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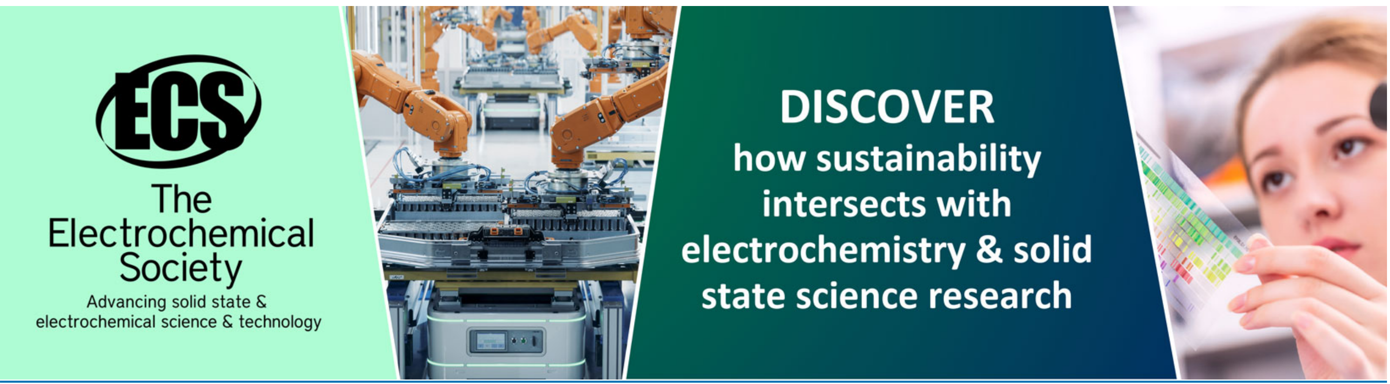

Emergence of parity time symmetric quantum critical phenomena

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Abstract. Emergence of different interesting and insightful phenomena at different length scale is the heart of quantum many-body system. We show that the physics of parity-time (PT) symmetry is one new addition to them. We show explicitly that the emergence of different topological excitation at different length scale for the PT symmetry system through the analysis of renormalization group (RG) flow lines. We observe that the higher order RG process favour the emergence of asymptotic freedom like behaviour and also show the effect of strong correlation on the emergent phases. Interestingly, the asymptotic freedom like behaviour is favoured by PT symmetry phase of the system. Moreover, we also derive the scaling relation for the couplings in RG equations. These findings can be tested experimentally in ultracold atoms.

1. Introduction

In quantum many-body physics, emergent phenomena are an essential aspect. In this view, fundamentally new kinds of phenomena emerge within the complex assemblies of particles which can not be anticipated from a priori knowledge of the microscopic laws of nature. One can raise the question at the fundamental level: what emergent principles and laws develop as we proceed from the microscopic scale to the macro scale? P. W. Anderson first introduced the concept of "emergent phenomena" into physics [1, 2]. The behavior of large and complex aggregation of elementary particles cannot be understood in terms of simple extrapolation properties of a few particles. Instead, at each level of complexity entirely new properties appear and the understanding of the new behavior triggers a new front of the research area.

Laughlin and Pines [3] have explained the emergent phenomena very nicely in their article on "The Theory of Everything". The emergent physical phenomena regulated by higher organizing principles have a property, namely their insensitivity to microscopic scale that is directly relevant to the broad question of what is knowable in the deepest sense of the term. One can also understand the emergent phenomena from the low energy excitations of conventional superconductors. It is completely generic and is characterized by a handful of parameters that may be obtained experimentally but not from the first principle.

It has been known since the early 1970s that renormalizability is an emergent property of ordinary matter either in stable quantum phases, such as the superconducting state, or at particular zero-temperature phase transitions between such states called quantum critical points. In either case the low-energy excitation spectrum becomes more and more generic and less and

less sensitive to microscopic details as the energy scale of the measurement is lowered, until in the extreme limit of low energy, all evidence of the microscopic equations vanishes away.
In this research paper, we study the emergent physics of parity-time (PT) symmetric quantum mechanical system in more advanced level. In the non-hermitian quantum system, PT symmetric quantum mechanics is an extension of conventional quantum mechanics into complex domain [4, 5, 6, 7, 8]. Initially, PT symmetry quantum mechanics has started as an interesting mathematical discovery and a good theoretical exercise in theoretical physics [4, 5, 6, 7, 8]. But in current literature, it has been expanded experimentally in different field of science like open quantum systems [8, 9], physics of gain and loss (as found in photonics [10, 11, 12]) or systems where the non-Hermiticity models the finite lifetime [13, 14], localization–delocalization of correlated many-body system [15, 16, 17], biological systems [18, 19, 20], Weyl semi-metals [21, 22, 23, 24, 25], topology and dissipation [26, 27] and PT symmetric circuit QED [28]. One of the most important feature of non-hermitian quantum mechanics is the exceptional point (EP) [3, 4, 5, 6, 29, 30]. This EP has a dramatic effect on the system, leading to the nontrivial physics with interesting counterintuitive features, which we will observe in this study.
We use the renormalization group (RG) method to study this PT symmetry quantum criticality problem. The mathematical structure and results of the RG theory are a significant conceptual advancement in the quantum field theory of both high-energy and condensed matter physics [31, 32] in the last several decades. The need for RG is really transparent in condensed matter physics. RG theory is a formalism that relates the physics at different length scales in condensed matter physics and the physics at different energy scales in high-energy physics [33, 34]. One of the successes of RG theory is in the study of classical and quantum Berezinskii–Kosterltz–Thouless (QBKT) transition [35] in XY model and topological insulator respectively. The classical two dimensional XY model was found to have a power-law correlation function at low-temperature regime and exponential at the high-temperature regime. The phase transition associated with it was not due to the well-known spontaneous symmetry breaking mechanism. Berezinskii [31], in the year 1971 and Kosterlitz and Thouless [32], in the year 1973 have explained this new kind of phase transition in terms of topological non-trivial vortex using the RG method. The high-temperature regime favours the thermal generation of vortices that unbounded, while in the low-temperature regime vortices are bounded and vortex and anti-vortex are always found in pairs. This binding and unbinding of vortex pairs facilitate the phase transition which is now called BKT transition. QBKT at zero temperature was also found to appear in the interacting helical liquid system at the one-dimensional edge of a two-dimensional topological insulator, coupled to an external magnetic field and s-wave superconductor. Through the RG flow analysis of this model, QBKT was identified between Luttinger liquid phase and Ising-ferromagnetic and gaped superconducting phase. The direction of the RG flow distinguishes between distinct quantum phases.
Motivation: We present emergent physics of quantum criticality at different length scales. We will see that the interplay between many-body correlations and PT symmetry leads to the emergence of quantum critical phenomena beyond the Hermitian paradigm of quantum many-body physics [36, 37]. We also raise the question of how the topology related to the PT-symmetry physics at different length-scales of the system emerges.
A successful part of the RG theory is the observation of asymptotic freedom for high-energy physics and the other is the Berezinskii–Kosterltz–Thouless transition physics for condensed matter physics, i.e, the two extreme ends of theoretical physics. In this study we are successful in unifying these two observations of RG theory in a single framework of physics PT symmetry for interacting quantum many body system.

2. The Model Hamiltonian and Renormalization Group Equation

We consider a class of one-dimensional quantum systems described by the sine-Gordon field theory

$$H = H_0 + V(\phi), \tag{1}$$

where H_0 is

$$H_0 = \left(\frac{hv}{2\pi}\right) \int dx[K(\partial_x\theta(x))^2 + \frac{1}{K}(\partial_x\phi(x))^2]. \tag{2}$$

The Hamiltonian, H gives a universal framework for describing one dimensional interacting bosons and fermionic system and $V(\phi(x))$ is the sine-Gordon potential. Here $\phi(x)$ is field with the parity operation, $P\phi P^{-1} = -\phi$. The $\theta(x)$ is the dual field of $\phi(x)$ and satisfies the commutation relation, $[\phi(x), \partial_x\theta(x')] = -i\pi\delta(x - x')$. H_0 is the Tomonoga Luttinger liquid (TLL) Hamiltonian. We will see that the value of K will play an important role to determine length-scale dependent quantum criticality for this model Hamiltonian system. The relevance of this term gives the gapped phase in the system, this transition from the gapless Luttinger liquid phase to the Mott insulating phase in the system.
A generalization to the PT symmetric case by adding an imaginary contribution [36, 37] to the potential term is as follows

$$V(\phi) = \frac{\alpha_r}{\pi} cos(2\phi) - \frac{i\alpha_i}{\pi} sin(2\phi), \tag{3}$$

where α_r and α_i are the real and imaginary part of the potential. The imaginary part of the potential, which introduces physics of spectral singularity occurs when the real and imaginary part of the potential are same [28].

We will observe that when α_r becomes relevant, a stable gapped phase, i.e, the fluctuation of ϕ gets suppressed. But when α_i becomes relevant, it facilities the fluctuations of ϕ. Then the imaginary of the potential profile are not the same as a consequence of it and the imaginary potential enhances the correlation of conjugate field θ. To get the correct physical picture of quantum criticality for this model Hamiltonian, RG study is essential [33]. The analytical expressions for the second and third order RG equations are the following:
The analytical expressions for the second order RG equations are [36, 37],

$$\begin{aligned}
\frac{dg_r}{dl} &= (2 - K)\, g_r \\
\frac{dg_i}{dl} &= (2 - K)\, g_i \\
\frac{dK}{dl} &= ({g_i}^2 - {g_r}^2)K^2.
\end{aligned} \tag{4}$$

The analytical expressions for the third order RG equations are,

$$\begin{aligned}
\frac{dg_r}{dl} &= (2 - K)\, g_r + 5{g_r}^3 - 5{g_i}^2 g_r \\
\frac{dg_i}{dl} &= (2 - K)\, g_i - 5{g_i}^3 + 5{g_r}^2 g_i \\
\frac{dK}{dl} &= ({g_i}^2 - {g_r}^2)K^2.
\end{aligned} \tag{5}$$

Here l is the logarithmic RG scale and $g_{r,i} = \frac{\alpha_{r,i} a^2}{hv}$ are the dimensionless coupling constants with a being a short distance cut-off and v is the velocity of the collective excitations.

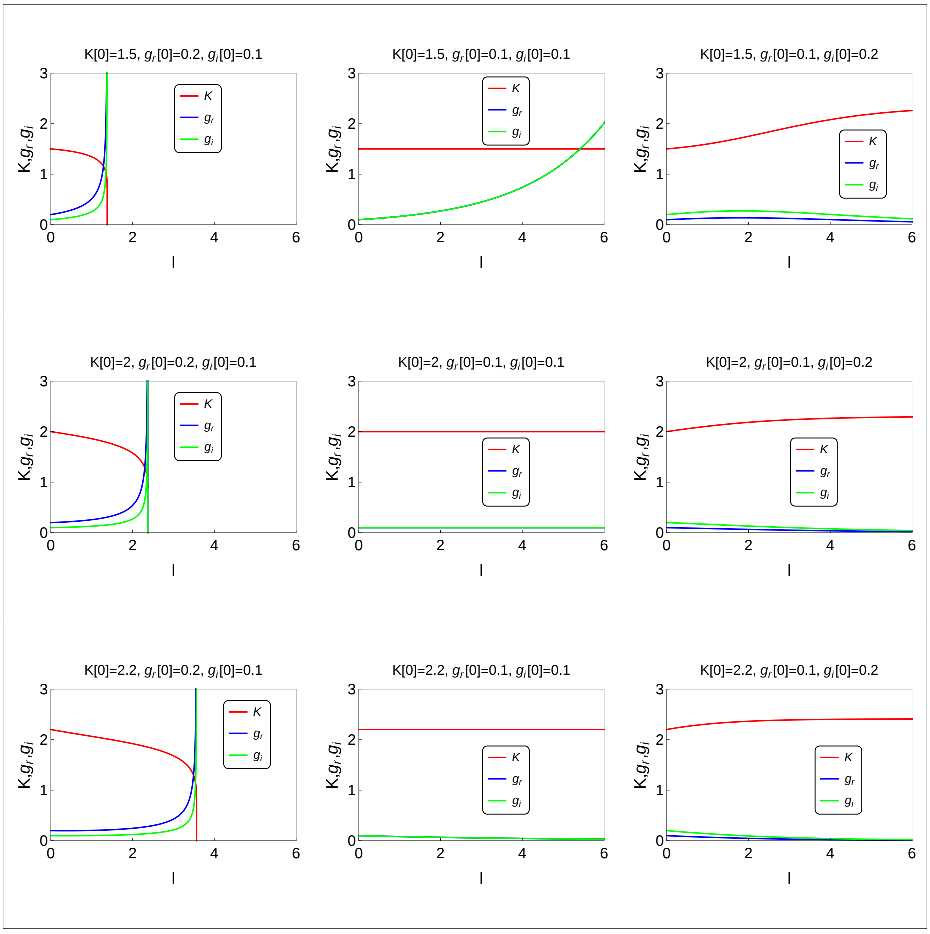

Figure 1. (Color online.) RG flow diagram for the couplings g_r, g_i and K with length scale (l) for third order RG equations (Eq. 5). Three different panels are for the different values of $K[0]$ (1.5, 2 and 2.2 are for the upper, middle and lower panels respectively) and three different figures in each panel for different values of $g_r[0]$ and $g_i[0]$. The initial values of $g_r[0], g_i[0]$ and $K[0]$ are depicted in the figure captions. In this plot we use the magnitude of the imaginary part of the coupling.

3. Derivation of scaling relation for the PT symmetry RG equations

It is well known that the critical theory is invariant under the rescaling. Then the singular part of the free energy density satisfies the following scaling relations,

$$f_s[g_r, g_i] = e^{-2l} f_s[e^{(2-K)l} g_r, e^{(2-K)l} g_i]. \tag{6}$$

The scale l can be fixed from the analytical relation, $e^{(2-K)l^*} g_r = 1$. Finally, after few steps of calculation, we arrive at

$$f_s[g_r, g_i] = {g_r}^{2/(2-K)} f_s[1, {g_r}^{-(2-K)/(2-K)} g_i]. \tag{7}$$

The scaling relation between the real and imaginary part of the sine-Gordon potential is

$${g_r}^{-1} g_i \sim 1. \tag{8}$$

It reveals from this analytical relation that the g_r and g_i are proportional to each other and are independent of K.

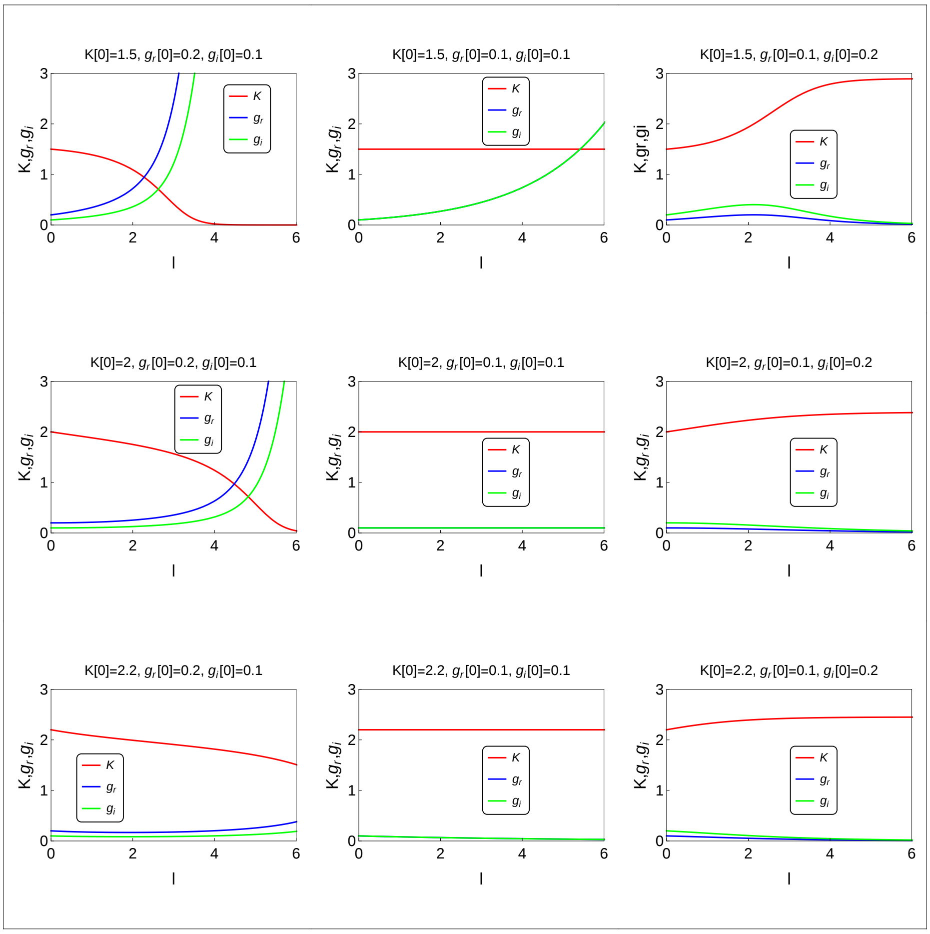

Figure 2. (Color online.) RG flow diagram for the couplings g_r, g_i and K with length scale (l) for second order RG equations (eq. 4). Three different panels are for the different values of $K[0]$ (1.5, 2 and 2.2 are for the upper, middle and lower panel respectively) and three different figures in each panel for different values of $g_r[0]$ and $g_i[0]$. The initial values of $g_r[0], g_i[0]$ and $K[0]$ are depicted in the figure captions. In this plot we use the magnitude of the imaginary part of the coupling.

4. Results along with physical interpretations

4.1. Emergence of quantum criticality: length scale dependent PT symmetry study from the whole set of RG equations

The author of Ref.[34] have solved the double frequencies dual field sine-Gordon Hamiltonian from the perspective of interacting Helical liquid and also for the topological states of interacting quantum matter. The most interesting features of the PT symmetry quantum criticality is that there are two couplings (g_r and g_i) but there is only single field $\phi(x)$. When the PT symmetry is broken, the fluctuation of the $\phi(x)$ facilitate the correlation of conjugate field. Therefore, PT symmetry broken phase generate the physics of sine-Gordon dual field theory for the system. This is the uniqueness of PT symmetry quantum criticality over the dual field double frequencies sine-Gordon field theory.

The authors of Ref.[38] and Ref.[39] have shown that in QCD, the gauge theory of quarks and gluons are asymptotically free, i.e., the coupling vanishes at very short distance (large momentum) and grows at large distance (small momentum). This allowed us to understand why quarks seemed to be free inside the nucleons in deep inelastic scattering and are confined at large distance. However, the present problem is not QCD. Here we solve a sine-Gordon model Hamiltonian with a generalization of PT symmetry. At the same time we are not interpreting our results in terms of quark and gluon physics, rather in terms of the real (g_r) and imaginary (g_i) part of the potential. Therefore, we interpret our results from the length-scale dependent asymptotic freedom like behaviour for this model Hamiltonian system.

Length-scale dependence study brings out the concept of asymptotic freedom like behaviour in the RG flow sense. This asymptotic freedom is a feature of all RG flows with a marginally relevant perturbation. We would like to explain it explicitly through the β function explanation. This can be written as $\beta_\lambda = \frac{d\lambda}{dlnL} = C\lambda^2$ (here β_λ is the β function of the RG theory from where one can predict the nature of the RG flow lines of coupling constant λ; in quantum field theory flow lines are defined in energy scale but here we define RG flow lines in length scale. Here λ is the coupling constant and $C > 0$ is a constant. As a result, the effective coupling is weak at short distances and strong at long distances. We use this behaviour of coupling constant of QCD in our present study.

Recently, asymptotic freedom like behaviour has found in different context of quantum matter system. The author of Ref.[39], has found the asymptotic freedom like behaviour for the topological state of interacting quantum matter. However, for the present study, we are interested to find the asymptotic freedom like behaviour for the PT symmetry system.

Now we present the emergent quantum criticality of PT symmetry physics at different length-scales of the system and also obtain the signature of asymptotic freedom like behaviour.

In Fig.1, we present the results for 3rd order RG equations from the whole set of RG equations. This figure consists of three panels for different initial values of $K[0]$. The upper, middle and lower panels are respectively for $K[0] = 1.5$, 2 and 2.2. Each panel consists of three figures for different values of the coupling $g_r[0]$ and $g_i[0]$. For each panel: left, middle and right figures are respectively for different initial values satisfying the conditions $g_r[0] > g_i[0]$, $g_r[0] = g_i[0]$ (spectral singularity condition) and $g_r[0] < g_i[0]$. It is found from this study that in the left figure for each panel, both the couplings increases with the length scale very sharply while the K decreases. This is the emergence of anomalous phase with the length scale, i.e., both of the couplings increase with in the length scale for the smaller values of K. We observe that as the initial values of $K[0]$ increases (as shown in the middle and lower panel), the variation of the couplings (g_r and g_i) with length scale are not sharp.

The middle figure for each panel is for $g_r[0] = g_i[0]$, i.e., at the spectral singularity point. We observe that K remains constant. For $K[0] = 1.5$ (the first panel), both the couplings increase with the length scale for the lower values of K, i.e., the system is in the anomalous phase region. As the value of K increases for the other panels both the couplings (g_r and g_i) decreases with length scale,i.e., there is no anomalous phase region for this phase. For this regime of parameter space system is in the Luttinger liquid phase.

In the right figures of each panel, K increases with the length scale for the smaller initial values of K otherwise it is saturate. But the RG flow lines for the couplings (g_r and g_i) are flowing off to zero. In this region, system is in the critical phase.

In Fig.2, we present the results of second order RG study for the same regime of parameter space as we did for third order (Fig.1). It reveals from this study , variation of the couplings constants with the length scale is not so sharp as we observed for third order RG. For the higher initial values of $K[0]$, almost there is no variation of K with length scale. The asymptotic freedom like behaviour disappears for the higher values of K. We observe that there is no evidence of asymptotic freedom for the higher values of $K[0](> 2.2)$ for the PT symmetry phase ($g_r[0] > g_i[0]$). Thus it is clear from this study that lower value $K[0]$ and also the higher initial value of $g_r[0]$ from the asymptotic like behaviour. The higher order RG process favours asymptotic freedom like behaviour.

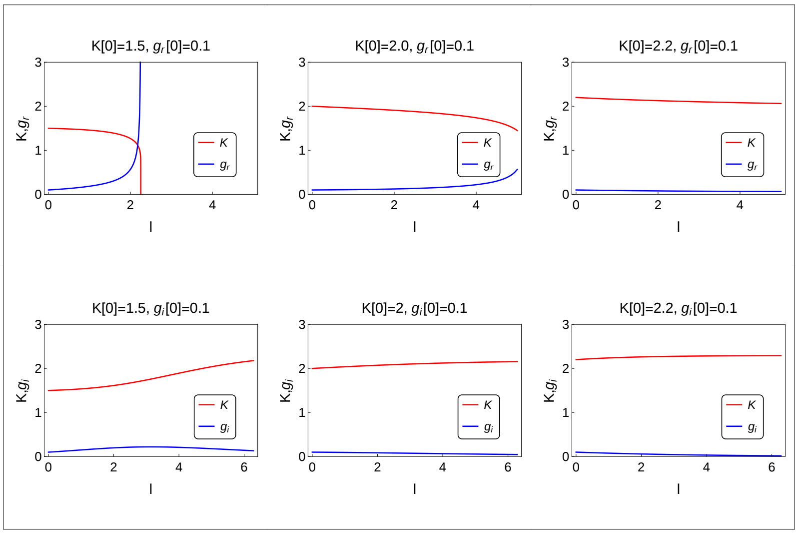

Figure 3. (Color online.) RG flow diagram for the couplings g_r, g_i and K with length scale (l) for the 3rd order RG equations (eq.8 and eq.9). The upper and lower panels are respectively for g_r,K and g_i,K with length scale (l). The initial values of $g_r[0], g_i[0]$ and $K[0]$ are depicted in the figures. In this plot we use the magnitude of the imaginary part of the coupling.

We have presented the scaling relation between the couplings g_r and g_i in Eq.6. We notice that this scaling relation is independent of K. Both of the couplings are proportional to each other, therefore the dependence of g_r and g_i for different initial values of $K[0]$ is not possible from this scaling relation, what we have found from the numerical solution of RG equations. One similarity that we have noticed from this study is that for a particular value $K[0]$ both of the couplings are showing the same behaviour with the length scale.

4.2. Emergence of quantum criticality: length scale dependent PT symmetry study from the conventional quantum Berezinskii-Kosterlitz-Thoulesss RG equations.

Here we present the four sets of conventional QBKT equations from the whole sets of RG equations upto second and third order terms. Second order QBKT RG equations are,

$$\begin{aligned} \frac{dg_r}{dl} &= (2-K)\, g_r \\ \frac{dK}{dl} &= -{g_r}^2 K^2. \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{dg_i}{dl} &= (2-K)\, g_i \\ \frac{dK}{dl} &= {g_i}^2 K^2. \end{aligned} \tag{10}$$

Third order QBKT RG equations are,

$$\begin{aligned} \frac{dg_r}{dl} &= (2-K)\, g_r + 5{g_r}^3 \\ \frac{dK}{dl} &= -{g_r}^2 K^2. \end{aligned} \tag{11}$$

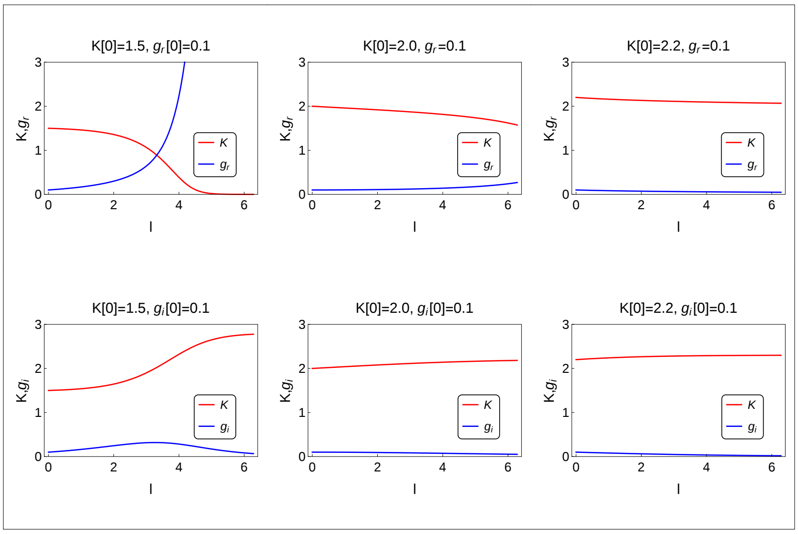

Figure 4. (Color online.) RG flow diagram for the couplings g_r, g_i and K with length scale (l) for the 2nd order RG equations. The upper and lower panels are respectively for g_r,K and g_i,K with length scale (l). The initial values of $g_r[0], g_i[0]$ and $K[0]$ are depicted in the figures. In this plot we use the magnitude of the imaginary part of the coupling.

$$\frac{dg_i}{dl} = (2 - K)\, g_i - 5{g_i}^3$$
$$\frac{dK}{dl} = {g_i}^2 K^2. \tag{12}$$

In Fig.3, we present the length scale dependent variation of couplings (g_r and g_i) for the third order quantum BKT equations. It consists of two panels, the upper and lower panels are respectively for g_r and g_i. Each panel consists of three figures for three different initial values of $K[0]$. The left, middle and the right figures are respectively for $K = 1.5, 2$ and 2.2. It reveals from this study that the K decreases rapidly with the length scale for the lower values of $K[0]$ (< 2) and it almost saturates for the higher initial values of $K[0]$. The coupling constant g_r increases as along as $K[0] < 2$ and decreases for $K[0] > 2$. In the lower panel we observe K increases with the length scale but the RG flow lines for the coupling constant, g_i goes to zero. In Fig.4, we present the length scale dependent variation of couplings in the quantum BKT equation with the second order terms. It consists of two panels where the upper and lower panels are respectively for g_r and g_i. Each panel consists of three figures for three different values of K. The left, middle and the right figures are respectively for $K = 1.5, 2$ and 2.2.
Thus it is clear from the studies of Fig. 3 and Fig. 4 that for only $K = 1.5$, g_r with l shows the asymptotic freedom like behaviour. This findings can be tested experimentally in ultracold atoms.

5. Discussion

We have observed that the presence of this quantum criticality for both the whole set renormalization group equations and also from the conventional quantum BKT equation. We have also presented the scaling equation for these coupling constants. We have observed the asymptotic freedom like behaviour favour for the PT symmetric phase only. We have also presented scaling relation for the coupling constants.

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