Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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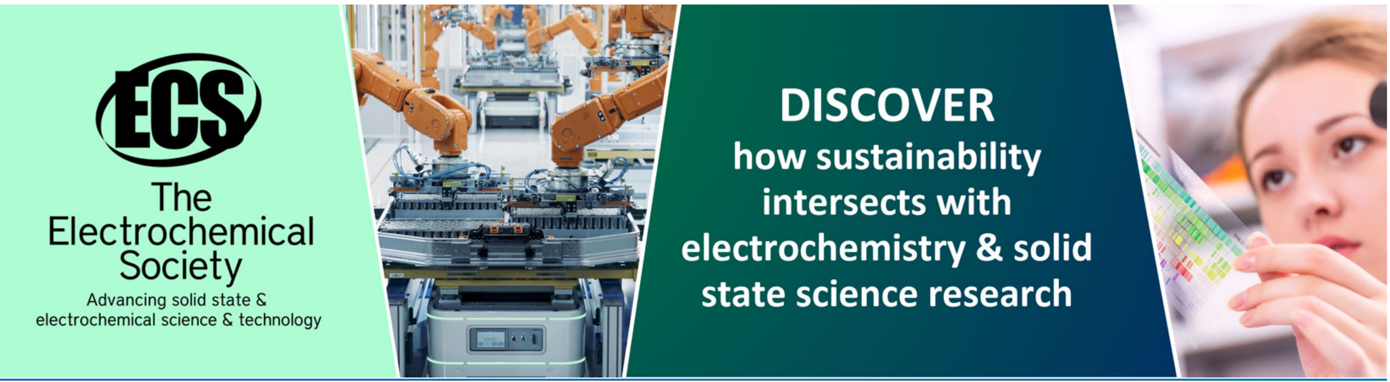
To cite this article: N Mohanapriya *et al* 2022 *J. Phys.: Conf. Ser.* **2157** 012007



On r-dynamic coloring of central vertex join of path, cycle with certain graphs

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Abstract. Let $G = (V, E)$ be a simple finite connected and undirected graph with n vertices and m edges. The n vertices are assigned the colors through mapping $c : V[G] \longrightarrow I^+$. An r-dynamic coloring is a proper k-coloring of a graph G such that each vertex of G receive colors in at least $\min\{deg(v), r\}$ different color classes. The minimum k such that the graph G has r-dynamic k coloring is called the r-dynamic chromatic number of graph G denoted as $\chi_r(G)$. Let G_1 and G_2 be a graphs with n_1 and n_2 vertices and m_1 and m_2 edges. The central vertex join of G_1 and G_2 is the graph $G_1 \dot{V} G_2$ is obtained from $C(G_1)$ and G_2 joining each vertex of G_1 with every vertex of G_2. The aim of this paper is to obtain the lower bound for r-dynamic chromatic number of central vertex join of path with a graph G, central vertex join of cycle with a graph G and r-dynamic chromatic number of $P_m \dot{V} P_n$, $P_m \dot{V} K_n$, $P_m \dot{V} C_n$, $C_m \dot{V} P_n$, $C_m \dot{V} K_n$ and $C_m \dot{V} C_n$ respectively.

1. Introduction

In this research, we use simple, finite, connected, and undirected graphs. Let $V(G)$ and $E(G)$ be the graph's vertex and edge sets, respectively and the maximum and minimum degree of the graph G is denoted as $\Delta(G)$ and $\delta(G)$ [6]. The neighborhood of a vertex in a graph G is denoted as $N_G(v)$. An r-dynamic coloring of a graph is assigning colors to the vertex such that (i) The coloring should be a proper coloring and (ii) for each vertex v , $|c(N_G(v))| \geq min\{r, deg(v)\}$, where $N_G(v)$ denotes the set of all vertices adjacent to v and $deg(v)$ its degree and r is a positive integer. The r-dynamic chromatic number [7] of a graph G is denoted by $\chi_r(G)$, is the minimum k such that G admits proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph G is studied by the name dynamic chromatic number in [1] - [5], [9]. Montgomery first demonstrated the r-dynamic coloring in [10]. Taherkhani et al. in [12] obtained the upper bound of regular graph. In [8] Jahfar T K et al introduced the new graph operation based on central graphs.

2. Preliminaries

The central graph of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G. The central graph of G is denoted by $C(G)$ [11], [13]. Let G_1 and G_2 be a graphs with n_1 and n_2 vertices and m_1 and m_2 edges. The central vertex join of G_1 and G_2 is the graph $G_1\dot{V}G_2$ is obtained from $C(G_1)$ and G_2 joining each vertex of G_1 with every vertex of G_2 . The central vertex join $G_1\dot{V}G_2$ has $(m_1 + n_1 + n_2)$ vertices and $\left(m_1 + m_2 + n_1n_2 + \frac{n_1(n_1-1)}{2}\right)$ edges [8]. A path graph P_n is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it.

3. Central Vertex Join of Path with some Graphs

In this section we obtain the lower bound for r-dynamic chromatic number of central vertex join of path with a graph G, r-dynamic chromatic number of central vertex join of path with path $P_m\dot{V}P_n$, path with complete graph $P_m\dot{V}K_n$ and path with cycle graph $P_m\dot{V}C_n$.

Lemma 1. *[9]* $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$.

Lemma 2. *Let P_m be a path on m vertices where $m \geq 4$ and G be a any finite, simple and connected graph with at least n vertices where $n \geq 2$ then the lower bound for the r-dynamic chromatic number of central vertex join of path P_m with G is given by*

$$\chi_r\left[P_m\dot{V}G\right] \geq \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor - 1, & r = 1 \\ m+2, & 2 \leq r \leq m \\ r+2, & m+1 \leq r \leq m+n-2 \\ m+n+2, & r \geq m+n-1 \end{cases}$$

Proof. Let $\{v_1, v_2, \cdots, v_m\}$ be the vertices of the path P_m and by the definition of central vertex join we are subdividing each edge $\{e_1, e_2, \cdots, e_{m-1}\}$ to produce a new set of $m-1$ vertices $\{w_1, w_2, \cdots, w_{m-1}\}$. Also let $\{u_1, u_2, \cdots, u_n\}$ be the n vertices of the graph G. The degree of each vertex v_i of P_m in $P_m\dot{V}G$ is $m+n-1$ and degree of w_i is 2.

Case: 1 When $r = 1$.

First color the vertices v_1, v_2, w_1, w_2 with the colors 1, 1, 2, 2 respectively. Now, the vertex v_3 cannot be colored with the colors 1 and 2 due to proper coloring criteria hence color it with a new color 3. Now color w_3, w_4 with color 2 and v_4 with color 3. Now v_5 cannot be provided with colors 1, 2 and 3 so we require a new color 4. Proceeding in a similar manner we require $\left\lfloor \frac{m+7}{2} \right\rfloor - 2$ colors for coloring the vertices of P_m and subdivided vertices. Thus the vertices of P_m are colored with sequence of colors $1, 1, 3, 3, 4, 4, \cdots, \left\lfloor \frac{m+7}{2} \right\rfloor - 2, \left\lfloor \frac{m+7}{2} \right\rfloor - 2$ and $w_i's$ with color 2. Now moving forward onto G which is a finite, simple and connected graph there must be at least an edge between any two vertices u and v of G. Now color u with 2 and by proper coloring criteria v has to be colored with a new color $\left\lfloor \frac{m+7}{2} \right\rfloor - 1$. Hence we require a minimum of $\left\lfloor \frac{m+7}{2} \right\rfloor - 1$ colors i.e., $\chi_r\left[P_m\dot{V}G\right] \geq \left\lfloor \frac{m+7}{2} \right\rfloor - 1$.

Case: 2 When $2 \leq r \leq m$.

For $r = 2$, color the vertices v_1, w_1 with 1 and 2. Now, each w_i has degree 2 hence for satisfying the 2-adjacency of w_1 we need to provide a new color 3 to the vertex v_2. Now provide the color 1 to w_2 and to satisfy its 2-adjacency provide the color 2 to v_3. Color w_3 with 3 and it is evident that we require a new color 4 for the v_4 for meeting its adjacency criteria. Proceeding in a similar manner we can see evidently that we require m different colors for this process. Now, none of the m colors can be provide to the vertices u, v of graph G, since proper coloring criteria will be violated hence provide them with two new colors m+1 and m+2 respectively. Now to put it into a simpler manner color v_i with i and assigning the colors $m, 1, 2, \cdots, m-2$ to the vertices $w_1, w_2, \cdots, w_{m-1}$. By this coloring we can see the the r-adjacencies of v_i from $r = 2$ till $r = m$ will be fulfilled. Thus $\chi_r\left[P_m\dot{V}G\right] \geq m+2$.

Case: 3 When $m+1 \leq r \leq m+n-2$.
Primarily provide the coloring mentioned in Case 2 to the vertices of P_m and subdivided vertices. Now when $r = m+1$ provide the colors $m+1, m+2, m+3 = r+2$ to any of the 3 vertices of G (for this G should neccesarily have at least 3 vertices else r=m+1 will be dealt in Case 4) for satisfying $r = m+1$-adjacency of $v_i's$; when $r = m+2$ provide the colors $m+1, m+2, m+3, m+4 = r+2$ to any of the 4 vertices of G (for this G should neccesarily have at least 4 vertices else r=m+2 will be dealt in Case 4) proceeding in the same way when $r = m+n-2$ provide the colors $m+1, m+2, m+3, m+n = r+2$ to the n vertices of G for satisfying the $r = m+n-2$-adjacency of $v_i's$. Hence $\chi_r\left[P_m\dot{V}G\right] \geq r+2$.

Case: 4 When $r \geq m+n-1$.
By Case 3 the vertices v_i will have $m+n-2$ differently colored neighbors. Now for satisfying the $r = m+n-1$-adjacency of v_1 provide the new color m+n+1 to the vertex w_1 and $r = m+n-1$-adjacency of v_2 provide the new color m+n+2 to the vertex w_2 as the color m+n+1 cannot be provided here. Now provide the colors m+n+1, m+n+2 alternatively to the remaining vertices $w_3, \cdots, w_{m-1}$. Hence $\chi_r\left[P_m\dot{V}G\right] \geq m+n+2$.

□

Theorem 3. *Let $m \geq 4$,$n \geq 3$ the r-dynamic chromatic number of central vertex join of path with path is*

$$\chi_r\left[P_m\dot{V}P_n\right] = \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor - 1, & r = 1 \\ m+2, & 2 \leq r \leq m \\ r+2, & m+1 \leq r \leq m+n-2 \\ m+n+2, & r = \Delta \end{cases}$$

Proof. The vertex set of central vertex join of path graph P_m with path graph P_n is given by $V\left[P_m\dot{V}P_n\right] = \{v_i, 1 \leq i \leq m\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq m-1\}$ and the edge set is $E\left[P_m\dot{V}P_n\right] = \{v_iw_i, 1 \leq i \leq m-1\} \cup \{w_iv_{i+1}, 1 \leq i \leq m-1\} \cup \{v_iv_j, 1 \leq i \leq m-2, 1+2 \leq j \leq m\} \cup \{v_iu_j, 1 \leq i \leq m, 1 \leq j \leq n\}$.The maximum and minimum degrees of $(P_m\dot{V}P_n)$ are $\Delta\left[P_m\dot{V}P_n\right] = m+n-1$ and $\delta\left[P_m\dot{V}P_n\right] = 2$. We prove the theorem in the following cases.

Case: 1 When $r = 1$ the r-dynamic coloring are as follows,
consider the mapping $c : V\left[(P_m\dot{V}P_n)\right] \to \{1, 2, \cdots, \lfloor \frac{m+7}{2} \rfloor - 1\}$

- $c(v_i) = \{3, 3, 4, 4, \cdots, \lfloor \frac{m+7}{2} \rfloor - 1, \lfloor \frac{m+7}{2} \rfloor - 1\}$, $1 \leq i \leq m$, when m is even
- $c(v_i) = \{3, 3, 4, 4, \cdots, \lfloor \frac{m+7}{2} \rfloor - 1\}$, $1 \leq i \leq m$, when m is odd
- $c(u_i) = \{1, 2, 1, 2, \cdots, 1, 2\}$, $1 \leq i \leq n$
- $c(w_i) = 2$, $1 \leq i \leq m-1$

This coloring provides the upper bound $\chi_r\left[P_m\dot{V}P_n\right] \leq \lfloor \frac{m+7}{2} \rfloor - 1$. By Lemma 3.2 we have the lower bound as $\chi_r\left[P_m\dot{V}P_n\right] \geq \lfloor \frac{m+7}{2} \rfloor - 1$. Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[P_m\dot{V}P_n\right] = \lfloor \frac{m+7}{2} \rfloor - 1$, when $r = 1$.

Case: 2 When $2 \leq r \leq m$, define the mapping $c : V\left[P_m\dot{V}P_n\right] \to \{1, 2, 3 \cdots, m+2\}$, the following coloring gives the upper bound of $P_m\dot{V}P_n$

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = \{m+1, m+2, \cdots, m+1, m+2\}$, $1 \leq i \leq n$
- $c(w_i) = \{m, 1, 2, \cdots m-2\}$, $1 \leq i \leq m-1$

By Lemma 3.2 we have the lower bound $\chi_r \left[P_m \dot{V} P_n\right] \geq m+2$. Hence we have $\chi_r \left[P_m \dot{V} P_n\right] = m+2$, when $2 \leq r \leq m$.

Case: 3 When $m+1 \leq r \leq m+n-2$. Consider the mapping $c : V\left[\left(P_m \dot{V} P_n\right)\right] \rightarrow \{1, 2, \cdots, r+2\}$. The r-dynamic coloring are as follows,

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = \{m+1, \cdots, r+2, m+1, \cdots, r+2, \cdots\}$, $1 \leq i \leq n$
- $c(w_i) = \{m, 1, 2, \cdots m-2\}$, $1 \leq i \leq m-1$

Hence the upper bound is $\chi_r \left[P_m \dot{V} P_n\right] \leq r+2$ and Lemma 3.2 provides the lower bound as $\chi_r \left[P_m \dot{V} P_n\right] \geq r+2$. Therefore $\chi_r \left[P_m \dot{V} P_n\right] = r+2$ when $m+1 \leq r \leq m+n-2$.

Case: 4 When $r = \Delta$ define the mapping $c : V\left[P_m \dot{V} P_n\right] \rightarrow \{1, 2, 3 \cdots, n+m+2\}$, the following coloring gives the upper bound of $P_m \dot{V} P_n$

- $c(u_i) = i$, $1 \leq i \leq n$
- $c(v_i) = \{n+1, n+2 \cdots, n+m\}$, $1 \leq i \leq m$
- $c(w_i) = \{n+m+1, n+m+2, n+m+1, n+m+2, \cdots, n+m+1, n+m+2\}$, $1 \leq i \leq m-1$

Lemma 3.2 provides the lower bound as $\chi_r \left[P_m \dot{V} P_n\right] \geq n+m+2$. Hence the r- adjacency condition fulfilled and therefore $\chi_r \left[P_m \dot{V} P_n\right] = n+m+2$, when $r = \Delta$.

□

Theorem 4. *Let $m \geq 3$, $n \geq 2$ the r-dynamic chromatic number of central vertex join of path graph P_m with complete graph K_n is*

$$\chi_r[P_m \dot{V} K_n] = \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3, & r = 1 \\ m+n, & 2 \leq r \leq \Delta - 1 \\ m+n+2, & r = \Delta \end{cases}$$

Proof. The vertex set of central vertex join of path graph P_m with complete graph K_n is given by $V\left[P_m \dot{V} K_n\right] = \{v_i, 1 \leq i \leq m\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq m-1\}$ and the edge set is $E\left[P_m \dot{V} K_n\right] = \{v_i w_i, 1 \leq i \leq m-1\} \cup \{w_i v_{i+1}, 1 \leq i \leq m-1\} \cup \{v_i v_j, 1 \leq i \leq m-2, 1+2 \leq j \leq m\} \cup \{v_i u_j, 1 \leq i \leq m, 1 \leq j \leq n\}$. The maximum and minimum degrees of $(P_m \dot{V} K_n)$ are $\Delta\left[P_m \dot{V} K_n\right] = m+n-1$ and $\delta\left[P_m \dot{V} K_n\right] = 2$. We divide the proof into three cases

Case: 1 When $r = 1$. By Lemma 3.2 we have the lower bound $\chi_r \left[P_m \dot{V} K_n\right] \geq \left\lfloor \frac{m+7}{2} \right\rfloor - 1$ but since we have complete graph in the place of G we require additional $n-2$ colors hence the lower bound transforms as $\chi_r \left[P_m \dot{V} K_n\right] \geq \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3$. Now define the mapping $c : V\left[P_m \dot{V} K_n\right] \rightarrow \{1, 2, 3, \cdots, \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3\}$.
We use the following colorings to show the upper bound,

- $c(u_i) = i, 1 \le i \le n$
- $c(v_i) = \left\{n+1, n+1, n+2, n+2, \cdots, \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3\right\}, 1 \le i \le m$, when m is odd
- $c(v_i) = \left\{n+1, n+1, n+2, n+2, \cdots, \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3, \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3\right\}, 1 \le i \le m$, when m is even
- $c(w_i) = 1, 1 \le i \le m-1$

Hence the r- adjacency condition is fulfilled and therefore
$\chi_r[P_m\dot{V}K_n] = \left\lfloor \frac{m+7}{2} \right\rfloor + n - 3$.

Case: 2 When $2 \le r \le \Delta - 1$, define the mapping $c : V\left[P_m\dot{V}K_n\right] \to \{1, 2, 3, \cdots m+n\}$ the upper bound is given by the following colorings

- $c(u_i) = i,\ 1 \le i \le n$
- $c(v_i) = n+i,\ 1 \le i \le m$
- $c(w_i) = i,\ 1 \le i \le m-1$

By the above coloring we have the upper bound as $\chi_r[P_m\dot{V}K_n] \le m+n$. Again since we have K_n in the place of G by Lemma 3.2 doesn't provide an efficient bound we are in requirement of more colors than in any other graph. We can easily see that we require at least $m+n$ in this case. Thus we have $\chi_r[P_m\dot{V}K_n] \ge m+n$. Hence the r- adjacency condition is fulfilled and
$\chi_r[P_m\dot{V}K_n] = m+n$, when $2 \le r \le \Delta - 1$.

Case :3 Proof same as Case 4 of Theorem 3.3.

□

Remark 5. *Let $m = 2$, $n \ge 2$ the r-dynamic chromatic number of central vertex join of path graph P_m with complete graph K_n is*

$$\chi_r[P_2\dot{V}K_n] = \begin{cases} n+1, & r = 1 \\ n+2, & 2 \le r \le \Delta - 1 \\ n+3, & r = \Delta \end{cases}$$

Proof. The vertex set of central vertex join of path graph P_2 with complete graph K_n is given by $V\left[P_2\dot{V}K_n\right] = \{v_i, 1 \le i \le 2\} \cup \{u_i, 1 \le i \le n\} \cup \{w_1\}$ and the edge set is
$E\left[P_2\dot{V}K_n\right] = \{v_1w_1, w_1v_2\} \cup \{v_iu_j, 1 \le i \le 2, 1 \le j \le n\}$. The maximum and minimun degrees of $\left[P_2\dot{V}K_n\right]$ are $\Delta\left[P_2\dot{V}K_n\right] = n+1$ and $\delta\left[P_2\dot{V}K_n\right] = 2$. We divide the proof into three cases

Case: 1 When $r = 1$, define the mapping $c : V\left[P_2\dot{V}K_n\right] \to \{1, 2, \cdots, n+1\}$.

- $c(u_i) = i, 1 \le i \le n$
- $c(v_1, v_2) = \{n+1, n+1\}$
- $c(w_1) = 1$

Hence the r- adjacency condition is fulfilled and therefore $\chi_r[P_2\dot{V}K_n] = n+1$, when $r = 1$.

Case: 2 When $2 \le r \le \Delta - 1$, define the mapping $c : V\left[\left(P_2\dot{V}K_n\right)\right] \to \{1, 2, 3, \cdots, n+2\}$ and we define the following colorings

- $c(u_i) = i,\ 1 \le i \le n$
- $c(v_1, v_2) = \{n+1, n+2\}$
- $c(w_1) = 1$

Hence $\chi_r[P_m\dot{V}K_n] = n+2$, when $2 \le r \le \Delta - 1$.

Case: 3 When $r = \Delta$, define the mapping $c : V\left[P_2\dot{V}K_n\right] \to \{1, 2, \cdots, n+3\}$ and the following colorings

- $c(u_i) = i,\ 1 \le i \le n$
- $c(v_i) = \{n+1, n+2\}$
- $c(w_1) = n+3$

Hence $\chi_r\left[P_m\dot{V}K_n\right] = n+3$, for $r = \Delta$.

□

Theorem 6. *Let $m, n \ge 3$ the r-dynamic chromatic number of central vertex join of path graph P_m with cycle graph C_n is*

$$\chi_r\left[P_m\dot{V}C_n\right] = \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor - 1, & r = 1\ n\ is\ even \\ \left\lfloor \frac{m+7}{2} \right\rfloor, & r = 1\ n\ is\ odd \\ m+2, & 2 \le r \le m,\ n\ \ is\ even \\ m+3, & 2 \le r \le m+1,\ n\ \ is\ odd \\ r+3, & r = m+2,\ n = 5 \\ r+2, & \begin{cases} m+1 \le r \le \Delta - 1, & n\ is\ even \\ m+2 \le r \le \Delta - 1, & n\ is\ odd\ and\ n \ne 5 \\ m+3 \le r \le \Delta - 1, & n = 5 \end{cases} \\ m+n+2, & r = \Delta \end{cases}$$

Proof. The vertex set of central vertex join of path graph P_m with cycle graph C_n is given by $V\left[P_m\dot{V}C_n\right] = \{v_i, 1 \le i \le m\} \cup \{u_i, 1 \le i \le n\} \cup \{w_i, 1 \le i \le m-1\}$ and the edge set is $E\left[P_m\dot{V}C_n\right] = \{v_iw_i, 1 \le i \le m-1\} \cup \{w_iv_{i+1}, 1 \le i \le m-1\} \cup \{v_iv_j, 1 \le i \le m-2, 1+2 \le j \le m\} \cup \{v_iu_j, 1 \le i \le m, 1 \le j \le n\}$. The maximum and minimun degrees of $(P_m\dot{V}C_n)$ are
$\Delta\left[P_m\dot{V}C_n\right] = m+n-1$ and $\delta\left[P_m\dot{V}C_n\right] = 2$. We prove the theorem in the following cases.

Case: 1 Proof same as Case of Theorem 3.3.

Case: 2 When $r = 1$ and n is odd , define the mapping $c : V\left[P_m\dot{V}C_n\right] \to \{1, 2, 3 \cdots, \lfloor \frac{m+7}{2} \rfloor\}$, the following coloring gives the upper bound of $P_m\dot{V}C_n$

- $c(u_i) = \{1, 2, 1, 2, \cdots, 1, 2, 3\}$, $1 \le i \le n$
- $c(v_i) = \{4, 4, 5, 5, \cdots, \lfloor \frac{m+7}{2} \rfloor\}$, $1 \le i \le m$
- $c(w_i) = 2$, $1 \le i \le m-1$

By Lemma 3.2 we have the lower bound $\chi_r\left[P_m\dot{V}C_n\right] \ge \lfloor \frac{m+7}{2} \rfloor - 1$ but since we have an odd cycle in G we require an additional color as odd cycle always requires 3 colors for proper coloring thus $\chi_r\left[P_m\dot{V}C_n\right] \ge \lfloor \frac{m+7}{2} \rfloor$. Hence $\chi_r\left[P_m\dot{V}C_n\right] = \lfloor \frac{m+7}{2} \rfloor$, when n is odd.

Case: 3 Proof same as Case 2 of Theorem 3.3.

Case: 4 When $2 \le r \le m+1$, n is odd , define the mapping $c : V\left[P_m\dot{V}C_n\right] \to \{1, 2, 3 \cdots, m+3\}$, the following coloring gives the upper bound of $P_m\dot{V}C_n$

- $c(u_i) = \{1, 2, 1, 2, \cdots, 1, 2, 3\}$, $1 \le i \le n$
- $c(v_i) = \{4, 5 \cdots, m+3\}$, $1 \le i \le m$

- $c(w_i) = \{2, 3, 4, \cdots, m\}$, $1 \le i \le m-1$

Thus we have the upper bound $\chi_r\left[P_m\dot{V}C_n\right] \le m+3$ for $2 \le r \le m+1$. By Lemma 3.2 we have the lower bound $\chi_r\left[P_m\dot{V}C_n\right] \ge m+2$ but since we have an odd cycle in G we require an additional color as odd cycle always requires 3 colors for proper coloring thus $\chi_r\left[P_m\dot{V}C_n\right] \ge m+3$ for $2 \le r \le m$ and by the same Lemma for $r = m+1$, $\chi_r\left[P_m\dot{V}C_n\right] \ge r+2 = m+3$. Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[P_m\dot{V}C_n\right] = m+3$, when n is odd.

Case: 5 When $r = m+2$ and $n = 5$, define the mapping $c : V\left[P_m\dot{V}C_n\right] \to \{1, 2, 3 \cdots, r+3 = m+5\}$, the following coloring gives the upper bound of $P_m\dot{V}C_n$

- $c(u_i) = i$, $1 \le i \le 5$
- $c(v_i) = \{5+1, 5+2 \cdots, 5+m\}$, $1 \le i \le m$
- $c(w_i) = i$, $1 \le i \le m-1$

The Lemma provides the bound $\chi_r\left[P_m\dot{V}C_n\right] \ge m+4$ but since C_5 require 5 different colors we have $\chi_r\left[P_m\dot{V}C_n\right] \ge m+5$. Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[P_m\dot{V}C_n\right] = r+3$, when $r = m+2$ and $n = 5$.

Case: 6 When $m+1 \le r \le \Delta-1$, n is even; $m+2 \le r \le \Delta-1$, n is odd and $m+3 \le r \le \Delta-1$, $n = 5$, define the mapping $c : V\left[P_m\dot{V}C_n\right] \to \{1, 2, 3 \cdots r+2\}$, the following coloring gives the upper bound of $P_m\dot{V}C_n$

Subcase: 1 When $m+3 \le r \le \Delta-1$, $n = 5$
Same as Case 5 of this theorem.

Subcase: 2 When $m+2 \le r \le \Delta-1$, n is odd and $n \ne 5$

- $c(v_i) = i$, $1 \le i \le m$
- $c(w_i) = \{m, 1, 2, 3, \cdots, m-2\}$, $1 \le i \le m-1$
- For the vertices u_i first provide the vertices with colors $\{m+1, m+2, \cdots, r+2\}$ in order and for the remaining vertices of u_i provide them with colors from the set $\{m+1, m+2, \cdots, r+2\}$ so that it is proper coloring and has 2- adjacency condition satisfied with u_i's.

Subcase: 3 when $m+1 \le r \le \Delta-1$, n is even

- $c(v_i) = i$, $1 \le i \le m$
- $c(w_i) = \{m, 1, 2, 3, \cdots, m-1\}$, $1 \le i \le m-1$
- $c(u_i) = \{m+1, m+2, \cdots, r+2, m+1, m+2, \cdots, r+2\}$, $1 \le i \le n$

By Lemma 3.2 we have the lower bound $\chi_r\left[P_m\dot{V}C_n\right] \ge r+2$. Hence we conclude that $\chi_r\left[P_m\dot{V}C_n\right] = r+2$ for all the subcases in Case 6.

Case: 7 Proof same as Case 4 of Theorem 3.3.

□

4. Central Vertex Join of Cycle with some Graphs

In this section we obtain the lower bound for r-dynamic chromatic number of central vertex join of cycle graph with a graph G, r-dynamic chromatic number of central vertex join of cycle graph with path graph $C_m\dot{V}P_n$, cycle graph with complete graph $C_m\dot{V}K_n$ and cycle graph with cycle graph $C_m\dot{V}C_n$.

Lemma 7. *Let C_m be a cycle on m vertices where $m \geq 4$ and G be a any finite, simple and connected graph with at least n vertices where $n \geq 2$ then the lower bound for the r-dynamic chromatic number of central vertex join of cycle C_m with G is given by*

$$\chi_r\left[C_m\dot{V}G\right] \geq \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor - 1, & r = 1 \\ m+2, & 2 \leq r \leq m \\ r+2, & m+1 \leq r \leq m+n-2 \\ m+n+2, & r \geq m+n-1 \text{ and } m \text{ is even} \\ m+n+3, & r \geq m+n-1 \text{ and } m \text{ is odd} \end{cases}$$

Proof. Let $\{v_1, v_2, \cdots, v_m\}$ be the vertices of the cycle C_m and by the definition of central vertex join we are subdividing each edge $\{e_1, e_2, \cdots, e_m\}$ to produce a new set of m vertices $\{w_1, w_2, \cdots, w_m\}$ where $e_i : 1 \leq i \leq m-1$ is the edge between the vertices v_i and v_{i+1} and e_m is edge between e_m and e_1. Also let $\{u_1, u_2, \cdots, u_n\}$ be the n vertices of the graph G. The degree of each vertex v_i of C_m in $(C_m\dot{V}G)$ is $m+n-1$ and degree of w_i is 2.
The proofs of Case 1, Case 2, Case 3 and Case 4 are similar to the ones of Lemma in Section 3.
Case: 5 When $r \geq m+n-1$ *and* m *is odd* .
As in Case 4 if we provide the pattern of assigning the colors m+n+1, m+n+2 alternatively to the vertices w_i it will end up with giving the color m+n+1 to the vertex w_m but by this the r-adjacency of the vertex v_1 will not be satisfied and the color m+n+2 cannot be assigned due to similar reason hence we require a new color m+n+3 for coloring. Hence $\chi_r\left[C_m\dot{V}G\right] \geq m+n+3$.

□

Theorem 8. *Let $m \geq 4$ and $n \geq 2$ the r - dynamic chromatic number of central vertex join of cycle graph C_m with path graph P_n is*

$$\chi_r\left[C_m\dot{V}P_n\right] = \begin{cases} \left\lfloor \frac{m+7}{2} \right\rfloor - 1, & r = 1, \\ m+2, & 2 \leq r \leq m \\ r+2, & m+1 \leq r \leq \Delta - 1 \\ m+n+2, & r = \Delta, m \text{ is even} \\ m+n+3, & r = \Delta, m \text{ is odd} \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_m with path graph P_n is given by $V\left[C_m\dot{V}P_n\right] = \{v_i, 1 \leq i \leq m\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq m\}$ and the edge set is
$E\left[C_m\dot{V}P_n\right] = \{v_iw_i, 1 \leq i \leq m\} \cup \{w_iv_{i+1}, 1 \leq i \leq m-1\} \cup \{w_mv_1\} \cup$
$\{v_iv_j, 1 \leq i \leq m-2, i+2 \leq j \leq m\} \cup \{v_iu_j, 1 \leq i \leq m, 1 \leq j \leq n\}$. The maximum and minimun degrees of $(C_m\dot{V}P_n)$ are $\Delta\left[C_m\dot{V}P_n\right] = m+n-1$ and $\delta\left[C_m\dot{V}P_n\right] = 2$. We prove the theorem in the following cases.

Case: 1 When $r = 1$, the r- dynamic coloring are as follows, consider the mapping $c : V\left[C_m\dot{V}P_n\right] \rightarrow \left\{1, 2, 3 \cdots, \left\lfloor \frac{m+7}{2} \right\rfloor - 1\right\}$, the following coloring gives the upper bound of $C_m\dot{V}P_n$

- $c(u_i) = \{1, 2, 1, 2, \cdots, 1, 2\}$, $1 \leq i \leq n$

- $c(v_i) = \{3, 3, 4, 4, \cdots, \lfloor \frac{m+7}{2} \rfloor - 1\}$, $1 \leq i \leq m$, when m is odd
- $c(v_i) = \{3, 3, 4, 4, \cdots, \lfloor \frac{m+7}{2} \rfloor - 1, \lfloor \frac{m+7}{2} \rfloor - 1\}$, $1 \leq i \leq m$, when m is even
- $c(w_i) = 2$, $1 \leq i \leq m$

By Lemma 4.1 we have the lower bound $\chi_r \left[C_m \dot{V} P_n\right] \geq \lfloor \frac{m+7}{2} \rfloor - 1$. Therefore $\chi_r \left[C_m \dot{V} P_n\right] = \lfloor \frac{m+7}{2} \rfloor - 1$, when $r = 1$.

Case: 2 When $2 \leq r \leq m$, define the mapping $c : V \left[C_m \dot{V} P_n\right] \to \{1, 2, 3 \cdots, m+2\}$, the following coloring gives the upper bound of $C_m \dot{V} P_n$

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = \{m+1, m+2, \cdots, m+1, m+2\}$, $1 \leq i \leq n$
- $c(w_i) = \{m, 1, 2, \cdots m-1\}$, $1 \leq i \leq m$

The lower bound follows from Lemma 4.1. Hence the r- adjacency condition fulfilled and therefore $\chi_r \left[C_m \dot{V} P_n\right] = m+2$, when $2 \leq r \leq m$.

Case: 3 When $m+1 \leq r \leq \Delta - 1$, define the mapping $c : V \left[C_m \dot{V} P_n\right] \to \{1, 2, 3 \cdots, r+2\}$, the following coloring gives the upper bound of $C_m \dot{V} P_n$

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = \{m+1, m+2, \cdots, r+2, m+1, m+2, \cdots, r+2, \cdots\}$, $1 \leq i \leq n$
- $c(w_i) = \{m, 1, 2, \cdots, m-1\}$, $1 \leq i \leq m$

From Lemma 4.1 we have the lower bound $\chi_r \left[C_m \dot{V} P_n\right] \geq r+2$. Hence the r- adjacency condition fulfilled and therefore $\chi_r \left[C_m \dot{V} P_n\right] = r+2$, when $m+1 \leq r \leq \Delta - 1$.

Case: 4 When $r = \Delta$, m is even, define the mapping $c : V \left[C_m \dot{V} P_n\right] \to \{1, 2, 3 \cdots, m+n+2\}$, the following coloring gives the upper bound of $C_m \dot{V} P_n$

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = m+i$, $1 \leq i \leq n$
- $c(w_i) = \{m+n+1, m+n+2, \cdots, m+n+1, m+n+2\}$, $1 \leq i \leq m$

By Lemma 4.1 we have the lower bound $\chi_r \left[C_m \dot{V} P_n\right] \geq m+n+2$. Hence the r- adjacency condition fulfilled and therefore $\chi_r \left[C_m \dot{V} P_n\right] = m+n+2$, when $r = \Delta$, m is even .

Case: 5 When $r = \Delta$, m is odd, define the mapping $c : V \left[C_m \dot{V} P_n\right] \to \{1, 2, 3 \cdots, m+n+3\}$, the following coloring gives the upper bound of $C_m \dot{V} P_n$

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = m+i$, $1 \leq i \leq n$
- $c(w_i) = \{m+n+1, m+n+2, \cdots, m+n+1, m+n+2, m+n+3\}$, $1 \leq i \leq m$

By Lemma 4.1 we have the lower bound $\chi_r \left[C_m \dot{V} P_n\right] \geq m+n+3$. Hence the r- adjacency condition fulfilled and therefore $\chi_r \left[C_m \dot{V} P_n\right] = m+n+3$, when $r = \Delta$, m is odd.

□

Remark 9. *Let $m = 3$, $n \geq 2$ the r-dynamic chromatic number of central vertex join of cycle graph C_3 with path graph P_n is*

$$\chi_r\left[C_3\dot{V}P_n\right] = \begin{cases} 3, & r = 1 \\ 5, & 2 \leq r \leq 4 \\ r+1, & 5 \leq r \leq \Delta \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_3 with path graph P_n is given by $V\left[C_3\dot{V}P_n\right] = \{v_i, 1 \leq i \leq 3\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq 3\}$ and the edge set is $E\left[C_3\dot{V}P_n\right] = \{v_iwi, 1 \leq i \leq 3\} \cup \{w_iv_{i+1}, 1 \leq i \leq 2\} \cup \{w_3v_1\} \cup \{v_iu_j, 1 \leq i \leq 3, 1 \leq j \leq n\}$. The maximum and minimun degrees of $(C_m\dot{V}P_n)$ are $\Delta\left[C_m\dot{V}P_n\right] = n+2$ and $\delta\left[C_m\dot{V}P_n\right] = 2$. We prove the theorem in the following cases.

Case: 1 When $r = 1$, define the mapping $c : V\left[C_3\dot{V}P_n\right] \to \{1,2,3\}$. The assignment of colors are as follows

- $c(v_i) = 1\ ,\ \ i = 1,2,3$
- $c(w_i) = 2\ ,\ \ i = 1,2,3$
- $c(u_i) = \begin{cases} 2,3,2,3,\cdots,2, & when \text{ n } is\ even \\ 2,3,2,3,\cdots,2,3 & when \text{ n } is\ odd \end{cases}$

Thus we require 3 colors that is $\chi_r\left[C_3\dot{V}P_n\right] = 3$, when $r = 1$.

Case: 2 When $2 \leq r \leq 4$, the r- dynamic coloring are as follows, consider the mapping $c : V\left[C_3\dot{V}P_n\right] \to \{1,2,3,4,5\}$. The assignment of colors are as follows

- $c(v_i) = i\ ,\ \ i = 1,2,3$
- $c(w_i) = \{3,1,2\}\ ,\ \ i = 1,2,3$
- $c(u_i) = \begin{cases} 4.5.4.5,\cdots,4, & when \text{ n } is\ odd \\ 4,5,4,5,\cdots,4,5, & when \text{ n } is\ even \end{cases}$

Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[C_3\dot{V}P_n\right] = 5$, when $2 \leq r \leq 4$.

Case: 3 When $5 \leq r \leq \Delta$, define the mapping $c : V\left[C_3\dot{V}P_n\right] \to \{1,2,3\cdots,r+1\}$. The assignment of colors are as follows

- $c(v_i) = i\ ,\ 1 \leq i \leq 3$
- $c(u_i) = \{m+1, m+2,\cdots,r+1, m+1, m+2,\cdots,r+1,\cdots\}\ ,\ 1 \leq i \leq n$
- $c(w_i) = \{3,1,2\}\ ,\ 1 \leq i \leq 3$

Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[C_3\dot{V}P_n\right] = r+1$, when $5 \leq r \leq \Delta$.

□

Theorem 10. *Let $m \geq 4$,$n \geq 2$ the r-dynamic chromatic number of cenntral vertex join of cycle graph C_m with complete graph K_n is*

$$\chi_r\left[C_m\dot{V}K_n\right] = \begin{cases} \left\lfloor\frac{m+7}{2}\right\rfloor + n - 3, & r = 1 \\ m+n, & 2 \leq r \leq \Delta - 1 \\ m+n+2, & r = \Delta,\ m\ is\ \ even \\ m+n+3, & r = \Delta,\ m\ is\ \ odd \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_m with complete graph K_n is given by $V\left[C_m\dot{V}K_n\right] = \{v_i, 1 \leq i \leq m\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq m\}$ and the edge set is $E\left[C_m\dot{V}K_n\right] = \{v_iw_i, 1 \leq i \leq m\} \cup \{w_iv_{i+1}, 1 \leq i \leq m-1\} \cup \{w_mv_1\} \cup \{v_iv_j, 1 \leq i \leq m-2, i+2 \leq j \leq m\} \cup \{v_iu_j, 1 \leq i \leq m, 1 \leq j \leq n\}$. The maximum and minimun degrees of $(C_m\dot{V}K_n)$ are $\Delta\left[C_m\dot{V}K_n\right] = m+n-1$ and $\delta\left[C_m\dot{V}K_n\right] = 2$. We divide the proof in the following two cases.

Case: 1 When $r = 1$. By Lemma 4.1 we have the lower bound $\chi_r\left[C_m\dot{V}K_n\right] \geq \lfloor\frac{m+7}{2}\rfloor - 1$ but since we have complete graph in the place of G we require additional $n-2$ colors hence the lower bound transforms as $\chi_r\left[C_m\dot{V}K_n\right] \geq \lfloor\frac{m+7}{2}\rfloor + n - 3$. Now define the mapping $c : V\left[C_m\dot{V}K_n\right] \rightarrow \{1, 2, 3, \cdots, \lfloor\frac{m+7}{2}\rfloor + n - 3\}$.
We use the following colorings to show the upper bound,

- $c(u_i) = i, 1 \leq i \leq n$
- $c(v_i) = \{n+1, n+1, n+2, n+2, \cdots, \lfloor\frac{m+7}{2}\rfloor + n - 3\}, 1 \leq i \leq m$, when m is odd
- $c(v_i) = \{n+1, n+1, n+2, n+2, \cdots, \lfloor\frac{m+7}{2}\rfloor + n - 3, \lfloor\frac{m+7}{2}\rfloor + n - 3\}, 1 \leq i \leq m$, when m is even
- $c(w_i) = 1, 1 \leq i \leq m$

Hence the r-adjacency condition fulfilled, $\chi_r\left[C_m\dot{V}K_n\right] = \lfloor\frac{m+7}{2}\rfloor + n - 3$, when $r = 1$.

Case: 2 When $2 \leq r \leq \Delta - 1$. Consider the mapping $c : V\left[C_m\dot{V}K_n\right] \rightarrow \{1, 2, \cdots, m+n\}$ the following coloring gives the upper bound of $C_m\dot{V}K_n$

- $c(v_i) = i$,$1 \leq i \leq m$
- $c(w_i) = \{m, 1, 2, \cdots, m-1\}$, $1 \leq i \leq m$
- $c(u_i) = m + i$, $1 \leq i \leq n$

By the above coloring we have the upper bound as $\chi_r[C_m\dot{V}K_n] \leq m+n$. Again since we have K_n in the place of G by Lemma 3.2 doesn't provide an efficient bound we are in requirement of more colors than in any other graph. We can easily see that we require at least $m+n$ in this case. Thus we have $\chi_r[C_m\dot{V}K_n] \geq m+n$. Hence the r- adjacency condition is fulfilled and
$\chi_r[C_m\dot{V}K_n] = m+n$, when $2 \leq r \leq \Delta - 1$.

Case: 3 Proof same as Case 4 of Theorem 4.2.

Case: 4 Proof same as Case 5 of Theorem 4.2.

□

Remark 11. *Let $m = 3$, $n \geq 2$ the r-dynamic chromatic number of central vertex join of cycle graph C_3 with complete graph K_n is*

$$\chi_r[C_3\dot{V}K_n] = \begin{cases} m+n-2, & r = 1 \\ m+n, & 2 \leq r \leq \Delta \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_3 with complete graph K_n is given by $V\left[C_3\dot{V}K_n\right] = \{v_i, 1 \leq i \leq 3\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq 3\}$ and the edge set is $E\left[C_3\dot{V}K_n\right] = \{v_iw_i, 1 \leq i \leq m\} \cup \{w_iv_{i+1}, 1 \leq i \leq 2\} \cup \{w_3v_1\} \cup \{v_iu_j, 1 \leq i \leq 3, 1 \leq j \leq n\}$.
The maximum and minimun degrees of $(C_3\dot{V}K_n)$ are $\Delta\left[C_3\dot{V}K_n\right] = m+n-1$ and $\delta\left[C_3\dot{V}K_n\right] = 2$.We divide the proof in the following two cases.

Case: 1 When $r = 1$. Define the mapping $c : V\left[C_3\dot{V}K_n\right] \to \{1, 2, 3, \cdots, m+n-2\}$.

- $c(v_i) = 1$, $1 \le i \le 3$
- $c(w_i) = 2$, $1 \le i \le 3$
- $c(u_i) = \{2, 3, 4, \cdots, m+n-2\}$,$1 \le i \le n$

Hence the r-adjacency condition fulfilled and $\chi_r\left[C_3\dot{V}K_n\right] = n+m-2$, when $r = 1$.

Case: 2 When $2 \le r \le \Delta$. Consider the mapping $c : V\left[C_3\dot{V}K_n\right] \to \{1, 2, \cdots, m+n\}$as follows

- $c(v_i) = i$,$1 \le i \le 3$
- $c(w_i) = \{3, 1, 2\}$, $1 \le i \le 3$
- $c(u_i) = m+i$, $1 \le i \le n$

Hence the r-adjacency condition fulfilled and $\chi_r\left[C_3\dot{V}K_n\right] \le m+n$, when $2 \le r \le \Delta$.

□

Theorem 12. *Let $m \ge 5$, $n \ge 3$ the r-dynamic chromatic number of cenntral vertex join of cycle graph C_m with cycle graph C_n is*

$$\chi_r\left[C_m\dot{V}C_n\right] = \begin{cases} \left\lfloor\frac{m+7}{2}\right\rfloor - 1, & r = 1, n \text{ is even} \\ \left\lfloor\frac{m+7}{2}\right\rfloor, & r = 1, n \text{ is odd} \\ m+2, & 2 \le r \le\ m, n \text{ is even} \\ m+3, & 2 \le r \le\ m+1, n \text{ is odd} \\ r+3, & r = m+2, n \text{ is odd} \\ r+2, & \begin{cases} m+1 \le r \le \Delta - 1, & n \text{ is even} \\ m+3 \le r \le \Delta - 1, & n \text{ is odd} \end{cases} \\ m+n+2, & r = \Delta,\ m \text{ is even} \\ m+n+3, & r = \Delta,\ m \text{ is odd} \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_m with complete graph C_n is given by $V\left[C_m\dot{V}C_n)\right] = \{v_i, 1 \le i \le 3\} \cup \{u_i, 1 \le i \le n\} \cup \{w_i, 1 \le i \le m\}$ and the edge set is
$E\left[C_m\dot{V}C_n\right] = \{v_iw_i, 1 \le i \le m\} \cup \{w_iv_{i+1}, 1 \le i \le m-1\} \cup \{w_mv_1\} \cup$
$\{v_iv_j, 1 \le i \le m-2, i+2 \le j \le m\} \cup \{v_iu_j, 1 \le i \le m, 1 \le j \le n\}$.
The maximum and minimun degrees of $\left(C_m\dot{V}C_n\right)$ are $\Delta\left[C_m\dot{V}C_n\right] = m+n-1$ and $\delta\left[C_m\dot{V}C_n\right] = 2$. We divide the proof in the following two cases.

Case: 1 Proof same as Case 1 of Theorem 4.2.

Case: 2 When $r = 1$ and n is odd , define the mapping $c : V\left[C_m\dot{V}C_n\right] \to \left\{1, 2, 3 \cdots, \left\lfloor\frac{m+7}{2}\right\rfloor\right\}$. To show the upper bound we assign colors as follows.

- $c(u_i) = \{1, 2, 1, 2, \cdots, 1, 2, 3\}$, $1 \le i \le n$
- $c(v_i) = \{4, 4, 5, 5, \cdots, \left\lfloor\frac{m+7}{2}\right\rfloor\}$, $1 \le i \le m$, when m is odd
- $c(v_i) = \{4, 4, 5, 5, \cdots, \left\lfloor\frac{m+7}{2}\right\rfloor, \left\lfloor\frac{m+7}{2}\right\rfloor\}$, $1 \le i \le m$, when m is even
- $c(w_i) = 2$, $1 \le i \le m-1$

By Lemma 4.1 we have the lower bound $\chi_r\left[C_m\dot{V}C_n\right] \ge \left\lfloor\frac{m+7}{2}\right\rfloor - 1$ but since we have an odd cycle in G we require an additional color as odd cycle always requires 3 colors for proper coloring thus $\chi_r\left[C_m\dot{V}C_n\right] \ge \left\lfloor\frac{m+7}{2}\right\rfloor$. Hence $\chi_r\left[C_m\dot{V}C_n\right] = \left\lfloor\frac{m+7}{2}\right\rfloor$, when n is odd.

Case: 3 Proof same as Case 2 of Theorem 4.2.

Case: 4 When $2 \leq r \leq m+1$ and n is odd. Define the mapping $c : V\left[C_m \dot{V} C_n\right] \rightarrow \{1, 2, 3, \cdots, m+3\}$. To show the upper bound we assign colors as follows.

- $c(v_i) = i\ ,\ \ 1 \leq i \leq m$
- $c(w_i) = \{m, 1, 2, \cdots, m-1\},\ \ 1 \leq i \leq m$
- $c(u_i) = \{m+1, m+2, m+1, m+2, \cdots, m+3\},\ 1 \leq i \leq n$

Thus we have the upper bound $\chi_r\left[C_m \dot{V} C_n\right] \leq m+3$ for $2 \leq r \leq m+1$. By Lemma 4.1 we have the lower bound $\chi_r\left[C_m \dot{V} C_n\right] \geq m+2$ but since we have an odd cycle in G we require an additional color as odd cycle always requires 3 colors for proper coloring thus $\chi_r\left[C_m \dot{V} C_n\right] \geq m+3$ for $2 \leq r \leq m$ and by the same Lemma for $r = m+1$, $\chi_r\left[C_m \dot{V} C_n\right] \geq r+2 = m+3$. Hence the r- adjacency condition fulfilled and therefore $\chi_r\left[C_m \dot{V} C_n\right] = m+3$, when n is odd.

Case: 5 When $r = m+2$ and n is odd. Define the mapping $c : V\left[C_m \dot{V} C_n\right] \rightarrow \{1, 2, 3, \cdots, r+3\}$. To show the upper bound we assign colors as follows.

- $c(v_i) = i\ ,\ \ 1 \leq i \leq m$
- $c(w_i) = \{m, 1, 2, \cdots, m-1\},\ \ 1 \leq i \leq m$
- For the vertices of u_i first provide them with colors $\{m+1, m+2, \cdots, r+3\}$ in order and then for the remaining vertices in u_i provide them with colors from the set $\{m+1, m+2, \cdots, r+3\}$ so that the coloring is proper and 2- adjacency condition satisfied within u_i's.

By Lemma 4.1 for $r = m+2$ we have the lower bound $\chi_r\left[C_m \dot{V} C_n\right] \geq r+2$ but since n is odd we are in requirement of an additional color thus $\chi_r\left[C_m \dot{V} C_n\right] \geq r+3$. Hence the r-adjacency condition fulfilled $\chi_r\left[C_m \dot{V} C_n\right] = r+3$,when n is odd.

Case: 6 When $m+1 \leq r \leq \Delta - 1$, n is even and $m+3 \leq r \leq \Delta - 1$, n is odd. Define the mapping $c : V\left[C_m \dot{V} C_n\right] \rightarrow \{1, 2, 3, \cdots, r+2\}$. To show the upper bound we assign colors as follows.

- $c(v_i) = i\ ,\ \ 1 \leq i \leq m$
- $c(w_i) = \{m, 1, 2, \cdots, m-1\},\ \ 1 \leq i \leq m$
- For the vertices of u_i first provide them with colors $\{m+1, m+2, \cdots, r+2\}$ in order and then for the remaining vertices in u_i provide them with colors from the set $\{m+1, m+2, \cdots, r+2\}$ so that the coloring is proper and 2- adjacency condition satisfied within u_i's.

The lower bound follows directly by Lemma 4.1 for the two subcases i.e., $\chi_r\left[C_m \dot{V} C_n\right] \geq r+2$ when $m+1 \leq r \leq \Delta - 1$, n is even and $m+3 \leq r \leq \Delta - 1$, n is odd. Hence $\chi_r\left[C_m \dot{V} C_n\right] = r+2\ ,\ \ m+1 \leq r \leq \Delta - 1$, n is even and $m+3 \leq r \leq \Delta - 1$, n is odd.

Case: 7 Proof same as Case 4 of Theorem 4.2.

Case: 8 Proof same as Case 5 of Theorem 4.2.

□

Remark 13. *Let $m = 4$,$n \geq 3$ the r-dynamic chromatic number of central vertex join of cycle graph C_4 with cycle graph C_n is*

$$\chi_r \left[C_4 \dot{V} C_n\right] = \begin{cases} 5, & r = 1, n \text{ is even} \\ 4, & r = 1, n \text{ is odd} \\ m+3, & 2 \leq r \leq m+1, n \text{ is odd} \\ m+2, & 2 \leq r \leq m, n \text{ is even} \\ r+3, & m+1 \leq r \leq \Delta - 1, \ n \text{ is even} \\ r+2, & m+2 \leq r \leq \Delta - 1, \ n \text{ is odd} \\ m+n+2, & r = \Delta \end{cases}$$

Proof. The vertex set of central vertex join of cycle graph C_4 with complete graph C_n is given by $V\left[C_4 \dot{V} C_n\right] = \{v_i, 1 \leq i \leq 4\} \cup \{u_i, 1 \leq i \leq n\} \cup \{w_i, 1 \leq i \leq 4\}$ and the edge set is $E\left[C_4 \dot{V} C_n\right] = \{v_i w_i, 1 \leq i \leq 4\} \cup \{w_i v_{i+1}, 1 \leq i \leq 3\} \cup \{w_4 v_1\} \cup \{v_1 v_3\} \cup \{v_2 v_4\}$ $\cup \{v_i u_j, 1 \leq i \leq 4, 1 \leq j \leq n\}$. The maximum and minimun degrees of $(C_4 \dot{V} C_n)$ are $\Delta\left[C_4 \dot{V} C_n\right] = n + 3$ and $\delta\left[C_4 \dot{V} C_n\right] = 2$.We prove the remark in the following cases.

Case: 1 When $r = 1$, n is even. Define the mapping $c : V\left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, 4, 5\}$ and the following coloring

- $c(v_1, v_2, v_3, v_4) = 1, 1, 2, 2$
- $c(w_i) = 3, \ 1 \leq i \leq 4$
- $c(u_i) = \{3, 4, 3, 4, \cdots\}, 1 \leq i \leq n - 1$
- $c(u_n) = 5$

Hence the r-adjacency condition fulfilled and $\chi_r\left[C_4 \dot{V} C_n\right] = 5$,when $r = 1$ and n is even.

Case: 2 When $r = 1$, n is odd. Define the mapping $c : V\left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, 4\}$ and the coloring are as follows

- $c(v_1, v_2, v_3, v_4) = 1, 1, 2, 2$
- $c(w_i) = 3, \ 1 \leq i \leq m$
- $c(u_i) = \{3, 4, 3, 4, \cdots\}, 1 \leq i \leq n$

Hence the r-adjacency condition fulfilled and $\chi_r\left[C_4 \dot{V} C_n\right] = 4$, when $r = 1$ and n is odd.

Case: 3 When $m = 4$, $2 \leq r \leq m + 1$, n is odd. Define the mapping $c : V\left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, \cdots, m + 3\}$ and the coloring are as follows

- $c(v_i) = i$, $1 \leq r \leq m$
- $c(w_i) = \{m, 1, 2, 3, \cdots, m - 1\}, \ 1 \leq i \leq m$
- $c(u_i) = \{m + 1, m + 2, m + 3 \cdots m + 1, m + 2, m + 3\}$, $1 \leq i \leq n$

Hence the r-adjacency condition fulfilled and $\chi_r\left[C_4 \dot{V} C_n\right] \leq m + 3$, when $2 \leq r \leq m + 1$, n is odd.

Case: 4 When $2 \leq r \leq m$, n is even. Define the mapping $c : V\left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, \cdots, m + 2\}$ and the coloring are as follows

- $c(v_i) = i$, $1 \leq r \leq m$
- $c(w_i) = \{m, 1, 2, 3, \cdots, m - 1\}, \ 1 \leq i \leq m$

- $c(u_i) = \{m+1, m+2, \cdots m+1, m+2\}$, $1 \leq i \leq n$

Hence the r-adjacency condition fulfilled and $\chi_r \left[C_4 \dot{V} C_n\right] = m+2$,when $2 \leq r \leq m$, n is even.

Case: 5 When $m+1 \leq r \leq \Delta - 1$, n is even. Define the mapping $c : V \left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, \cdots, r+3\}$ and the coloring are as follows

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(w_i) = m+i$, $1 \leq i \leq m$
- For the vertices of u_i first provide them with colors $\{m+1, m+2, \cdots, r+3\}$ in order and then for the remaining vertices in u_i provide them with colors from the set $\{m+1, m+2, \cdots, r+3\}$ so that the coloring is proper and 2- adjacency condition satisfied within u_i's.

Hence the r-adjacency condition fulfilled and $\chi_r \left[C_4 \dot{V} C_n\right] = r+3$,when $m+1 \leq r \leq \Delta-1$, n is even.

Case: 6 When $m+2 \leq r \leq \Delta - 1$, n is odd. Define the mapping $c : V \left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, \cdots, r+2\}$ and the coloring are as follows

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(w_i) = m+i$, $1 \leq i \leq m$
- For the vertices of $u_i$first provide them with colors $\{m+1, m+2, \cdots, r+2\}$ in order and then for the remaining vertices in u_i provide them with colors from the set $\{m+1, m+2, \cdots, r+2\}$ so that the coloring is proper and 2- adjacency condition satisfied within u_i's.

Hence the r-adjacency condition fulfilled and $\chi_r \left[C_4 \dot{V} C_n\right] = r+2$,when n is odd.

Case: 7 When $r = \Delta$. Define the mapping $c : V \left[C_4 \dot{V} C_n\right] \to \{1, 2, 3, \cdots, m+n+2\}$ and the coloring are as follows

- $c(v_i) = i$, $1 \leq i \leq m$
- $c(u_i) = m+i$, $1 \leq i \leq n$
- $c(w_i) = \{m+n+1, m+n+2, m+n+1, m+n+2\}$, $1 \leq i \leq m$

Hence the r-adjacency condition fulfilled and $\chi_r \left[C_4 \dot{V} C_n\right] = m+n+2$,When $r = \Delta$.

□

5. Conclusion

In this paper we have attained the r- dynamic chromatic number of central vertex join of path graph with path graph, complete graph and cycle graph. Also cycle graph with path graph, complete graph and cycle graph. We can extend this work for central edge join for the same graphs.

Acknowledgments

The authors express their gratitude to the anonymous reviewers for their thorough reading, insightful remarks, and helpful suggestions that improved the content of this manuscript.

Journal of Physics: Conference Series **2157** (2022) 012007 doi:10.1088/1742-6596/2157/1/012007

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