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Edge odd graceful of alternate snake graphs

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Edge odd graceful of alternate snake graphs

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Abstract. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, and the number of edges q. An edge odd graceful labeling of G is a bijection $f : E(G) \longrightarrow \{1,3,5,\dots,2q-1\}$ so that induced mapping $f^+ : V(G) \longrightarrow \{0,1,2,\dots,2q-1\}$ given by $f^+(x) = \sum_{xy\in E(G)} f(xy)$ $(mod\ 2q)$ is injective. A graph which admits an edge odd graceful labeling is called edge odd graceful. An alternate triangular snake graph $A(C_3^m)$ is a graph obtained from a path $u_1u_2u_3 \dots u_{2m}$ by joining every u_{2i-1} and u_{2i} to a new vertex $v_i, 1 \le i \le m$. An alternate quadrilateral snake graph $A(C_4^m)$ is a graph obtained from vertices $u_1, u_2, u_3, \dots, u_{2m}$ by joining every u_{2i-1} and u_{2i} to two vertices v_i and $w_i, 1 \le i \le m$, and joining every u_{2i} to u_{2i+1} with $1 \le i \le m-1$. In this paper, we show that alternate triangular snake and alternate quadrilateral snake graphs are edge odd graceful.

1. Introduction

In this paper, we follow Hartsfield and Ringel [1] for the basic notations, definitions, and terminology. Thus, if G is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively; $p = |V(G)|$ and $q = |E(G)|$ denote the number of vertices and the number of edges of G, respectively. We only consider finite and simple graphs, all graphs have finite number of vertices and finite number of edges, have no loops nor multiple edges.

A labeling of a graph G is an assignment of labels to the vertices or edges or both subject. If only the vertices (or the edges) of G are labeled, the resulting graph is called vertex labeled graph (or edge labeled graph). In a vertex labeling of a graph, traditionally, we label distinct vertices with distinct labels [2]. Many kinds and results of graph labeling can be found in Chartrand et.al [2] and Gallian [3]. These two books are very good resource for graph labeling.

One kind of labeling that has been studied in the literature is graceful labeling. Let G be nonempty graph of order p and size q. The graph G has a graceful labeling if there is an injective function $f: V(G) \to \{0,1,2,\dots,p\}$ such that the induced $f': E(G) \to \{1,2,3,\dots,q\}$ given by $f'(uv) = |f(u) - f(v)|$ for every $E(G)$ is bijective [2]. In [4] it is mentioned that in 1985, Lo introduced a new graph labeling i.e. edge graceful labeling. A graph G with p vertices and q edges which admits edge graceful graph if there exists a bijection $f: E(G) \to \{0,1,2,\dots,q\}$ that the induced function $f^+ : V(G) \longrightarrow \sum_{xy\in E(G)} f(xy)$ (mod p). A graph which admits an edge graceful labeling is called an edge graceful graph.

In 2009, Solaraiju and Chitra introduced edge odd graceful labeling [5]. An edge odd graceful labeling of G is a bijection$f : E(G) \longrightarrow \{1,3,5,\dots,2q-1\}$so that induced mapping $f^+ : V(G) \longrightarrow \{0,1,2,\dots,2q-1\}$ given by $f^+(x) = \sum_{xy\in E(G)} f(xy)$ (mod $2q$) is injective. A graph which admits an edge odd graceful labeling is said to be edge odd graceful. They proved that Huffman tree P_n^+ for $n \geq 2$, bistar graph $B_{n,n}$ for n odd, graph $< K_{1,n}: 2 >$ for n odd, and double star graph $K_{1,n,n}$ for even n, are edge odd graceful graph [5]. Since then, edge odd graceful labeling of many types of graphs were studied, see for example [4,6-9].

In this paper we study edge odd graceful labeling of alternate triangular snake and alternate quadrilateral snake graphs. The two graphs can be found in [10] and [11], respectively. An alternate triangular snake graph $A(C_3^m)$ is a graph obtained from a path $u_1 u_2 u_3 \dots u_{2m}$ by joining every u_{2i-1} and u_{2i} to a new vertex v_i, $1 \leq i \leq m$. Here we use a definition of an alternate quadrilateral snake graph as follows. An alternate quadrilateral snake graph $A(C_4^m)$ is a graph obtained from vertices $u_1, u_2, u_3, \dots, u_{2m}$ by joining every u_{2i-1} and u_{2i} to two vertices v_{2i} and v_{2i-1}, $1 \leq i \leq m$, and joining every u_{2i} to u_{2i+1}, $1 \leq i \leq m-1$. It is known that alternate triangular snake and alternate quadrilateral snake graphs admit several labeling, see [10] and [11]. In this paper, we show that each of the two graphs admits an edge odd graceful labeling.

2. Main Results

In this section, we show that alternate triangular snake graphs and alternate quadrilateral snake graphs are edge odd graceful.

Theorem 1. *Let m be a positive integer. The alternate triangular snake graph $A(C_3^m)$ is edge odd graceful.*

Proof. Let graph $G = A(C_3^m)$ be a graph with a vertex set

$$V(G) = \{u_i \mid 1 \leq i \leq 2m\} \cup \{v_i \mid 1 \leq i \leq m\}$$

and an edge set

$$E(G) = \{u_{2i-1}v_i, u_{2i-1}u_{2i}, v_i u_{2i} \mid 1 \leq i \leq m\} \cup \{u_{2i}u_{2i+1} \mid 1 \leq i \leq m-1\}$$

Figure 1 shows graph $A(C_3^m)$.

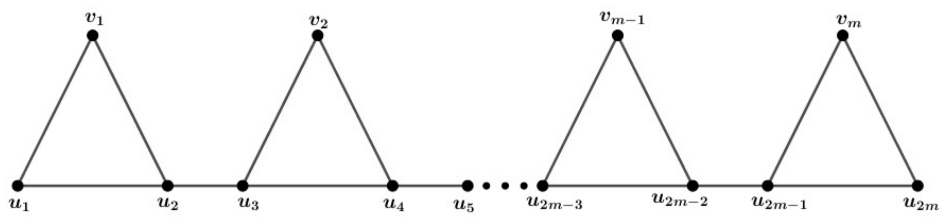

Figure 1. Graph $A(C_3^m)$.

The number of vertices of G is

$$p = |V(G)| = 3m,$$

and the number of edges of G is

$$q = |E(G)| = 4m - 1.$$

and hence,

$$2q = 8m - 2$$

Define $f : E(G) \longrightarrow \{1,3,5, \dots, (2q-1)\}$ by :

For every $1 \leq i \leq 2m-1$,

$$f(u_i u_{i+1}) = 2i - 1$$

And for every $1 \leq i \leq m$,

$$f(u_{2i-1}v_i) = 8m - 4i + 1$$
$$f(u_{2i}v_i) = 8m - 4i - 1$$

as figure 2.

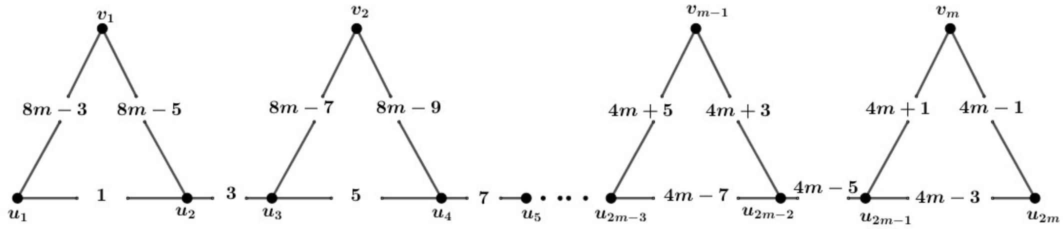

Figure 2. Edge Labeling of $A(C_3^m)$.

It is obvious that

$$\{f(u_i u_{i+1}) | 1 \leq i \leq 2m-1\} = \{1, 3, 5, \ldots, 4m-3\},$$
$$\{f(u_{2i-1} v_i) | 1 \leq i \leq m\} = \{4m+1, 4m+5, 4m+9, \ldots, 8m-3\},$$
$$\{f(u_{2i} v_i) | 1 \leq i \leq m\} = \{4m-1, 4m+3, 4m+7, \ldots, 8m-5\},$$

and

$$f : E(G) \longrightarrow \{1, 3, 5, \ldots, 2q-1\}$$

is bijective, where $2q-1 = 8m-3$.

We will show that the induced function $f^+ : V(G) \longrightarrow \{0, 1, 2, \ldots, (2q-1)\}$, defined by

$$f^+(x) = \sum_{xy \in E(G)} f(xy) \ (mod\ 8\text{m}-2)$$

is injective. It is obvious that

$$f^+(u_1) = 8m-2 \equiv 0\ mod\ (8m-2)$$
$$f^+(u_{2m}) \equiv 8m-4\ mod\ (8m-2)$$

For every $1 \leq i \leq m-1$,

$$\begin{aligned} f^+(u_{2i}) &= f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i}v_i) \\ &= (4i-3) + (4i-1) + (8m-4i-1) \\ &= 8m+4i-5 \\ &\equiv 4i-3 \ (\text{mod } 8m-2), \end{aligned}$$

where

$$1 \leq 4i-3 \leq 4m-7$$

Similarly,

$$\begin{aligned} f^+(u_{2i+1}) &= f(u_{2i}u_{2i+1}) + f(u_{2i+1}u_{2i+2}) + f(u_{2i+1}v_{i+1}) \\ &= (4i-1) + (4i+1) + (8m-4i-3) \\ &= 8m+4i-3 \\ &\equiv 4i-1 (\text{mod } 8m-2), \end{aligned}$$

where

$$3 \leq 4i-1 \leq 4m-5$$

For every $1 \leq i \leq m$, we have

$$\begin{aligned} f^+(v_i) &= f(u_{2i-1}v_i) + f(u_{2i}v_i) \\ &= (8m-4i+1) + (8m-4i-1) \\ &= 16m-8i \\ &\equiv 8m-8i+2 \ (\text{mod } 8m-2), \\ &= 4(2m-2i)+2 \ (\text{mod } 8m-2), \end{aligned}$$

where

$$2 \leq 4(2m-2i)+2 \leq 8m-6$$

We can see that all the vertex labels are distinct and then f^+ is injective.
This completes the proof of the theorem. ■

For example, Figure 3 shows an edge odd graceful labeling of $A(C_3^8)$ and $A(C_3^7)$

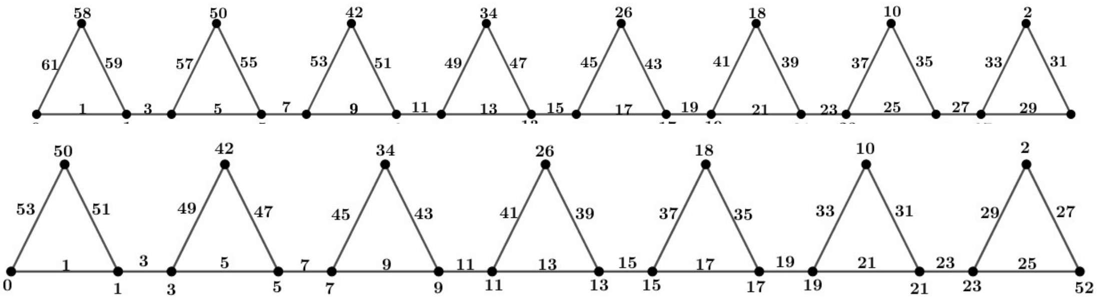

Figure 3. Edge Odd Graceful Graph of $A(C_3^8)$ and $A(C_3^7)$.

Our first result is on alternate quadrilateral snake graphs $A(C_4^m)$, $m \geq 2$. When $m = 1$, $A(C_4^m) = C_4^1$ is a cycle C_4. It is easy to see that the cycle C_4 is not edge odd graceful.

Theorem 2. *Let* m *be a positive integer, for* $m \geq 2$.*The alternate quadrilateral snake graph* $A(C_4^m)$ *is an edge odd graceful.*

Proof. Let graph $G = A(C_4^m)$ be a graph with a vertex set

$$V(G) = \{u_i |\ 1 \leq i \leq m+1\} \cup \{v_i |\ 1 \leq i \leq 2m-1\}$$

And an edge set

$$E(G) = \{u_{2i-1}v_{2i}, u_{2i}v_{2i}, u_{2i-1}v_{2i-1}, u_{2i}v_{2i-1} | 1 \leq i \leq m\} \cup \{u_{2i}v_{2i+1} | 1 \leq i \leq m-1\}$$

As figure 4.

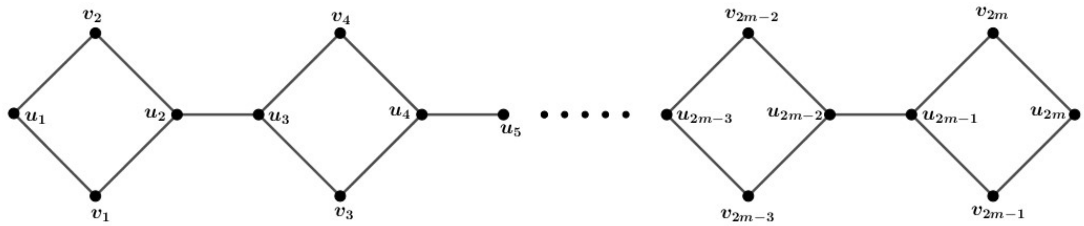

Figure 4. Graph $A(C_4^m)$.

The number of vertices of G is

$$p = |V(G)| = 4m,$$

and the number of edges of G is

$$q = |E(G)| = 5m - 1.$$

and hence,

$$2q = 10m - 2$$

Define $f : E(G) \longrightarrow \{1,3,5, \ldots, (2q-1)\}$ by:

For every $1 \leq i \leq m-1$,

$$f(u_{2i}u_{2i+1}) = 8m + 2i - 1$$

For every $1 \leq i \leq m$,

$$f(u_{2i-1}v_{2i}) = 4i - 3,$$
$$f(u_{2i}v_{2i}) = 4m + 4i - 3,$$
$$f(u_{2i-1}v_{2i-1}) = 4m + 4i - 1,$$
$$f(u_{2i}v_{2i-1}) = 4i - 1.$$

As figure 5.

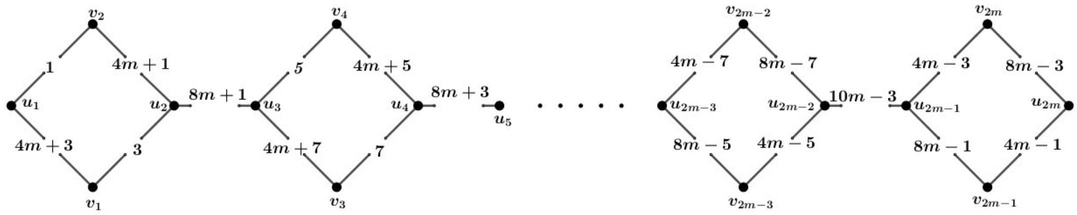

Figure 5. Edge labeling of $A(C_4^m)$.

It obvious that

$$\{f(u_{2i}u_{2i+1})|1 \leq i \leq m-1\} = \{8m+1, 8m+3, 8m+5, \dots, 10m-3\},$$
$$\{f(u_{2i-1}v_{2i})|1 \leq i \leq m\} = \{1, 5, 9, \dots, 4m-3\},$$
$$\{f(u_{2i}v_{2i})|1 \leq i \leq m\} = \{4m+1, 4m+5, 4m+9, \dots, 8m-3\},$$
$$\{f(u_{2i-1}v_{2i-1})|1 \leq i \leq m\} = \{4m+3, 4m+7, 4m+11, \dots, 8m-1\},$$
$$\{f(u_{2i}v_{2i-1})|1 \leq i \leq m\} = \{3, 7, 11, \dots, 4m-1\},$$

and

$$f: E(G) \longrightarrow \{1, 3, 5, \dots, 2q-1\}$$

is bijective, where $2q-1 = 10m-3$.

We will show that the induced function$f^+: V(G) \longrightarrow \{0, 1, 2, \dots, (2q-1)\}$, defined by

$$f^+(x) = \sum_{xy \in E(G)} f(xy)\ (mod\ 10\text{m}-2)$$

is injective. It is obvious that

$$f^+(u_1) = 4m+4 \text{ (mod } 10m-2)$$
$$f^+(u_{2m}) = 12m-4.$$
$$\equiv 2m-2 \text{ (mod } 10m-2).$$

For every $1 \leq i \leq m-1$, we have

$$\begin{aligned} f^+(u_{2i}) &= f(u_{2i}v_{2i}) + f(u_{2i}v_{2i}) + f(u_{2i}u_{2i+1}) + \\ &= (4m+4i-3) + (4i-1) + (8m+2i-1) \\ &= 12m+10i-5 \\ &\equiv 2m+10i-3 \text{ (mod } 10m-2). \end{aligned}$$

and

$$\begin{aligned} f^+(u_{2i+1}) &= f(u_{2i}u_{2i+1}) + f(u_{2i+1}v_{2i+2}) + f(u_{2i+1}v_{2i+1}) \\ &= (4m+2i-1) + (4i+1) + (8m+4i+3) \\ &= 12m+10i+3 \\ &\equiv 2m+10i+5 \text{ (mod } 10m-2), \end{aligned}$$

For every $1 \leq i \leq m$, we have

$$\begin{aligned} f^+(v_{2i}) &= f(u_{2i-1}v_{2i}) + f(u_{2i}v_{2i}) \\ &= (4i-3) + (4m+4i-3) \\ &= 4m+8i-6 \text{ (mod } 10m-2), \end{aligned}$$

and

$$\begin{aligned} f^+(v_{2i-1}) &= f(u_{2i-1}v_{2i-1}) + f(u_{2i}v_{2i-1}) \\ &= (4m+4i-1) + (4i-1) \\ &= 4m+8i-2 \text{ (mod } 10m-2). \end{aligned}$$

Note that the value of $f^+(x)$ is odd if and only if degree of x is odd. Further,

$$\{f^+(u_{2i})|1 \leq i \leq m-1\} = \{12m+5, 12m+15, 12m+25, \dots, 22m-15\},$$
$$\{f^+(u_{2i+1})|1 \leq i \leq m-1\} = \{12m+13, 2m+23, 12m+33, \dots, 12m-7\},$$
$$\{f^+(v_{2i})|1 \leq i \leq m-1\} = \{4m+2, 4m+10, 4m+18, \dots, 12m-6\},$$
$$\{f^+(v_{2i-1})|1 \leq i \leq m-1\} = \{4m+6, 4m+14, 4m+28, \dots, 12m-2\},$$
$$\{f^+(u_1), f^+(u_1)\} = \{4m+4, 12m-4\}.$$

Suppose for some integers $1 \le i < j \le m-1$,

$$f^{+}(u_{2i}) \equiv f^{+}(u_{2j}) \pmod{10m-2}.$$

Then $1 \le j-i \le m-2$, and

$$12m + 10i - 5 \equiv 12m + 10j - 5 \pmod{10m-2},$$
$$10(j-i) = k(10m-2),$$

for some positive integer k; which is impossible since

$$k(10m-2) = 10k(m-1) + 8k > 10(j-i).$$

Thus $f^{+}(u_{2i}) \not\equiv f^{+}(u_{2j}) \pmod{10m-2}$ when $i \neq j$. By using the similar argument, it can be shown that all the vertex labels are distinct, f^{+} is injective.
This completes the proof of the theorem. ■

For example, Figure 6 shows edge odd graceful labeling of alternate quadrilateral snake graph $A(C_4^5)$ and $A(C_4^7)$

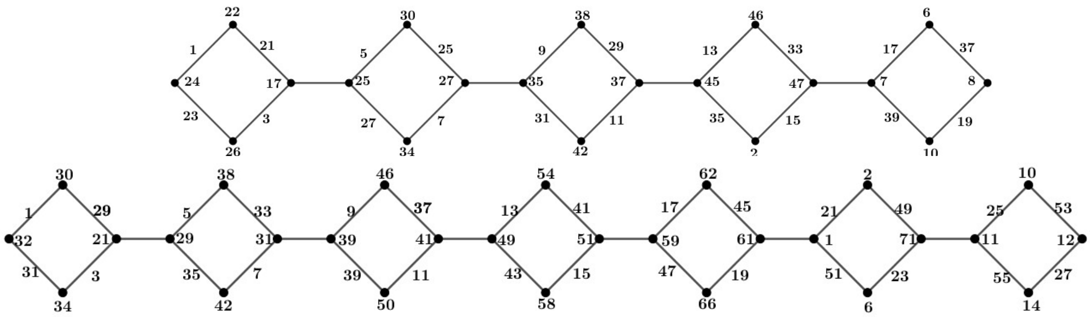

Figure 6. Edge odd graceful graph of $A(C_4^5)$ and $A(C_4^7)$.

3. Conclusions

We have shown that alternate triangular snake graph $A(C_3^m)$ and alternate quadrilateral snake graph $A(C_4^m)$, for any$m \ge 2$, are edge odd graceful. However, the problems are still open for graphs $A(C_n^m)$ when m, n is a positive integer, $n > 4$. We can try to find out wether this graph is edge odd gracefulor not.

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References

[1] Hartsfield N and Ringel G 1994 *Pearls in Graph Theory* (London: Academic Press)
[2] Chartrand G, Egan C and Zhang P 2019 *How to Label a Graph* (Switzerland: Springer International Publishing)
[3] Gallian J A 2020 A Dynamic Survey of Graph Labeling *Electron. J. Comb.* 8–342
[4] Daoud S N 2017 Edge Odd Graceful Labeling of Some Path and Cycle Related Graphs *AKCE Int. J. Graphs Comb.***14** 178–203
[5] Solairaju A and Chithra K 2009 Edge - Odd Graceful Graphs *Electron. Notes Discret. Math.* **33** 15–20
[6] Jeba Jesintha J and Ezhilarasi Hilda K 2016 Shell Butterfly Graphs are Edge Odd Gracaeful **109** 159–66
[7] Daoud S N 2019 Edge odd graceful labeling of cylinder and torus grid graphs *IEEE Access* **7** 10568–92
[8] Seoud M and Salim M 2016 Further results on edge - odd graceful graphs *Turkish J. Math.* **40** 647–56

[9] Solairaju A, Subbulakshmi S and Kokila R 2017 Various Labelings for Ladder, Cycle Merging with Fan, and Open Staircase Graph **13** 1347–55
[10] Sunoj B S and Mathew Varkey T K 2017 Square Difference Prime Labeling for Some Snake Graphs *Glob. J. Pure Appl. Math.* **13** 1083–9
[11] Vaghela U and Parmar D 2020 Difference Perfect Square Cordial Labeling of Snake Graphs Difference Perfect Square Cordial Labeling of Snake Graphs *Zeichen J.* **6** 54–5