient dominating set of graph G. The efficient domination number of graph G is the minimum cardinality of an efficient dominating set of graph G and denoted by $\gamma_e(G)$ [20]. Bange, et al [6] presented that if graph G has an efficient dominating set, at that point the cardinality of any efficient dominating set equals the domination number of G .Therefore, the cardinality of all efficient dominating sets are the same. Several previous results on this topic are studied in [9, 8, 12, 24, 25, 16].

Birgham et al.[10] combined the concept of dominating set and metric dimension which is stated as the resolving dominating set. The resolving domination number is its minimum cardinality and denoted by $\gamma_r G$. A resolving dominating set of graph G is a subset Y of $V(G)$ which is not only a ominating set of G but also determine every vertex of G by its distance representation in respect to every vertex in Y. We refer to [1, 21, 23, 34] for some previous results. Several types of resolving domination number are resolving strong domination number, resolving perfect domination number, and resolving efficient domination number, see [28, 24].

Recently, Hakim, et al. [20] constructed a new stereotype by combining the theory of resolving set and the theory of efficient dominating set which is stated as the resolving efficient dominating set. Z is called the resolving efficient dominating set of any graph G if it is not only statisfy the characteristic of efficient set and dominating set, but also $r(x|Z) \neq r(y|Z)\ \forall\ x, y \in G$. $\gamma_{re}G$ denotes the resolving efficient domination number which is the minimum cardinality of the resolving efficient dominating set of graph G. Farther, There are also results on resolving strong domination number and resolving perfect domination number in [16, 5].

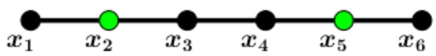

Figure 1. Efficient dominating set of P_6.

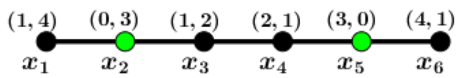

Figure 2. Resolving efficient dominating set of P_6.

In this paper, all graphs studied are the comb product graph. Consider we use two finite, connected and undirected graphs denoted by G_1 and G_2. The notation $G_1 \triangleright G_2$ denotes a new graph obtained by gaining a copy of graph G_1 and some duplicates of graph G_2 as many as the vertices in the graph G_1 and implanting the ith copy of graph G_2 to the ith vertex of graph G_1 which is stated as the comb product of graph G_1 and graph G_2 [17].

2. Previous Results
In this section, we provide several propositions and theorems as previous results. Those are,

(i) Birgham [10], has found the dimension of path and complete graph. For every positive interger greater than or equal to 2, $dim(K_n) = n - 1$ and $dim(P_n) = 1$

(ii) [36]$dim(W_m) = \begin{cases} 3, & \text{if } m \in \{3, 6\} \\ 2, & \text{if } m \in \{4, 5\} \\ \lfloor \frac{2n+2}{5} \rfloor, & \text{if otherwise} \end{cases}$

(iii) If G is a connected graph with order greater than or equal to 2 and o is a leaf of P_n, then $dim(G \triangleright_o P_n) \geq dim(G)$ [29].

(iv) [3] $\gamma(P_n) = \lceil \frac{n}{3} \rceil$

(v) Umilasari [33], found that
$\gamma(P_m \triangleright P_n) = \begin{cases} m\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 0(mod3) \text{ and } n \equiv 2(mod3) \\ n\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1(mod3) \end{cases}$

(vi) For every graph G, $max\{\gamma(G), dim(G)\} \leq \gamma_r(G) \leq \gamma(G) + dim(G)$ [10].

(vii) Hakim, et al. [20] have found the exact value of some comb product of special graph i.e. $\gamma_{re}(K_n \triangleright C_3) = \gamma_{re}(K_n \triangleright P_3) = n$, and $\gamma_{re}(W_n \triangleright P_3) = \gamma_{re}(W_n \triangleright C_3) = \gamma_{re}(S_n \triangleright P_2) = n + 1$

3. Results

We discovered the resolving efficient domination number of path and some comb products of special graph in this section. Those are $P_m \triangleright P_n$, $S_m \triangleright P_n$, and $K_m \triangleright P_n$. The followings bellow are theorems that will be utilized.

Theorem 1 *Let P_n be a path, for $n \geq 4$, $\gamma_{re}(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. The vertex set of P_n is $V(P_n) = \{x_a : 1 \leq a \leq n\}$ and its set of edge is $E(P_n) = \{x_a x_{a+1} : 1 \leq a \leq n-1\}$. The order and size of P_n respectively are $|V(P_n)| = n$ and $|E(P_n)| = n - 1$. We choose three subsets of P_n, namely $D = \{x_a : 1 \leq a \leq n : a \equiv 1(mod\ 3)\}$ for $n \equiv 1, 2(mod\ 3)$, and $D = \{x_a : 2 \leq a \leq n-1 : a \equiv 2(mod\ 3)\}$ for $n \equiv 0(mod\ 3)$ and we have $|D| = \lceil \frac{n}{3} \rceil$. We will show that D is resolving efficient dominating set with the minimum cardinality by the following steps.

First, we will show that D satisfies the characteristic of efficient dominating set. For any $x_a, x_b \in D$, $d(x_a, x_b) \geq 3$, so it means $|N(x_a \in V(P_n) - D) \cap D| = 1$ and every vertex $x_a \in V(P_n) - D$ is dominated by exactly one vertex $x_a \in D$. Thus, we stated that subset D is an efficient dominating set.

Second, we will show that subset D we have choose also satisfies the characteristic of resolving set, that the representation of each vertex is different one another. The distance representation of any vertex $x_a \in V(P_n)$ respect to D is provided in Table 1.

Table 1. The distance representation of all vertex of graph P_n to D.

v	$r(v\|D)$	condition
x_a	$(\underbrace{\|a-k\|, \ldots, \|a-k\|}_{\lceil \frac{n}{3} \rceil})$	$k \equiv 1(mod\ 3), 1 \leq k \leq n$ for $n \equiv 1, 2(mod\ 3)$
x_a	$(\underbrace{\|a-k\|, \ldots, \|a-k\|}_{\lceil \frac{n}{3} \rceil})$	$k \equiv 2(mod\ 3), 2 \leq k \leq n-1$ for $n \equiv 0(mod\ 3)$

We can see that D is a resolving set based on the representation of each vertex provided in Table 1 because each vertex has distinct representation one another. Thus, we stated that D is a resolving set of P_n.

Third, we will show that D is the resolving efficient dominating set with the minimum cardinality. Supposing $|D_1| < \lceil \frac{n}{3} \rceil$, so we have $|D_1| = \lceil \frac{n}{3} \rceil - 1$. Here are the conditions that might occur

(i) For $n \equiv 0(mod\ 3)$, any vertex $\{x_a\ : a \equiv 2(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a-1}, x_a, x_{a+1}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set

(ii) For $n \equiv 1(mod\ 3)$
- vertex $x_1 \notin D_1 \rightarrow \exists\ x_1,\ x_2$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set
- vertex $x_n \notin D_1 \rightarrow \exists\ x_{n-1},\ x_n$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set
- any vertex $\{x_a :\ a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a-1},\ x_a, x_{a+1}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set

(iii) For $n \equiv 2(mod\ 3)$
- vertex $x_1 \notin D_1 \rightarrow \exists\ x_1,\ x_2$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set
- any vertex $\{x_a :\ a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a-1},\ x_a, x_{a+1}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set.

Subset D_1 doesn't statisfy the characteristic of efficient dominating set, it contradicts the resolving efficient dominating set. Therefore D with the minimum cardinality $|D| = \lceil \frac{n}{3} \rceil$ must be the efficient dominating set of P_n. □

Theorem 2 *For every positive interger $m, n \geq 2$,*

$$\gamma_{re}(P_m \triangleright P_n) = \begin{cases} m\lceil \frac{n}{3} \rceil & if\ n \equiv 0(mod\ 3)\ and\ n \equiv 2(mod\ 3) \\ m\lfloor \frac{n}{3} \rfloor + \lceil \frac{m}{3} \rceil & if\ n \equiv 1(mod\ 3), m \equiv 1, 2(mod\ 3) \\ \lceil \frac{m}{2} \rceil \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{3} \rceil & if\ n \equiv 1(mod\ 3), m \equiv 0(mod\ 3). \end{cases}$$

Proof. Graph $P_m \triangleright P_n$ is a comb product graph of one graph P_m and m copies graph P_n where the vertices $x_{a,1}$ are the sticking point on the graph P_m. The vertex set is $V(P_m \triangleright P_n) = \{x_{a,b} : 1 \leq a \leq m : 1 \leq b \leq n\}$, and the edge set is $E(P_m \triangleright P_n) = \{x_{a,1}x_{a+1,1} : 1 \leq a \leq m-1\} \cup \{x_{a,b}x_{a,b+1} : 1 \leq i \leq m : 1 \leq j \leq n-1\}$. The order and size respectively are $|V(P_m \triangleright P_n)| = mn$ and $|E(P_m \triangleright P_n)| = mn - 1$. We choose subset D as below

$$D = \begin{cases} \{x_{a,b} : 1 \leq a \leq m : 2 \leq b \leq n : b \equiv 2(mod\ 3)\} & \text{for } n \equiv 0, 2(mod\ 3) \\ \\ \{x_{a,b} : 1 \leq a \leq m-1 : 3 \leq b \leq n-1 : & \text{for } n \equiv 1(mod\ 3), m \equiv 0(mod\ 3) \\ \quad a \equiv 1(mod\ 2) : b \equiv 0(mod\ 3)\} \cup \{x_{a,b} : \\ \quad 2 \leq a \leq m : 1 \leq b \leq n : b \equiv 1(mod\ 3) :\} \\ \quad a \equiv 0(mod\ 2)\} \\ \\ \{x_{a,b} : 1 \leq a \leq m : 1 \leq b \leq n : a, b \equiv 1(mod\ 3)\} & \text{for } n \equiv 1(mod\ 3), m \equiv 1, 2(mod\ 3) \\ \quad \cup \{x_{a,b} : 2 \leq a \leq m : 3 \leq b \leq n-1 : \\ \quad a \not\equiv 1(mod\ 3) : b \equiv 0(mod\ 3)\} \end{cases}$$

We have $|D| = m\lceil \frac{n}{3} \rceil$ for $n \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$, while $|D| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{m}{3} \rceil$ for $n \equiv 1(mod\ 3)$. We will show that D is resolving efficient dominating set with the minimum cardinality by the following steps.

First, we will show that D satisfies the characteristic of efficient dominating set. For any $x_{k,l}, x_{m,n} \in D,\ d(x_{k,l}, x_{m,n}) \geq 3$, so it means $|N(x_{a,b} \in V(P_m \triangleright P_n) - D) \cap D| = 1$ and every vertex $x_{a,b} \in V(P_m \triangleright P_n) - D$ is dominated by exactly one vertex in D. Thus, we stated that subset D is an efficient dominating set.

Second, we will show that subset D we have choose also satisfies the characteristic of resolving set, that the representation of each vertex is different one another. To find out whether the representation of each vertex respect of the element in D is different from one another, we can see the distance function of any two vertices in $P_m \triangleright P_n$ as follows.

$$d(x_{(a,b)}x_{(k,l)}) = \begin{cases} |b-l| & \text{if } a = k \\ |a-k| + |b-l| & \text{if } a \neq k \end{cases}$$

Based on the subset D we have choose and the distance function of any two vertices, we know that the representation of each vertex respect of the elements in D must be different one another. Thus, D satisfies the characteristic of resolving set

Third, we show that D is resolving efficient dominating set with the minimum cardinality and we have four cases. Suppose that $|D_1| < m\lceil \frac{n}{3} \rceil$, so we have $|D_1| = m\lceil \frac{n}{3} \rceil - 1$. The first case is when $n \equiv 0(mod\ 3)$, the condition that might occur is any vertex $\{x_{a,b} : b \equiv 2(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. The second case is when $n \equiv 2(mod\ 3)$, here are some conditions that might occur

- Any vertex $x_{a,n} \notin D_1 \rightarrow \exists\ x_{a,n}, x_{a,n-1}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set
- Any vertex $x_{a,2} \notin D_1 \rightarrow \exists\ x_{a,1}, x_{a,2}, x_{a,3}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set
- Any vertex $\{x_{a,b} : b \equiv 2(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, we can't say that D_1 is an efficient dominating set.

The third case is when $n \equiv 1(mod\ 3)$ and $m \equiv 1, 2(mod\ 3)$, suppose that $|D_1| < m\lfloor \frac{n}{3} \rfloor + \lceil \frac{m}{3} \rceil$, so we have $|D_1| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{m}{3} \rceil - 1$. Here are the conditions that might occur

(i) For $m \equiv 1(mod\ 3)$

- Any vertex $\{x_{a,b} : 2 \leq a \leq m-1 : 3 \leq b \leq n-1 : a \not\equiv 1(mod\ 3) : b \equiv 0(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.
- Any vertex $\{x_{a,1} : 1 \leq a \leq m : a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a-1,1}, x_{a,1}, x_{a+1,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $x_{a,n} : 1 \leq a \leq m : a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,n}, x_{a,n-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 1 \leq a \leq m : 1 < b < n : a \equiv 1(mod\ 3) : b \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

(ii) For $m \equiv 2(mod\ 3)$

- Any vertex $\{x_{a,b} : 2 \leq a \leq m : 3 \leq b \leq n-1 : a \not\equiv 1(mod\ 3) : b \equiv 0(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.
- Any vertex $\{x_{a,1} : 1 \leq a \leq m-1 : a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a-1,1}, x_{a,1}, x_{a+1,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $x_{a,n} : 1 \leq a \leq m-1 : a \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,n}, x_{a,n-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 1 \leq a \leq m-1 : 1 < b < n : a \equiv 1(mod\ 3) : b \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

The fourth case is when is when $n \equiv 1(mod\ 3)$ and $m \equiv 0(mod\ 3)$, supposing that $|D_1| < \lceil \frac{m}{2} \rceil \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{3} \rceil$, so we have $|D_1| = \lceil \frac{m}{2} \rceil \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{3} \rceil - 1$. The conditions that might occur are

- Any vertex $\{x_{a,b} : 1 \leq a \leq m : 3 \leq b \leq n-1 : a \not\equiv 2(mod\ 3) : b \equiv 0(mod\ 3)\} \notin D_1 \to \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.
- Any vertex $\{x_{a,1} : a \equiv 2(mod\ 3)\} \notin D_1 \to \exists\ x_{a-1,1}, x_{a,1}, x_{a+1,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $x_{a,n} \notin D_1 \to \exists\ x_{a,n}, x_{a,n-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 2 \leq a \leq m : 1 \leq b < n : a \equiv 2(mod\ 3) : b \equiv 1(mod\ 3)\} \notin D_1 \to \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

Subset D_1 doesn't statisfy the characteristic of efficient dominating set for $P_m \triangleright P_n$, it contradicts the resolving efficient dominating set. Therefore, $|D| = m\lceil \frac{n}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $P_m \triangleright P_n$ when $n \equiv 0, 2(mod\ 3)$, $|D| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{m}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $P_m \triangleright P_n$ when $n \equiv 1(mod\ 3)$ and $m \equiv 1, 2(mod\ 3)$, and $|D| = \lceil \frac{m}{2} \rceil \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $P_m \triangleright P_n$ when $n \equiv 1(mod\ 3)$ and $m \equiv 0(mod\ 3)$. □

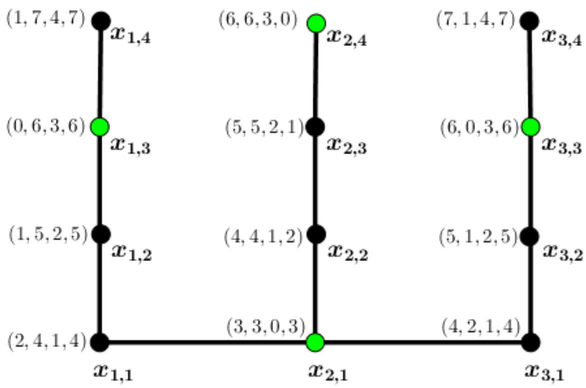

Figure 3. The resolving efficient dominating set of $P_3 \triangleright P_4$.

Theorem 3 *For every positive interger $m, n \geq 2$,*

$$\gamma_{re}(S_m \triangleright P_n) = \begin{cases} (m+1)\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 0, 2(mod\ 3) \\ m\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1(mod\ 3). \end{cases}$$

Proof. Graph $S_m \triangleright P_n$ is a comb product graph of one graph S_m and $m+1$ copies of graph P_n where the vertices x_a are the sticking point on the graph S_m. The vertex set is $V(S_m \triangleright P_n) = \{x_a : 1 \leq a \leq m+1\} \cup \{x_{a,b} : 1 \leq a \leq m+1 :\ 1 \leq b \leq n-1\}$, and the edge set is $E(S_m \triangleright P_n) = \{x_1 x_a : 2 \leq a \leq m+1\} \cup \{x_a x_{a,1} :\ 1 \leq a \leq m+1\} \cup \{x_{a,b} x_{a,b+1} : 1 \leq a \leq m+1 :\ 1 \leq b \leq n-2\}$. The order and size respectively are $|V(S_m \triangleright P_n)| = mn + n$ and $|E(S_m \triangleright P_n)| = mn + n - 1$. We choose subset D as below

$$D = \begin{cases} \{x_{a,b} : 1 \le a \le m+1 : 1 \le b \le n-1 : b \equiv 1(mod\ 3)\} & \text{for } n \equiv 0, 2(mod\ 3) \\ \{x_1\} \cup \{x_{1,b} : 3 \le b \le n-1 : b \equiv 0(mod\ 3)\} \cup & \text{for } n \equiv 1(mod\ 3) \\ \{x_{a,b} : 2 \le a \le m+1 : 2 \le b \le n-2 : b \equiv 2(mod\ 3)\} & \end{cases}$$

We have $|D| = (m+1)\lceil \frac{n}{3} \rceil$ for $n \equiv 0, 2(mod\ 3)$ and $|D| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil$ for $n \equiv 1(mod\ 3)$. We will show that D is resolving efficient dominating set with the minimum cardinality by the following steps.

First, we will show that D satisfies the characteristic of efficient dominating set. For any $x_{k,l}, x_{m,n} \in D$, $d(x_{k,l}, x_{m,n}) \ge 3$, so it means $|N(x_{a,b}, x_a \in V(S_m \triangleright P_n) - D) \cap D| = 1$ and every vertex $x_{a,b}, x_a \in V(S_m \triangleright P_n) - D$ is dominated by exactly one vertex in D. Thus, we stated that subset D is an efficient dominating set.

Second, we will show that subset D we have choose also satisfies the characteristic of resolving set. To find out whether the representation of each vertex respect of the elements in D is different one another, we can see the distance function of any two vertices in $S_m \triangleright P_n$ as follows.

$$d(x_{a,b}x_1) = b+1$$

$$d(x_a x_1) = \begin{cases} 0 & \text{if } a = 1 \\ 1 & \text{if } a \ne 1 \end{cases}$$

$$d(x_a x_{(k,l)}) = \begin{cases} l & \text{if } a = k \\ l+1 & \text{if } a = 1, k \ne 1 \ or \ k = 1, a \ne 1 \\ l+2 & \text{if } a \ne k \end{cases}$$

$$d(x_{(a,b)} x_{(k,l)}) = \begin{cases} j+l+1 & \text{if k=1} \\ |b-l| & \text{if } a = k \\ b+l+2 & \text{if } a \ne k \end{cases}$$

Based on the subset D we have choose and the distance function of any two vertices, we know that the representation of each vertex respect of the elements in D must be different one another's. Thus, D satisfies the characteristic of resolving set.

Third, we show that D is resolving efficient dominating set with the minimum cardinality. Supposing $|D_1| < (m+1)\lceil \frac{n}{3} \rceil$, so we have $|D_1| = (m+1)\lceil \frac{n}{3} \rceil - 1$ for $n \equiv 0, 2(mod\ 3)$ and $|D_1| < m\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil$, so we have $|D_1| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil - 1$ for $n \equiv 1(mod\ 3)$. Here are the conditions that might occur

(i) For $n \equiv 0(mod\ 3)$
- any vertex $\{x_{a,1}\} \notin D_1 \rightarrow \exists\ x_a, x_{a,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 is not the efficient dominating set.
- any vertex $\{x_{a,b} : b \equiv 1(mod\ 3) : 4 \le b \le n-2\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

(ii) For $n \equiv 1(mod\ 3)$
- Vertex $x_1 \notin D_1 \rightarrow \exists\ x_{1,1}, x_1, x_a$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Vertex $\{x_{1,b} : b \equiv 0(mod\ 3) : 3 \le b \le n-1\} \notin D_1 \rightarrow \exists\ x_{1,b-1}, x_{1,b}, x1, b+1$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 2 \le a \le m+1 : 2 \le b \le n-2 : b \equiv 2(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

(iii) For $n \equiv 2(mod\ 3)$
- Vertex $x_{a,1} \notin D_1 \rightarrow \exists\ x_a, x_{a,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set

- Vertex $x_{a,n-1} \notin D_1 \rightarrow \exists\, x_{a,n-1}, x_{a,n-2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 1 \leq b \leq n-1 : b \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\, x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set

Subset D_1 doesn't statisfy the characteristic of efficient dominating set for $S_m \triangleright P_n$, it contradicts the resolving efficient dominating set. Therefore, $|D| = (m+1)\lceil \frac{n}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $S_m \triangleright P_n$ when $n \equiv 0, 2(mod\ 3)$, and $|D| = m\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $S_m \triangleright P_n$ when $n \equiv 1(mod\ 3)$. □

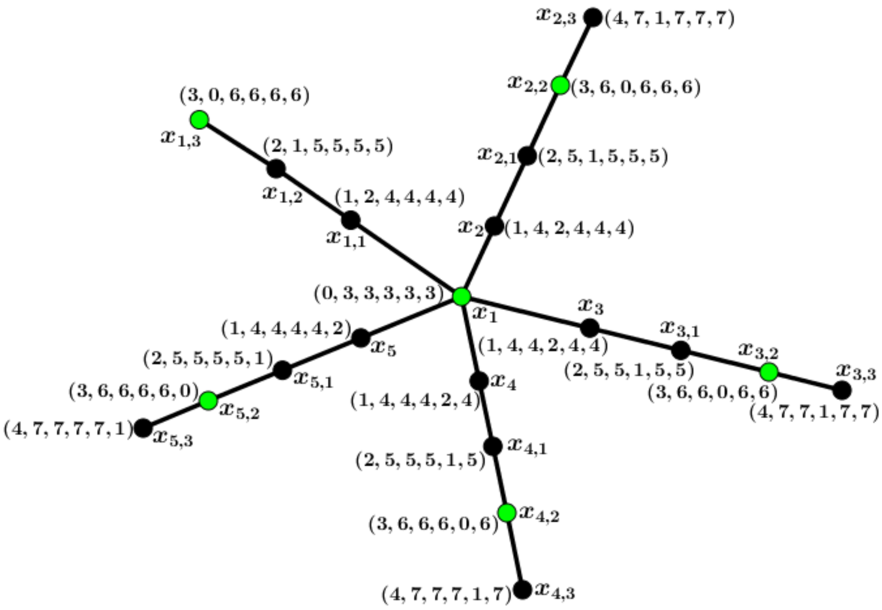

Figure 4. The resolving efficient dominating set of $S_4 \triangleright P_4$.

Theorem 4 *For every positive interger $m, n \geq 2$,*

$$\gamma_{re}(K_m \triangleright P_n) = \begin{cases} m\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 0(mod\ 3) \text{ and } n \equiv 2(mod\ 3) \\ \frac{m(n-1)}{3} + 1, & \text{if } n \equiv 1(mod\ 3). \end{cases}$$

Proof. Graph $K_m \triangleright P_n$ is a comb product graph of a graph K_m and m copies graph P_n where the vertices x_a are the sticking point on the graph K_m. The vertex set is $V(K_m \triangleright P_n) = \{x_a : 1 \leq a \leq m\} \cup \{x_{a,b} : 1 \leq a \leq m : 1 \leq b \leq n-1\}$, and the edge set is $E(K_m \triangleright P_n) = \{x_a x_b : 1 \leq a \leq m-1 : a+1 \leq b \leq m : a \neq b\} \cup \{x_a x_{a,1} : 1 \leq a \leq m\} \cup \{x_{a,b} x_{a,b+1} : 1 \leq a \leq m : 1 \leq b \leq n-2\}$. The order and size respectively are $|V(K_m \triangleright P_n)| = mn$ and $|E(K_m \triangleright P_n)| = \frac{m(m+2n-3)}{2}$. We choose subset D as below

$$D = \begin{cases} \{x_{a,b} : 1 \leq a \leq m : 1 \leq b \leq n-1 : b \equiv 1(mod\ 3)\} & \text{for } n \equiv 0, 2(mod\ 3) \\ \\ \{x_1\} \cup \{x_{1,b} : 3 \leq b \leq n : b \equiv 0(mod\ 3)\} \cup & \text{for } n \equiv 1(mod\ 3) \\ \{x_{a,b} : 2 \leq a \leq m : 2 \leq b \leq n-2 : b \equiv 2(mod\ 3)\} & \end{cases}$$

We have $|D| = m\lceil \frac{n}{3} \rceil$ for $n \equiv 0, 2(mod\ 3)$, and $|D| = \frac{m(n-1)}{3} + 1$ for $n \equiv 1(mod\ 3)$. We will show that D is resolving efficient dominating set with the minimum cardinality by the following steps.

First, we will show that D satisfies the characteristic of efficient dominating set. For every $x_{k,l}, x_{m,n} \in D$, $d(x_{k,l}, x_{m,n}) \geq 3$, so it means $|N(x_{a,b}, x_a \in V(K_m \triangleright P_n) - D) \cap D| = 1$ and every vertex $x_{a,b}, x_a \in V(K_m \triangleright P_n) - D$ is dominated by exactly one vertex in D. Thus, we stated that subset D is an efficient dominating set.

Second, we will show that subset D we have choose also satisfies the characteristic of resolving set. To find out whether the distance representation of each vertex respect of the elements in D is different one another, we can see the distance function of any two vertices in $K_m \triangleright P_n$ as follows.

$$d(x_a x_k) = \begin{cases} 1 & \text{if } a = k \\ 0 & \text{if } a \neq k \end{cases}$$

$$d(x_a x_{(k,l)}) = \begin{cases} l & \text{if } a = k \\ l+1 & \text{if } a \neq k \end{cases}$$

$$d(x_{(a,b)} x_{(k,l)}) = \begin{cases} |b-l| & \text{if } a = k \\ b+l+1 & \text{if } a \neq k \end{cases}$$

Based on the subset D we have choose and the distance function of any two vertices, we know that the representation of each vertex respect of the elements in D must be different one another's. Thus, D satisfies the characteristic of resolving set.

Third, we show that D is resolving efficient dominating set of $K_m \triangleright P_n$ with the minimum cardinality. Supposing $|D_1| < m\lceil \frac{n}{3} \rceil$, so we have $|D_1| = m\lceil \frac{n}{3} \rceil - 1$ for $n \equiv 0, 2(mod\ 3)$ and $|D_1| < \frac{m(n-1)}{3} + 1$, so we have $|D_1| = \frac{m(n-1)}{3}$ for $n \equiv 1(mod\ 3)$. Here are some conditions that might occur

(i) For $n \equiv 0(mod\ 3)$
- any vertex $\{x_{a,1}\} \notin D_1 \rightarrow \exists\ x_a, x_{a,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.
- any vertex $\{x_{a,b} : b \equiv 1(mod\ 3) : 4 \leq b \leq n-2\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

(ii) For $n \equiv 1(mod\ 3)$
- Vertex $x_1 \notin D_1 \rightarrow \exists\ x_{1,1}, x_a$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Vertex $x_{1,n-1} \notin D_1 \rightarrow \exists\ x_{1,n-1}, x_{1,n-2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 2 \leq a \leq m : 2 \leq b \leq n-2 : b \equiv 2(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.
- Any vertex $\{x_{1,b} : 3 \leq b < n-1 : b \equiv 0(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{1,b+1}, x_{1,b}, x_{1,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set.

(iii) For $n \equiv 2(mod\ 3)$
- Vertex $x_{a,1} \notin D_1 \rightarrow \exists\ x_a, x_{a,1}, x_{a,2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Vertex $x_{a,n-1} \notin D_1 \rightarrow \exists\ x_{a,n-1}, x_{a,n-2}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set
- Any vertex $\{x_{a,b} : 1 < b < n-1 : b \equiv 1(mod\ 3)\} \notin D_1 \rightarrow \exists\ x_{a,b+1}, x_{a,b}, x_{a,b-1}$ which are not dominated by D_1. Therefore, D_1 isn't the efficient dominating set

Subset D_1 doesn't statisfy the characteristic of efficient dominating set, it contradicts the resolving efficient dominating set. Therefore $|D| = m\lceil \frac{n}{3} \rceil$ must be the minimum cardinality of resolving efficient dominating set of $K_m \triangleright P_n$ when $n \equiv 0, 2(mod\ 3)$ and $|D| = \frac{m(n-1)}{3} + 1$ when $n \equiv 1(mod\ 3)$. $\square$

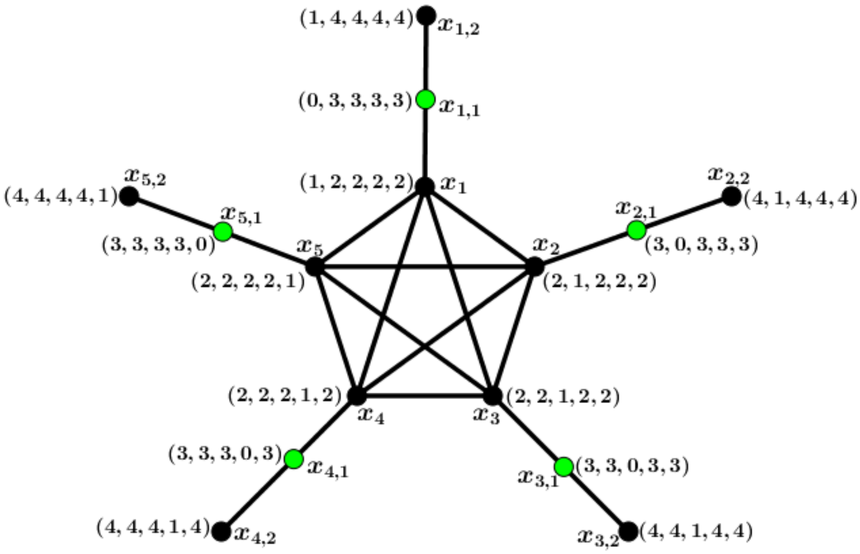

Figure 5. The resolving efficient dominating set of $K_5 \triangleright P_3$.

4. Conclusion

In this paper, we have investigated and analyzed the exact values of resolving efficient domination number on path and some comb product of special graph, namely $P_m \triangleright P_n$, $S_m \triangleright P_n$, and $K_m \triangleright P_n$. Since this topic is a new research in combining study of metric dimension and efficient dominating, there are still lots problems related to this topic have not been discovered yet. Consequently, we propose some of the following open problems.

Open Problem 1 *Let G be any connected graph, determine the exact value of its resolving efficient domination number.*

Open Problem 2 *Determine the exact value of resolving efficient domination number of other special graph and its operation, such as joint, cartesian and etc.*

Acknowledgement

We gratefully acknowledge the support from CGANT Research Group and LP2M University of Jember of the year 2022.

References

[1] Alfarisi R, Dafik and Kristiana A I, 2019, Resolving domination number of graphs, *Discrete Mathematics, Algorithms and Applications* **11** 06 1950071.
[2] Alfarisi R, Dafik, Kristiana A I and I H Agustin, 2019, Non-Isolated Resolving Number of Graph with Pendant Edges, *Journal of Interconnection Networks* **19** 02 1950003.
[3] Alikhani S and Peng Y, 2009, Dominating sets and domination polynomials of paths, *International journal of Mathematics and mathematical sciences*, 2009.
[4] Arumugam S and Mathew V, 2012, The fractional metric dimension of graphs, *Discrete Mathematics* **312** 9 1584-1590.
[5] Aziza M O, Kristiana A I, Alfarisi R, Wardani D A R et al, 2021, On resolving perfect dominating number of comb product of special graphs, *Journal of Physics: Conference Series* **1832** 1 012019.
[6] Bange D W, 1988, Efficient dominating sets in graphs, *Applications of Discrete Mathematics.*
[7] Boutin D L, 2009,Determining sets, resolving sets, and the exchange property, *Graphs and Combinatorics* **25** 6 789-806.
[8] Brandstädt A and Mosca R, 2019, On efficient domination for some classes of H-free bipartite graphs, *Discrete Applied Mathematics* **270** 58-67.

[9] Brandstädt A et al, 2014, A note on efficient domination in a superclass of P5-free graphs, *Information Processing Letters* **114** 7 357-359.
[10] Brigham R C, Chartrand G and Dutton R D, and Zhang P, 2003, Resolving domination in graphs, *Mathematica Bohemica* **128** 1 25-36.
[11] Cáceres J, Hernando C, Mora M, Pelayo I M, and Puertas M L, 2012, On the metric dimension of infinite graphs, *Discrete Applied Mathematics* **160** 18 2618-2626.
[12] Çalışkan C, Miklavič T and Özkan S, 2019, Domination and efficient domination in cubic and quartic Cayley graphs on abelian groups, *Discrete Applied Mathematics* **271** 15-24.
[13] Chartrand G and Lesniak L, 1986, Graphs and Digraphs, Wadsworth & Brooks.
[14] Chartrand G, Eroh L, Jhonson M A, and Oellermann O R, 2000, Resolvability in graphs and the metric dimension of a graph, *Discrete Applied Mathematics* **105** 1-3 99-113.
[15] Chartrand G, Haynes T W, Henning M A and Zhang P, 2019, From Domination to Coloring: Stephen Hedetniemi's Graph Theory and Beyond, *Springer Nature*.
[16] Dafik, Slamin, Agustin I H, Retnowardani D A, Kurniawati E Y, 2021, On the resolving strong domination number of graphs: a new notion, *Journal of Physics: Conference Series* **1836** 1 012019.
[17] Darmaji and Alfarisi R, 2017, On the partition dimension of comb product of path and complete graph, *AIP Conference Proceedings* **1867** 1 020038.
[18] Deng Y, Sun Y, Liu Q and Wang H, 2017, Efficient dominating sets in circulant graphs, *Discrete Mathematics* **340** 7 1503-1507.
[19] Du D and Wan P, 2012, Connected dominating set: theory and applications, **105** Springer Science & Business Media.
[20] Hakim R A, Tirta I M, Prihandini R M, Agustin I H, et al, 2021, On resolving efficient domination number of comb product of special graphs, *Journal of Physics: Conference Series* **1832** 1 012018.
[21] Hayyu A N, Tirta I M, Adawiyah R, Prihandini R M et al, 2020, Resolving domination number of helm graph and it's operation, *Journal of Physics: Conference Series* **1465** 1 012022.
[22] Hernando C, Mora M, Pelayo I M, Seara C, Cáceres J and Puertas M L, 2005, On the metric dimension of some families of graphs, *Electronic notes in discrete mathematics* **22** 129-133.
[23] Kurniawati S, Wardani D A R, Albirri E R et al, 2020, On resolving domination number of friendship graph and its operation, *Journal of Physics: Conference Series* **1465** 1 012019.
[24] Lin M C, Mizrahi M J and Szwarcfiter J L, 2015, Efficient and Perfect domination on circular-arc graphs, *Electronic Notes in Discrete Mathematics* **50** 307-312.
[25] Obradović N, Peters J and Ružić G, 2007, Efficient domination in circulant graphs with two chord lengths, *Information Processing Letters* **102** 6 253-258.
[26] Oellermann O R and Peters-Fransen J, 2007, The strong metric dimension of graphs and digraphs, *Discrete Applied Mathematics* **155** 3 356-364.
[27] Okamoto F, Phinezy B and Zhang P, 2010, The local metric dimension of a graph, *Mathematica Bohemica* **135** 3 239-255.
[28] Rautenbach D, 2000, Bounds on the strong domination number, *Discrete Mathematics* **215** 1-3 201-212.
[29] Saputro S W, Mardiana N and Purwasih I A, 2013, The metric dimension of comb product graphs, *Graph Theory Conference in honor of Egawa's 60th birthday* 1-2.
[30] Shahida A T, and Sunitha M S, 2017, On the metric dimension of join of a graph with empty graph (op), *Electronic Notes in Discrete Mathematics* **63** 435-445.
[31] Shanmukha B, Sooryanarayana B and Harinath K S, 2002, Metric Dimension of Wheels, *Far East J. Appl. Math* **8** 3 217-229.
[32] Siddiqui H M A and Imran M, 2014, Computing the metric dimension of wheel related graphs, *Applied mathematics and computation* **242** 3 624-632.
[33] Umilasari R, 2015, *Bilangan dominasi jarak dua pada graf-graf hasil operasi korona dan comb*, PhD thesis, Institut Teknologi Sepuluh Nopember.
[34] Wangguway Y, Wardani A R, Alfarisi R et al, 2020, On resolving domination number of special family of graphs, *Journal of Physics: Conference Series* **1465** 1 012015.
[35] Yero I G, 2016, Vertices, edges, distances and metric dimension in graphs, *Electronic Notes in Discrete Mathematics* **55** 191-194.
[36] Yero I G, Kuziak D and Rodríguez-Velázquez J A, 2011, On the metric dimension of corona product graphs, *Computers & Mathematics with Applications* **61** 9 2793-2798.