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# Conformal bridge transformation and $\mathcal{PT}$ symmetry

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**Abstract.** The conformal bridge transformation (CBT) is reviewed in the light of the  $\mathcal{PT}$  symmetry. Originally, the CBT was presented as a non-unitary transformation (a complex canonical transformation in the classical case) that relates two different forms of dynamics in the sense of Dirac. Namely, it maps the asymptotically free form into the harmonically confined form of dynamics associated with the  $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$  conformal symmetry. However, as the transformation relates the non-Hermitian operator  $i\hat{D}$ , where  $\hat{D}$  is the generator of dilations, with the compact Hermitian generator  $\hat{J}_0$  of the  $\mathfrak{sl}(2, \mathbb{R})$  algebra, the CBT generator can be associated with a  $\mathcal{PT}$ -symmetric metric. In this work we review the applications of this transformation for one- and two-dimensional systems, as well as for systems on a cosmic string background, and for a conformally extended charged particle in the field of Dirac monopole. We also compare and unify the CBT with the Darboux transformation. The latter is used to construct  $\mathcal{PT}$ -symmetric solutions of the equations of the KdV hierarchy with the properties of extreme waves. As a new result, by using a modified CBT we relate the one-dimensional  $\mathcal{PT}$ -regularized asymptotically free conformal mechanics model with the  $\mathcal{PT}$ -regularized version of the de Alfaro, Fubini and Furlan system.

## 1. Introduction

The very fact that the properties of various complex systems can be related with the properties of a free particle and obtained from it in elegant ways is just amazing. A good example of this is the connection between the free particle and the KdV hierarchy, based on the covariance of the Lax representation with respect to Darboux transformations [1]. The stationary Schrödinger equation for a one-dimensional free particle enters the game when the operators of the auxiliary spectral problem in the Lax pair representation are taken with a zero potential identified as a trivial solution of the KdV equation. Then, the iterative application of the Darboux transformation to the linear equations associated with the Lax representation combined with the Darboux dressing of the Lax operator allows ones to generate multi-soliton solutions of the equations of the KdV hierarchy. A Schrödinger system of the auxiliary spectral problem with the obtained multi-soliton potential is reflectionless being almost isospectral to the free particle, and its states are generated from the eigenstates of the free particle Hamiltonian operator by the Darboux transformation [2, 3]. At least some of reflectionless systems are converted by periodization into the finite-gap quantum systems [4, 5, 6], and their potentials can be promoted to the cnoidal type solutions by using, again, the Darboux covariance of the Lax representation [3]. One can introduce soliton defects propagating in a crystalline background with the help of the same Darboux transformations [3]. By the Miura transformation, intimately related to



supersymmetry, one also can relate the free particle system with the modified KdV equation [7]. Using the same methods, one can construct  $\mathcal{PT}$ -regularized Calogero type quantum models with the exotic properties, whose potentials can be transformed into complex  $\mathcal{PT}$ -symmetric solutions of the equations of the KdV hierarchy [8, 9].

On the other hand, an important characteristic of the free particle in arbitrary case of  $d$ -dimensional Euclidean space is that this non-relativistic system is described by the  $\mathfrak{so}(2, 1)$  conformal symmetry. The non-relativistic conformal symmetry occurs naturally in a wide variety of physical phenomena, and attracted recently a lot of attention in the context of non-relativistic AdS/CFT correspondence [10, 11, 12, 13], black hole physics [14, 15, 16], cosmology [17, 18, 19, 20], AdS/CDM correspondence [21, 22, 23, 24] and QCD confinement [25, 26], to name a few. The non-relativistic conformal symmetry of the free particle and its generalizations lie in the base of the so-called conformal bridge transformation (CBT) [27, 28, 29, 30] by which the dynamics and symmetries of an asymptotically free conformally invariant system can be mapped into those of the associated in a certain way harmonically trapped system. This corresponds to the picture described in the Dirac seminal article [31], where different forms of dynamics are studied by choosing, in the general case, a linear combination of the generators of a given symmetry as the Hamiltonian of the system. In the original work [27], it is shown that in one dimension the CBT relates the free particle and the two-body Calogero model with, respectively, the harmonic oscillator and the conformal mechanical model of de Alfaro, Fubini and Furlan (AFF) [32]. In two dimensions, the free particle system can be related with a variety of systems such as the isotropic and anisotropic harmonic oscillators, the Landau problem, and the exotic family of rotationally invariant harmonic oscillators [27, 30]. As the transformation is based on the algebraic arguments, it can be applied to systems in any conformally-invariant space-time and gauge backgrounds. In this way, it was employed to study the dynamics and hidden symmetries in backgrounds of the Dirac monopole [28] and cosmic string [29]. One of the goals of this article is to review how this transformation works and the scope of its applications.

In comparison with the Darboux transformation, which relates almost isospectral one-dimensional systems by means of finite order differential operators, the CBT is generated by a non-local operator, whose realization is not restricted by a space dimension, and it relates the systems with essentially different spectra in a non-trivial way. In fact, it is expected that the possibilities to connect the systems by the CBT expand with increasing the number of dimensions and with the conformally invariant change of the space-time metric. Additionally, some hints on a possible close relationship of the CBT with  $\mathcal{PT}$ -symmetric systems [33, 34, 35, 36] were indicated in the original works [27, 28, 29, 30]. They are based on the fact that at the quantum level, the transformation is realized by a non-unitary operator  $\hat{\mathcal{S}}$  that transforms a non-Hermitian operator  $i\hat{D}$ , where  $\hat{D}$  is a generator of dilations, into the Hermitian compact generator of the  $\mathfrak{sl}(2, \mathbb{R})$  symmetry which has a real discrete spectrum. In this work, we show that the connection between the one-dimensional free particle and the harmonic oscillator corresponds to a particular example of the  $\mathcal{PT}$ -symmetric Swanson models studied in [37, 38, 39, 40]. In this way our Hermitian generator  $\hat{\mathcal{S}}$  of the CBT can be related to a  $\mathcal{PT}$ -symmetric metric operator. Furthermore, since the applications of our CBT touch the systems in the spaces  $\mathbb{R}^d$  with  $d \geq 1$ , and the models in different geometric and gauge backgrounds, new possibilities are opened for connecting  $\mathcal{PT}$ -symmetric systems with models that reveal interesting physical properties such as quantum anomalies, Bose-Einstein condensation, gauge symmetries, etc.

The paper is organized as follows. In Sec. 2 we present the basic properties of the CBT at the classical and quantum levels, and establish its connection with  $\mathcal{PT}$ -symmetric systems. In Sec. 3 we consider the explicit applications of the CBT to one-dimensional systems. As a new result we present the connection between a one-parametric family of the  $\mathcal{PT}$ -regularized perfectly invisible zero-gap Calogero type systems with a  $\mathcal{PT}$ -symmetric version of the AFF conformal mechanics. We use the relation of the former family with the free-particle by means

of Darboux transformations based on a scale-invariant higher-order differential equation to build the complete set of the spectrum-generating ladder operators for the latter system. We also consider there a reinterpretation of the CBT from the point of view of the  $\mathcal{PT}$ -symmetric Swanson model. In Sec. 4 we consider the isotropic and anisotropic CBT in  $d$ -dimensions, as well as the generation of the exotic rotationally invariant harmonic oscillator in two dimensions. In Sec. 5 we study the application of the CBT for systems in two different backgrounds which correspond to the cosmic string and Dirac monopole. In Sec. 6 we show how the Darboux transformation applied appropriately to the  $\mathcal{PT}$ -regularized Calogero type systems allows us to produce complex  $\mathcal{PT}$ -symmetric solutions of the equations of the KdV hierarchy which reveal the properties typical for extreme waves. In Sec. 7 we conclude with discussion of some interesting open problems and further generalizations of the CBT in the light of the  $\mathcal{PT}$  symmetry.

## 2. Conformal bridge transformation

### 2.1. Classical case

Consider the classical  $\mathfrak{so}(2, 1)$  algebra

$$\{D_0, H_0\} = H_0, \quad \{D_0, K_0\} = -K_0, \quad \{K_0, H_0\} = 2D_0, \quad (1)$$

without specifying the concrete form of the generators. Identifying  $H_0$  as a Hamiltonian of a particular classical system, one sees that  $D_0$  and  $K_0$  cannot be true, not depending explicitly on time, integrals of motion. They, however, can easily be promoted to the dynamical, explicitly depending on time integrals of motion in the sense of the evolution equation  $\dot{A} = \{A, H\} + \frac{\partial A}{\partial t} = 0$ ,

$$K_0 \rightarrow K = K(t) = T_{H_0}(t)(K_0), \quad D_0 \rightarrow D = D(t) = T_{H_0}(t)(D_0). \quad (2)$$

Here,  $T_{H_0}(t)$  indicates the Hamiltonian flux in a phase space,

$$\exp(\gamma F) \star f(q, p) := f(q, p) + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \{F, \underbrace{\{\dots, \{F, f\} \dots\}}_n\} := T_F(\gamma)(f), \quad (3)$$

that is a canonical transformation. Obviously,  $T_{H_0}(t)(H_0) = H_0$ , and  $H_0$ ,  $D$  and  $K$  satisfy the same  $\mathfrak{so}(2, 1)$  algebra, in which  $D$  and  $K$  are identified as generators of dilations and special conformal transformations, respectively.

The real and complex linear combinations

$$\mathcal{J}_0 = \frac{1}{2}(\omega^{-1}H_0 + \omega K_0), \quad \mathcal{J}_{\pm} = \mathcal{J}_1 \pm i\mathcal{J}_2 = -\frac{1}{2}(\omega^{-1}H_0 - \omega K_0 \pm 2iD_0), \quad (4)$$

satisfy the classical  $\mathfrak{sl}(2, \mathbb{R})$  algebra,

$$\{\mathcal{J}_0, \mathcal{J}_{\pm}\} = \mp i\mathcal{J}_{\pm}, \quad \{\mathcal{J}_-, \mathcal{J}_+\} = -2i\mathcal{J}_0. \quad (5)$$

A constant  $\omega$  of dimension of frequency is introduced to compensate the dimensions of the generators  $H_0$  and  $K_0$ .  $H_0$  has a nature of a non-compact (parabolic) generator of the  $\mathfrak{so}(2, 1)$  algebra [41, 42], and so, the corresponding classical system can be asymptotically free (like, e.g. a free particle, or conformal mechanics model).  $\mathcal{J}_0$  is a compact (elliptic) generator of  $\mathfrak{sl}(2, \mathbb{R})$ , and represents a harmonically trapped (confined) version of the system  $H_0$ . The quantity  $2\omega\mathcal{J}_0$  can be considered as a Hamiltonian of such a system, which corresponds to another form of dynamics with respect to the same conformal symmetry  $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$ . Then  $\mathcal{J}_{\pm}$  can be promoted to the dynamical integrals of motion for the harmonically trapped system by the analog of the transformation given by Eqs. (2) and (3) with  $H_0$  and  $t$  changed for  $2\omega\mathcal{J}_0$  and  $\tau$ .

The quantities  $2\omega\mathcal{J}_0$  and  $\mathcal{J}_\pm$  generate the Newton-Hooke symmetry of the harmonically trapped system [43, 44, 45, 46, 47].

Consider now the transformation

$$\mathfrak{S} : (H_0, D, K) \rightarrow (-\omega\mathcal{J}_-, -i\mathcal{J}_0, \omega^{-1}\mathcal{J}_+), \quad (6)$$

where we assume that  $\mathcal{J}_\pm$  are the dynamical integrals of motion with respect to the evolution generated by  $2\omega\mathcal{J}_0$ . It is an internal automorphism of the conformal algebra  $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$  generated by the composition of the canonical transformations

$$\mathcal{T}(\tau, \beta, \gamma, \delta, t) = T_{2\omega\mathcal{J}_0}(\tau) \circ T_{\beta\gamma\delta} \circ T_H(-t), \quad (7)$$

where

$$T_{\beta\gamma\delta} := T_{\omega K_0}(\beta) \circ T_{\omega^{-1}H_0}(\tfrac{1}{2}\delta) \circ T_{D_0}(\gamma) = T_{\omega K_0}(\beta) \circ T_{D_0}(\gamma) \circ T_{\omega^{-1}H_0}(\delta), \quad (8)$$

$$\beta = -i, \quad \gamma = -\ln 2, \quad \delta = i. \quad (9)$$

In this composition, the first transformation  $T_H(-t)$  removes dependence on  $t$  in the dynamical integrals  $D$  and  $K$ . The second transformation relates the  $t = 0$  generators with the generators  $\mathcal{J}_0$  and  $\mathcal{J}_\pm$  of the  $\mathfrak{sl}(2, \mathbb{R})$  algebra, taken at  $\tau = 0$ . The last transformation  $T_{2\omega\mathcal{J}_0}(\tau)$  restores the  $\tau$  dependence. The independent of the evolution parameters transformation  $T_{\beta\gamma\delta}$  is equivalent to

$$T_{\beta\gamma\delta} = T_{G_1}(\varepsilon), \quad G_1 := \omega^{-1}H_0 - \omega K_0, \quad \varepsilon = \tfrac{\pi}{4}i. \quad (10)$$

Since the parameters  $\beta$ ,  $\delta$  and  $\varepsilon$  are pure imaginary, this canonical transformation is of an unusual, complex form from the point of view of the conventional classical mechanics. It transforms, particularly, the  $\mathfrak{so}(2, 1)$  hyperbolic [41, 42] generator  $D_0$  multiplied by  $2i\omega$  into the compact real  $\mathfrak{sl}(2, \mathbb{R})$  generator  $\mathcal{J}_0$  multiplied by  $2\omega$ . This picture corresponds to the change of the form of dynamics in the sense of Dirac [31]. Both, the asymptotically free and the harmonically confined, forms of dynamics are associated to the conformal symmetry, and are related one to another by the described classical conformal bridge transformation [27, 29].

One can easily check that for the complex  $\mathfrak{so}(2, 1)$  automorphism (6) the following relation

$$\mathfrak{S}^2 = \mathfrak{S} \circ \mathfrak{S} : (\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2) \rightarrow (-\mathcal{J}_0, \mathcal{J}_1, -\mathcal{J}_2) \quad (11)$$

is valid in terms of the  $\mathfrak{sl}(2, \mathbb{R})$  generators, that is a rotation by  $\pi$  about  $\mathcal{J}_1$ . Therefore, (6) is the fourth order root of the  $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$  identity automorphism,  $\mathfrak{S}^4 = 1$ . In terms of the  $\mathfrak{so}(2, 1)$  generators its eigenelement of eigenvalue 1 is  $G_1 = \omega^{-1}H_0 - \omega K_0$ , see Eq. (10), while linear combinations  $G_{+i} = -2D_0 + \omega^{-1}H_0 + \omega K_0$  and  $G_{-i} = 2D_0 + \omega^{-1}H_0 + \omega K_0$  are eigenelements of eigenvalues  $+i$  and  $-i$ , respectively. There is no eigenelement of the automorphism (6) of eigenvalue  $-1$  to be linear in the  $\mathfrak{so}(2, 1)$  generators. The quadratic eigenelement of eigenvalue  $-1$  is  $a(G_{+i})^2 + b(G_{-i})^2$  with arbitrary coefficients  $a$  and  $b$ . The  $\mathfrak{so}(2, 1)$  classical Casimir  $C = H_0K_0 - D_0^2 = \frac{1}{4}G_{+i}G_{-i}$  is an eigenelement of automorphism  $\mathfrak{S}$  of eigenvalue 1.

Given a particular system described by some symmetry algebra (that can be of a non-linear,  $W$  type), in which conformal symmetry  $\mathfrak{so}(2, 1)$  appears as a subalgebra, we can always apply to this system the classical (and quantum, see below) conformal bridge transformation to relate it to its harmonically confined version. As a result, the integrals of one system can be mapped into integrals of another. Explicit examples of this are discussed in Secs. 4 and 5. Note that in this picture, the classical dynamics of the harmonically trapped system, which is generated by the compact Hamiltonian  $2\omega\mathcal{J}_0$ , corresponds to the Hamiltonian flow generated by the complex quantity  $2i\omega D_0$  in the asymptotically free system described by the Hamiltonian  $H_0$ .

## 2.2. Quantum version

Consider now the quantum  $\mathfrak{so}(2, 1)$  algebra

$$[\hat{D}, \hat{H}_0] = i\hbar \hat{H}_0, \quad [\hat{D}, \hat{K}] = -i\hbar \hat{K}, \quad [\hat{K}, \hat{H}_0] = 2i\hbar \hat{D}. \quad (12)$$

Here

$$\hat{D} = e^{-i\frac{\hat{H}_0}{\hbar}t} \hat{D}_0 e^{i\frac{\hat{H}_0}{\hbar}t}, \quad \hat{K} = e^{-i\frac{\hat{H}_0}{\hbar}t} \hat{K}_0 e^{i\frac{\hat{H}_0}{\hbar}t}, \quad (13)$$

and we just note that acting on a solution of the time-dependent Schrödinger equation  $\Psi(t) = e^{-i\frac{\hat{H}_0}{\hbar}t} \Psi(0)$ , a generic dynamical integral operator  $\hat{A} = e^{-i\frac{\hat{H}_0}{\hbar}t} \hat{A}_0 e^{i\frac{\hat{H}_0}{\hbar}t}$  produces  $\hat{A}(t)\Psi(t) = e^{-i\frac{\hat{H}_0}{\hbar}t} \hat{A}_0 \Psi(0)$ . In this work we consider only stationary eigenstates and linear combinations of them, and for this reason, at the quantum level we suppose  $\hat{D} = \hat{D}_0$  and  $\hat{K} = \hat{K}_0$ , bearing in mind that the time dependence can be reconstructed by application of the corresponding time-evolution operator.

By introducing the rescaled by  $\hbar$  quantum analogs of the linear combinations (4),

$$\hat{\mathcal{J}}_0 = \frac{1}{2\omega\hbar}(\hat{H}_0 + \omega^2 \hat{K}), \quad \hat{\mathcal{J}}_{\pm} = \hat{\mathcal{J}}_1 \pm i\hat{\mathcal{J}}_2 = -\frac{1}{2\omega\hbar}(\hat{H}_0 - \omega^2 \hat{K} \pm 2i\omega \hat{D}), \quad (14)$$

we produce the quantum  $\mathfrak{sl}(2, \mathbb{R})$  algebra

$$[\hat{\mathcal{J}}_0, \hat{\mathcal{J}}_{\pm}] = \pm \hat{\mathcal{J}}_{\pm}, \quad [\hat{\mathcal{J}}_-, \hat{\mathcal{J}}_+] = 2\hat{\mathcal{J}}_0. \quad (15)$$

The quantum conformal bridge transformation (CBT) [27, 29, 30] is a similarity transformation

$$\hat{\mathfrak{S}}(\hat{H}_0)\hat{\mathfrak{S}}^{-1} = -\omega\hbar\hat{\mathcal{J}}_-, \quad \hat{\mathfrak{S}}(i\hat{D})\hat{\mathfrak{S}}^{-1} = \hbar\hat{\mathcal{J}}_0, \quad \hat{\mathfrak{S}}(\hat{K})\hat{\mathfrak{S}}^{-1} = \hbar\omega^{-1}\hat{\mathcal{J}}_+, \quad (16)$$

generated by the non-unitary, Hermitian operator

$$\hat{\mathfrak{S}} = e^{-\frac{\omega}{\hbar}\hat{K}} e^{\frac{i}{\hbar}\ln(2)\hat{D}} e^{\frac{\hat{H}_0}{\hbar\omega}} = \exp\left[\frac{\pi}{4\hbar}\left(\omega^{-1}\hat{H}_0 - \omega\hat{K}\right)\right]. \quad (17)$$

Notice that all the operators  $\hat{H}_0$ ,  $\hat{K}$  and  $i\hat{D}$ , to which the CBT (16) is applied, as well as the CBT generator  $\hat{\mathfrak{S}}$  itself commute with the  $\mathcal{PT}$  operator, i.e. they are  $\mathcal{PT}$ -symmetric<sup>1</sup>.

Relations (16) imply that

$$\hat{D}|\lambda\rangle = i\hbar\lambda|\lambda\rangle \Rightarrow \hat{\mathcal{J}}_0(\hat{\mathfrak{S}}|\lambda\rangle) = \lambda\hat{\mathfrak{S}}|\lambda\rangle, \quad (18)$$

$$\hat{H}_0|E\rangle = E|E\rangle \Rightarrow \hat{\mathcal{J}}_-(\hat{\mathfrak{S}}|E\rangle) = -\frac{E}{\hbar\omega}\hat{\mathfrak{S}}|E\rangle. \quad (19)$$

Then, to have a physical eigenstate  $\hat{\mathfrak{S}}|\lambda\rangle$  of  $\hat{\mathcal{J}}_0$ , the formal state  $|\lambda\rangle$  must obey the following conditions:

- I. The series  $\exp\left(\frac{\hat{H}_0}{2\hbar\omega}\right)|\lambda\rangle = \sum_{n=0}^{\infty} \frac{1}{n!(2\hbar\omega)^n} (\hat{H}_0)^n |\lambda\rangle$  has to reduce to a finite number of terms; this means that  $|\lambda\rangle$  should be a Jordan state of the operator  $\hat{H}_0$  corresponding to zero energy<sup>2</sup>.

<sup>1</sup> Here  $\mathcal{P}$  is a space reflection operator,  $\mathcal{P}x = -x\mathcal{P}$ ,  $\mathcal{P}^2 = 1$ , and a complex conjugation operator  $\mathcal{T}$  is defined by  $\mathcal{T}z = -z^*\mathcal{T}$ ,  $\mathcal{T}^2 = 1$ , where  $z \in \mathbb{C}$  is an arbitrary complex number.

<sup>2</sup> The wave functions of generalized Jordan states corresponding to energy  $\lambda$  satisfy relations of the form  $P(\hat{H})\Omega_\lambda = \psi_\lambda$ , where  $\hat{H}\psi_\lambda = \lambda\psi_\lambda$  and  $P(\eta)$  is a polynomial [48, 49, 50]. Here we consider the Jordan states satisfying the relations  $(\hat{H})^\ell\Omega_\lambda = \lambda\psi_\lambda$  with  $\lambda = 0$  for a certain natural number  $\ell$ .



II. If a wave function  $\Omega_\lambda = \langle x|\lambda \rangle$  is a rank  $n$  Jordan state of  $\hat{H}$  of zero energy,  $\hat{H}^n \Omega_\lambda = 0$ , then  $\Omega_\lambda$ , as well as  $(\hat{H}_0)^k \Omega_\lambda$ ,  $k = 1, \dots, n-1$ , must not have poles and have to be single-valued.

On the other hand, the eigenvectors  $|E\rangle$  (physical, or non-physical, with complex eigenvalues in general case) of  $\hat{H}_0$  are transformed into eigenvectors of the lowering operator  $\hat{\mathcal{J}}_-$  of the  $\mathfrak{sl}(2, \mathbb{R})$  algebra. Therefore, the resulting eigenstates in (19) are the coherent states of  $\mathfrak{sl}(2, \mathbb{R})$  in the sense of Perelomov [51].

The formalism related to the quantum CBT admits a reinterpretation in the context of the  $\mathcal{PT}$  symmetry [37, 39]. Indeed, the second relation in (16) can be written as

$$\hat{\mathcal{S}} \hat{H}_{\mathcal{PT}} = \hat{H}_{HC} \hat{\mathcal{S}}, \quad (20)$$

where  $\hat{H}_{\mathcal{PT}} = 2i\omega\hbar\hat{D}$  is a  $\mathcal{PT}$ -invariant operator, while  $\hat{H}_{HC} = 2\omega\hbar\hat{\mathcal{J}}_0$  is a Hermitian operator. From here we see that the Hermitian operator  $\hat{\mathcal{S}}$ , being the generator of the complex automorphism of the conformal algebra  $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$ , intertwines a non-Hermitian, but  $\mathcal{PT}$ -invariant Hamiltonian with a Hermitian one. By multiplying this relation from the left by  $\hat{\mathcal{S}}$  we obtain  $\hat{\mathcal{S}} \hat{H}_{\mathcal{PT}} = \hat{H}_{\mathcal{PT}}^\dagger \hat{\mathcal{S}}$ , where  $\hat{\mathcal{S}} = \hat{\mathcal{S}}^\dagger$ , and we have taken into account the relation  $\hat{D} = -\hbar\hat{\mathcal{J}}_2$  and the quantum analog of (11). This implies that if  $\hat{H}_{HC}$  represents a well defined quantum system with real eigenvalues and normalizable eigenfunctions, then  $\hat{H}_{\mathcal{PT}}$  has a real spectrum with corresponding eigenstates of finite but not positive definite norm under the indefinite scalar product  $\langle \lambda_1 | \hat{\mathcal{S}} | \lambda_2 \rangle$ . With respect to this inner product, operator  $\hat{H}_{\mathcal{PT}}$  is pseudo-Hermitian [35].

### 3. Applications of CBT to one-dimensional systems

In this section, based on [27], we apply the quantum CBT to one-dimensional systems. For the sake of simplicity, we use here the units  $\hbar = m = 1$ . In each of the examples, we consider the symmetry operators, the eigenstates, and the rank  $n$  Jordan states of zero energy corresponding to an asymptotically free system. By applying the CBT, we get the symmetry operators, the eigenstates and the coherent states of the corresponding harmonically confined models.

In subsection 3.3, we give a reinterpretation of the CBT by comparing its construction with the Swanson  $\mathcal{PT}$ -symmetric system [37, 39] in correspondence with Eq. (20) and related comments there. This will allow us, particularly, to generalize the construction of ref. [37, 39] to the case of the AFF conformal mechanics model.

#### 3.1. Example 1: The free particle - harmonic oscillator relation

Let us start with the one-dimensional free particle symmetry generators

$$\hat{H} = -\frac{1}{2} \frac{d^2}{dx^2}, \quad \hat{D} = -\frac{i}{2} \left[ x \frac{d}{dx} + \frac{1}{2} \right], \quad \hat{K} = \frac{1}{2} x^2, \quad (21)$$

$$\hat{p} = -i \frac{d}{dx}, \quad \hat{x} = x. \quad (22)$$

They produce the one-dimensional Schrödinger symmetry [52]. The eigenstates and eigenvalues of  $\hat{H}$  are

$$\psi_\kappa = e^{i\kappa x}, \quad E = \frac{1}{2} \hbar^2 \kappa^2, \quad \kappa \in \mathbb{R}. \quad (23)$$

The functions  $\langle x|\lambda \rangle$  that satisfy the two conditions specified in Sec. 2.2 correspond to

$$\Omega_n(x) = \langle x|\lambda \rangle = x^n, \quad n = 0, 1, 2, \dots \quad (24)$$

The set of states (24) as a whole is invariant under the action of the symmetry generators (21), (22),

$$\hat{H} \Omega_n = -\frac{1}{2} n(n-1) \Omega_{n-2}, \quad \hat{K} \Omega_n = \frac{1}{2} \Omega_{n+2}, \quad 2i\hat{D} \Omega_n = (n + \frac{1}{2}) \Omega_n, \quad (25)$$

$$\hat{p} \Omega_n = -in \Omega_{n-1}, \quad \hat{x} \Omega_n = \Omega_{n+1}. \quad (26)$$

Via the repeated application of  $\hat{H}$  to  $\Omega_n$ , we arrive at the functions  $\Omega_0 = 1$  (if  $n$  is even) or  $\Omega_1 = x$  (if  $n$  is odd), which are the (physical and non-physical) zero energy solutions of the free particle stationary Schrödinger equation. So,  $\Omega_n(x)$  are the rank  $[n/2] + 1$  Jordan states of zero energy of the free particle,  $(\hat{H})^{[n/2]+1}\Omega_n = 0$ , where  $[.]$  denotes an integer part. We see that  $\Omega_n$  are common formal eigenfunctions of the operators  $2i\hat{D}$  and  $(\hat{H})^{[n/2]+1}$ .

According to Eq. (16), the application of the CBT to the free particle's  $\mathfrak{so}(2, 1)$  generators gives us the  $\mathfrak{sl}(2, \mathbb{R})$  generators of the one-dimensional harmonic oscillator system,

$$\hat{H}_{\text{os}} = 2\omega\hat{J}_0 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2}{2}x^2, \quad \hat{J}_{\pm} = \frac{1}{4\omega} \left[ \frac{d^2}{dx^2} + \omega^2x^2 \mp \omega(x\frac{d}{dx} + \frac{1}{2}) \right]. \quad (27)$$

Along with them, we obtain the Heisenberg generators (ladder operators)

$$\hat{a}^{\pm} = \sqrt{\frac{\omega}{2}} \left( x \mp \frac{1}{\omega} \frac{d}{dx} \right), \quad [\hat{a}^-, \hat{a}^+] = 1, \quad (28)$$

according to the relations  $\hat{\mathfrak{S}}(\hat{p})\hat{\mathfrak{S}}^{-1} = -i\sqrt{\omega}\hat{a}^-$  and  $\hat{\mathfrak{S}}(\hat{x})\hat{\mathfrak{S}}^{-1} = \sqrt{\frac{1}{\omega}}\hat{a}^+$ . From here one deduces that this non-unitary (similarity) transformation can be identified as the fourth order root of the space reflection operator  $\mathcal{P}$ ,

$$\begin{aligned} \hat{\mathfrak{S}} : (x, \hat{p}, \hat{a}^+, \hat{a}^-) &\rightarrow (\hat{a}^+, -i\hat{a}^-, -i\hat{p}, x), & \hat{\mathfrak{S}}^2 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) &\rightarrow (-i\hat{p}, -ix, -\hat{a}^-, \hat{a}^+), \\ \hat{\mathfrak{S}}^4 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) &\rightarrow (-x, -\hat{p}, -\hat{a}^+, -\hat{a}^-). \end{aligned} \quad (29)$$

For the sake of simplicity, we set  $\omega = 1$  in (29). Notice here that the action of the CBT on generators of the Heisenberg algebra corresponds to the eighth order root of the identity automorphism,  $\hat{\mathfrak{S}}^8 = 1$ <sup>3</sup>.

The application of the operator  $\hat{\mathfrak{S}}$  to the functions (24) gives us

$$\hat{\mathfrak{S}}\Omega_n = \mathcal{N}_n\psi_n, \quad \psi_n = \frac{1}{\sqrt{2^n n!}} \left( \frac{\omega}{\pi} \right)^{\frac{1}{4}} H_n(\sqrt{\omega}x) e^{-\frac{\omega x^2}{2}}, \quad \mathcal{N}_n = \frac{(2\pi)^{\frac{1}{4}}}{\omega^{\frac{n}{2} + \frac{1}{4}}} \sqrt{n}, \quad (30)$$

where we have used the Weierstrass transformation [55, 27]. Additionally, by acting from the left by the operator  $\hat{\mathfrak{S}}$  on both sides of Eqs. (25) and (26) one gets the well known relations

$$\hat{H}_{\text{os}}\psi_n = \omega(n + \frac{1}{2})\psi_n, \quad \hat{J}_{\pm}\psi_n = \sqrt{(n \pm \beta_{\pm})(n + \beta_{\pm} \pm 1)}\psi_{n \pm 2}, \quad \beta_{\pm} = \frac{1 \pm 1}{2}, \quad (31)$$

$$\hat{a}^-\psi_n = \sqrt{n}\psi_{n-1}, \quad \hat{a}^+\psi_n = \sqrt{n+1}\psi_{n+1}. \quad (32)$$

Finally, the action of the operator  $\hat{\mathfrak{S}}$  on the free particle eigenstates of the form (23), being simultaneously eigenfunctions of  $\hat{p}$ , produces

$$\phi(x, \kappa) = \hat{\mathfrak{S}} e^{i\frac{\kappa x}{\sqrt{2}}} = 2^{\frac{1}{4}} \exp\left(-\frac{\omega^2}{2}x^2 + \frac{\kappa^2}{4\omega} + i\kappa x\right) = \left(\frac{2\pi}{\omega}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \left(\frac{ik}{\sqrt{2\omega}}\right)^n \frac{\psi_n}{\sqrt{n!}}. \quad (33)$$

These are the coherent states [56] that satisfy the relation  $\hat{a}^-\phi(x, \kappa) = \frac{i\kappa}{\sqrt{2\omega}}\phi(x, \kappa)$ . Under the time evolution these functions take the form

$$\phi(x, \kappa, t) = e^{-i\hat{H}_{\text{os}}t}\phi(x, \kappa) = e^{-i\frac{\omega t}{2}}\phi(x, \kappa e^{-i\omega t}). \quad (34)$$

To have the over-complete set of coherent states of the quantum harmonic oscillator, we allow the parameter  $\kappa$  to take complex values. In the same vein one can show that the free particle

<sup>3</sup> See the comments related to Eq. (35) below, and refs. [53, 54] where the automorphism of the one-dimensional Heisenberg group is discussed in the context of the Stone-von Neumann theorem and the properties of the Fourier transform.



Gaussian wave packets are mapped into the squeezed states of the harmonic oscillator, see ref. [27].

Finally, we note that the described CBT formalism is close with the unitary transformation between the coordinate and the Fock-Bargmann representations. In fact, this last representation can be obtained if we formally replace the spacial variable  $x$  with the complex variable  $z$  in operators (21), (22) as well as in the Jordan states (24), and as an additional step, substitute the usual  $L^2(\mathbb{R})$  scalar product for the inner product

$$(\psi_1, \psi_2) = \frac{1}{\pi} \int_{\mathbb{R}^2} \overline{\psi_1(z)} \psi_2(z) e^{-\bar{z}z} d^2z, \quad d^2z = d(\operatorname{Re} z) d(\operatorname{Im} z). \quad (35)$$

The kernel of the integral transformation, which is a unitary transformation from the  $L^2(\mathbb{R})$  Hilbert space to the Fock-Bargmann space, can be related to the considered CBT, for further details see ref. [27].

### 3.2. Example 2: The one-dimensional Calogero model - AFF model relation

The two-particle Calogero model admits a separation of variables in terms of the relative coordinate and the coordinate of the center of mass, which has a free dynamics. The corresponding  $\mathfrak{so}(2, 1)$  symmetry generators associated with the relative coordinate  $x > 0$  are defined on the positive real half-line  $\mathbb{R}^+$ , and they correspond to

$$\hat{H}_\nu = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{2x^2}, \quad \hat{D} = -\frac{i}{2} \left[ x \frac{d}{dx} + \frac{1}{2} \right], \quad \hat{K} = \frac{1}{2} x^2, \quad (36)$$

where we assume that  $\nu > -\frac{1}{2}$  [50]. The eigenstates and eigenvalues of  $\hat{H}_\nu$ , are given by

$$\psi_{\kappa, \nu}(x) = \sqrt{x} J_{\nu+1/2}(\kappa x), \quad E = \frac{1}{2} \hbar^2 \kappa^2, \quad \kappa > 0, \quad (37)$$

where  $J_\alpha$  are the Bessel functions of the first kind. Besides, the functions  $\langle x | \lambda \rangle = \Omega_{n, \nu}$ , that are well defined on  $\mathbb{R}^+$  and are the zero energy Jordan states of  $\hat{H}_\nu$  correspond to

$$\Omega_{n, \nu} = x^{2n+\nu+1}. \quad (38)$$

These functions satisfy the following set of equations,

$$\hat{H}_\nu \Omega_{n, \nu} = -n(2n + 2\nu + 1) \Omega_{n-1, \nu}, \quad \hat{K} \Omega_{n, \nu} = \frac{1}{2} \Omega_{n+1, \nu}, \quad (39)$$

$$2i\hat{D} \Omega_{n, \nu} = (2n + \nu + \frac{3}{2}) \Omega_{n, \nu}. \quad (40)$$

In the same vein as in the previous subsection, one can see that after the repeated application of  $\hat{H}_\nu$  one gets the zero energy solution  $\Omega_{0, \nu} = x^{\nu+1}$ , which is a regular function on  $\mathbb{R}^+$ .

On the other hand, the conformal symmetry generators of the AFF model [32],

$$\hat{H}_\nu^{AFF} = 2\omega \hat{\mathcal{J}}_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{2x^2} + \frac{\omega^2}{2} x^2, \quad (41)$$

$$\hat{\mathcal{J}}_\pm = \frac{1}{4\omega} \left[ \frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2 \mp \omega \left( x \frac{d}{dx} + \frac{1}{2} \right) \right], \quad (42)$$

are obtained by applying the CBT to generators (36). In the same way, the normalized eigenstates of  $\hat{H}_\nu^{AFF}$  correspond to

$$\hat{\mathcal{G}} \Omega_{n, \nu} = \mathcal{N}_{n, \nu} \psi_{n, \nu}, \quad \psi_{n, \nu} = \sqrt{\frac{2\omega^{\nu+\frac{3}{2}} n!}{\Gamma(n+\nu+\frac{3}{2})}} x^{\nu+1} L_n^{(\nu+\frac{1}{2})}(\omega x^2) e^{-\frac{\omega x^2}{2}}, \quad (43)$$

where  $\mathcal{N}_{n,\nu} = (-1)^n \left(\frac{2}{\omega}\right)^{\frac{\nu}{2}+n} \sqrt{\omega^{-1}n!\Gamma(n+\nu+\frac{3}{2})}$ . They satisfy equations

$$\hat{H}_\nu^{AFF} \psi_{n,\nu} = E_{n,\nu} \psi_{n,\nu}, \quad E_{n,\nu} = \omega(2n + \nu + \frac{3}{2}), \quad (44)$$

$$\hat{\mathcal{J}}_- \psi_{n,\nu} = -\sqrt{n(n+\nu+\frac{1}{2})} \psi_{n-1,\nu}, \quad \hat{\mathcal{J}}_+ \psi_{n,\nu} = -\sqrt{(n+1)(n+\nu+\frac{3}{2})} \psi_{n+1,\nu}, \quad (45)$$

that are obtained directly from the application of  $\hat{\mathfrak{S}}$  to equations (39), (40).

On the other hand, the application of the operator  $\hat{\mathfrak{S}}$  to eigenstates (37) of the system  $\hat{H}_\nu$  yields

$$\hat{\mathfrak{S}} \psi_{\kappa,\nu}(\frac{1}{\sqrt{2}}x) = 2^{\frac{1}{4}} e^{-\frac{1}{2}x^2 + \frac{1}{4}\kappa^2} \sqrt{x} J_{\nu+1/2}(\kappa x) := \phi_\nu(x, \kappa), \quad (46)$$

that are the states satisfying the relation  $\hat{\mathcal{J}}_- \phi_\nu(x, \kappa) = -\frac{1}{4}\kappa^2 \phi_\nu(x, \kappa)$ . By changing the parameter  $\kappa$  for the complex parameter  $z$ , one obtains coherent states that are eigenstates of operator  $\hat{\mathcal{J}}_-$  with complex eigenvalue  $-\frac{1}{4}z^2$ . By using the evolution operator  $\exp(-it\hat{H}_\nu^{AFF})$ , the time-dependent coherent states are obtained,

$$\phi_\nu(x, z, t) = 2^{1/4} \sqrt{x} J_{\nu+1/2}(z(t)x) e^{-x^2/2 + z^2(t)/4 - it}, \quad z(t) = z e^{-it}. \quad (47)$$

### 3.3. Example 3: CBT and $\mathcal{PT}$ -regularized conformal systems

The conformal symmetry operators of the  $\mathcal{PT}$ -regularized Calogero systems, which are defined for  $x \in \mathbb{R}$ , are given by [8, 9]

$$\hat{H}_{\alpha,\nu} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{2(x+i\alpha)^2}, \quad \hat{D}_\alpha = -\frac{i}{2} \left[ (x+i\alpha) \frac{d}{dx} + \frac{1}{2} \right], \quad (48)$$

$$\hat{K}_\alpha = \frac{1}{2} (x+i\alpha)^2. \quad (49)$$

Notice that  $\hat{H}_{\alpha,\nu}$  and  $\hat{K}_\alpha$  are  $\mathcal{PT}$ -symmetric,  $[\mathcal{PT}, \hat{H}_{\alpha,\nu}] = [\mathcal{PT}, \hat{K}_\alpha] = 0$ , while  $\hat{D}_\alpha$  is the  $\mathcal{PT}$ -odd,  $\mathcal{PT}\hat{D}_\alpha = -\hat{D}_\alpha\mathcal{PT}$ , operators. They are obtained by application of the complex translation  $x \rightarrow x + i\alpha$ ,  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 0$ , generated by the Hermitian operator

$$\hat{I}_\alpha = e^{-\alpha\hat{p}} = e^{i\alpha\frac{d}{dx}}, \quad (50)$$

to the generators of the Hermitian Calogero system (36),

$$\hat{I}_\alpha(\hat{H}_{0,\nu})\hat{I}_{-\alpha} = \hat{H}_{\alpha,\nu}, \quad \hat{I}_\alpha(\hat{D}_0)\hat{I}_{-\alpha} = \hat{D}_\alpha, \quad \hat{I}_\alpha(\hat{K}_0)\hat{I}_{-\alpha} = \hat{K}_\alpha, \quad (51)$$

supplemented by extension of the domain for the position variable to the entire real line, that we imply in the rest of this subsection. In the same way, the eigenstates of  $\hat{H}_{\alpha,\nu}$  are formally obtained by application of operator  $\hat{I}_\alpha$  to the states (37), and to their corresponding linearly independent partners, which are given by the Neumann functions  $Y_\nu(\kappa x)$ . On the other hand, the rank  $[n/2] + 1$  Jordan states of zero energy are given by

$$\Omega_{n,\nu}^\alpha = \hat{I}_\alpha \Omega_{n,\nu} = (x+i\alpha)^{2n+\nu+1}, \quad \Xi_{n,\nu}^\alpha = \hat{I}_\alpha \Xi_{n,\nu} = (x+i\alpha)^{2n-\nu}, \quad (52)$$

where  $\Xi_{n,\nu} = x^{2n-\nu}$  is obtained by the transformation  $\rho: \nu \rightarrow -\nu - 1$  (with respect to which generators (36) are invariant) over functions (38) [50]. Relations analogous to (39), (40) for functions  $\Omega_{n,\nu}^\alpha$  are generated by applying the operator  $\hat{I}_\alpha$ . Additional transformation  $\rho$  then yields

$$\hat{H}_{\alpha,\nu} \Xi_{n,\nu}^\alpha = -n(2n-2\nu-1) \Xi_{n-1,\nu}^\alpha, \quad \hat{K}_\alpha \Xi_{n,\nu}^\alpha = \frac{1}{2} \Xi_{n+1,\nu}^\alpha, \quad (53)$$

$$2i\hat{D}_\alpha \Xi_{n,\nu}^\alpha = (2n-\nu+\frac{1}{2}) \Xi_{n,\nu}^\alpha. \quad (54)$$

In the special case  $\nu = m$  with  $m = 1, 2, \dots$ , the system  $\hat{H}_{\alpha,m}$  can be obtained from the free particle system on  $\mathbb{R}$  by the Darboux transformation of the order  $m$  [57, 8]. As a consequence, each such system possesses a hidden symmetry described by the Darboux-dressed generators of the translations and Galilean boosts of the free particle,

$$\hat{\mathcal{P}}_{\alpha,m} = \hat{\mathbb{A}}_{\alpha,m}^- \hat{p} \hat{\mathbb{A}}_{\alpha,m}^+, \quad \hat{\mathcal{X}}_{\alpha,m} = \hat{\mathbb{A}}_{\alpha,m}^- (\hat{x} + i\alpha) \hat{\mathbb{A}}_{\alpha,m}^+, \quad (55)$$

$$\hat{\mathbb{A}}_{\alpha,m}^- = \hat{A}_{\alpha,m}^- \dots \hat{A}_{\alpha,1}^-, \quad \hat{\mathbb{A}}_{\alpha,m}^+ = \hat{A}_{\alpha,1}^+ \dots \hat{A}_{\alpha,m}^+, \quad \hat{A}_{\alpha,m}^\pm = \mp \frac{1}{\sqrt{2}} \left( \frac{d}{dx} \pm \frac{m}{x+i\alpha} \right). \quad (56)$$

The order  $(2m+1)$  differential operator  $\hat{\mathcal{P}}_{\alpha,m}$  is the analog of the Lax-Novikov integral in the quantum reflectionless and finite-gap systems whose potentials are snapshots of the corresponding multi-soliton and cnoidal-type solutions to the KdV equation [7, 3]. Here the operators  $\hat{\mathbb{A}}_{\alpha,m}^\pm$  are the higher order intertwining operators that connect the free particle Hamiltonian  $\hat{H}_{\alpha,0} = \hat{H}$  with the  $\mathcal{PT}$  regularized Calogero Hamiltonian  $\hat{H}_{\alpha,m}$ ,

$$\hat{\mathbb{A}}_{\alpha,m}^- \hat{H}_{\alpha,0} = \hat{H}_{\alpha,m} \hat{\mathbb{A}}_{\alpha,m}^-, \quad \hat{\mathbb{A}}_{\alpha,m}^+ \hat{H}_{\alpha,m} = \hat{H}_{\alpha,0} \hat{\mathbb{A}}_{\alpha,m}^+. \quad (57)$$

The  $\mathcal{PT}$ -symmetric system  $\hat{H}_{\alpha,m}$  is regular on a real line, and due to its relation to the free particle via the intertwining relations (57), it turns out to be perfectly invisible system with a unique  $L^2(\mathbb{R})$  integrable state  $\Xi_{0,m}$  of zero energy at the very edge of the doubly degenerate continuous part of the spectrum. The integral  $\hat{\mathcal{P}}_{\alpha,m}$  separates the states of the same energy in the doubly degenerate continuous part of the spectrum as well as detects the unique bound state of the system  $\hat{H}_{\alpha,m}$  by annihilating it [8, 9].

The commutation relations between the symmetry generators (55) and  $\hat{H}_{\alpha,m}$ ,  $\hat{D}_\alpha$  and  $\hat{K}_\alpha$  produce an extended non-linear algebra, that includes, particularly, the Lie algebraic relations

$$[\hat{H}_{\alpha,m}, \hat{\mathcal{P}}_{\alpha,m}] = 0, \quad [\hat{H}_{\alpha,m}, \hat{\mathcal{X}}_{\alpha,m}] = -i\hat{\mathcal{P}}_{\alpha,m}, \quad (58)$$

$$[\hat{D}_\alpha, \hat{\mathcal{P}}_{\alpha,m}] = \frac{i}{2}(2m+1)\hat{\mathcal{P}}_{\alpha,m}, \quad [\hat{D}_\alpha, \hat{\mathcal{X}}_{\alpha,m}] = \frac{i}{2}(2m-1)\hat{\mathcal{X}}_{\alpha,m}. \quad (59)$$

The action of operators (55) on Jordan states (52) with  $\nu = m$  yields

$$\hat{\mathcal{P}}_{\alpha,m} \Xi_{n,m}^\alpha \propto \Omega_{n-2m-1,m}^\alpha, \quad \hat{\mathcal{X}}_{\alpha,m} \Xi_{n,m}^\alpha \propto \Omega_{n-2m,m}^\alpha, \quad (60)$$

$$\hat{\mathcal{P}}_{\alpha,m} \Omega_{n,m}^\alpha \propto \Xi_{n,m}^\alpha, \quad \hat{\mathcal{X}}_{\alpha,m} \Omega_{n,m}^\alpha \propto \Xi_{n,m}^\alpha, \quad (61)$$

$$\ker \hat{\mathcal{P}}_{\alpha,m} = \text{span}\{\Xi_{0,m}^\alpha, \dots, \Xi_{2m,m}^\alpha\}, \quad \ker \hat{\mathcal{X}}_{\alpha,m} = \text{span}\{\Xi_{0,m}^\alpha, \dots, \Xi_{2m-1,m}^\alpha\}. \quad (62)$$

To relate a non-Hermitian  $\mathcal{PT}$ -symmetric asymptotically free system like  $\hat{H}_{\alpha,m}$  with its confined version, we introduce an extended CBT operator

$$\hat{\mathfrak{S}}_\alpha = \hat{I}_\alpha \hat{\mathfrak{S}}_0 \hat{I}_{-\alpha}, \quad \hat{\mathfrak{S}}_\alpha^{-1} = \hat{I}_\alpha \hat{\mathfrak{S}}_0^{-1} \hat{I}_{-\alpha}, \quad (63)$$

which in this case yields

$$\hat{\mathfrak{S}}_\alpha (\hat{H}_{\alpha,m}) \hat{\mathfrak{S}}_\alpha^{-1} = -\omega \hat{\mathcal{J}}_{-, \alpha}, \quad \hat{\mathfrak{S}}_\alpha (i\hat{D}_\alpha) \hat{\mathfrak{S}}_\alpha^{-1} = \hat{\mathcal{J}}_{0, \alpha}, \quad \hat{\mathfrak{S}}_\alpha (\hat{K}_\alpha) \hat{\mathfrak{S}}_\alpha^{-1} = \frac{1}{\omega} \hat{\mathcal{J}}_{+, \alpha}, \quad (64)$$

$$\hat{\mathcal{J}}_{0, \alpha} = \frac{1}{2\omega} (\hat{H}_\alpha + \omega^2 \hat{K}_\alpha), \quad \hat{\mathcal{J}}_{\pm, \alpha} = -\frac{1}{2\omega} (\hat{H}_\alpha - \omega^2 \hat{K}_\alpha \pm 2i\omega \hat{D}_\alpha). \quad (65)$$

Here we have  $\hat{\mathcal{J}}_{0, \alpha} = \hat{I}_\alpha \hat{\mathcal{J}}_{0,0} \hat{I}_{-\alpha}$  and  $\hat{\mathcal{J}}_{\pm, \alpha} = \hat{I}_\alpha \hat{\mathcal{J}}_{\pm,0} \hat{I}_{-\alpha}$ , and this implies that  $\hat{\mathcal{J}}_{+, \alpha}$  is not the Hermitian conjugation (with respect to the usual inner product on  $\mathbb{R}$ ) of  $\hat{\mathcal{J}}_{-, \alpha}$ . Operators  $(\hat{\mathcal{J}}_{0,0}, \hat{\mathcal{J}}_{\pm,0})$  correspond to the AFF model generators (41), which are singular for  $x \in \mathbb{R}$ .

On the other hand,

$$\hat{H}_{\alpha,\nu}^{AFF} = \hat{I}_\alpha(\hat{H}_{0,\nu}^{AFF})\hat{I}_{-\alpha} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{2(x+i\alpha)^2} + \frac{\omega^2}{2}(x+i\alpha)^2, \quad (66)$$

is the non-Hermitian Hamiltonian of the  $\mathcal{PT}$ -regularized AFF model with arbitrary value of the parameter  $\nu > -1/2$ . Note that for  $\nu = 0$  we recover the  $\mathcal{PT}$ -symmetric one-dimensional harmonic oscillator with  $x$  displaced for imaginary constant  $i\alpha$  being the simplest case of non-Hermitian systems introduced in [33]. The corresponding eigenstates of (66) are

$$\psi_{n,\nu}^\alpha \propto \hat{\mathfrak{S}}_\alpha \Omega_{n,\nu}^\alpha, \quad \phi_{n,\nu}^\alpha \propto \hat{\mathfrak{S}}_\alpha \Xi_{n,\nu}^\alpha, \quad (67)$$

whose explicit form is

$$\psi_{n,\nu}^\alpha = (x+i\alpha)^{\nu+1} e^{-\frac{\omega(x^2-\alpha^2)}{2} + i\omega\alpha x} L_n^{(\nu+\frac{1}{2})}(\omega(x+i\alpha)^2), \quad \phi_{n,\nu}^\alpha = \psi_{n,-\nu-1}^\alpha. \quad (68)$$

These functions satisfy the eigenvalue equations

$$\hat{H}_{\alpha,\nu}^{AFF} \psi_{n,\nu}^\alpha = E_{n,\nu} \psi_{n,\nu}^\alpha, \quad E_{n,\nu} = \omega(2n + \nu + \frac{3}{2}), \quad (69)$$

$$\hat{H}_{\alpha,\nu}^{AFF} \phi_{n,\nu}^\alpha = \mathcal{E}_{n,\nu} \phi_{n,\nu}^\alpha, \quad \mathcal{E}_{n,\nu} = \omega(2n - \nu + \frac{1}{2}), \quad (70)$$

and they are  $L^2(\mathbb{R})$  normalizable, being of the form of a regular polynomial times a Gaussian term, see Figs. 1a and 1b. The notable here is that we have two different towers of states, where the distance between two consecutive energy levels in each tower is given by  $\Delta E = E_{n,\nu} - E_{n-1,\nu} = \mathcal{E}_{n,\nu} - \mathcal{E}_{n-1,\nu} = 2\omega$ . On the other hand,  $\delta E = E_{n,\nu} - \mathcal{E}_{n,\nu} = \omega(2\nu + 1)$ . The last relation means that when  $\nu = \ell - \frac{1}{2}$ ,  $\delta E = \ell \Delta E$ ,  $\ell = 1, 2, \dots$ , one could conclude that there emerges a double degeneracy in the spectrum because of the relation  $E_{s-\ell, \ell-\frac{1}{2}} = \mathcal{E}_{s, \ell-\frac{1}{2}}$  with  $s \geq \ell$ . However, due to the Laguerre polynomial identity

$$\frac{(-\eta)^s}{s!} L_\ell^{(s-\ell)}(\eta) = \frac{(-\eta)^\ell}{\ell!} L_s^{(\ell-s)}(\eta), \quad (71)$$

one can deduce that  $\psi_{s-\ell, \ell-\frac{1}{2}} \propto \phi_{s, \ell-\frac{1}{2}}$ , and so, such a double degeneracy does not really exist. For a similar phenomenon observed earlier in the Darboux transformations of the AFF model see ref. [50]. On the other hand, when  $\nu = m$  one has  $\delta E = \omega(2m + 1)$ . In this case the levels of the tower  $E_{n,m}$  appear in the middle between the levels corresponding to  $\mathcal{E}_{n,m}$ , and the resulting spectrum is divided in two parts. One part corresponds to a semi-infinite equidistant part with energy levels separated by  $\omega$ . In another, finite part, equidistant separation between energy levels is  $\Delta E = 2\omega$ , see Figs. 1c-1e.

The action of the operators  $\hat{\mathcal{J}}_{\alpha,\pm}$  is obtained via the application of  $\hat{\mathfrak{S}}_\alpha$ , that yields

$$\mathcal{J}_{\alpha,\pm} \psi_{n,\nu}^\alpha \propto \psi_{n\pm 1,\nu}^\alpha, \quad \mathcal{J}_{\alpha,\pm} \phi_{n,\nu}^\alpha \propto \phi_{n\pm 1,\nu}^\alpha, \quad \ker \mathcal{J}_{\alpha,-} = \text{span}\{\psi_{0,\nu}^\alpha, \phi_{0,\nu}^\alpha\}. \quad (72)$$

This tells us that the states associated to each tower of energy levels can be produced by the  $\mathfrak{sl}(2, \mathbb{R})$  generators starting from any fixed state, and also shows that there is no way to relate the states from the two towers when  $\nu$  is not integer. In the integer case  $\nu = m$  we have the operators

$$\hat{\mathcal{A}}_{\alpha,m} = \hat{\mathfrak{S}}_\alpha(\hat{\mathcal{P}}_{\alpha,m})\hat{\mathfrak{S}}_\alpha^{-1}, \quad \hat{\mathcal{B}}_{\alpha,m} = \hat{\mathfrak{S}}_\alpha(\hat{\mathcal{X}}_{\alpha,m})\hat{\mathfrak{S}}_\alpha^{-1}, \quad (73)$$

$$[\hat{H}_{\alpha,m}^{AFF}, \hat{\mathcal{A}}_{\alpha,m}] = -\omega(2m+1)\hat{\mathcal{A}}_{\alpha,m}, \quad [\hat{H}_{\alpha,m}^{AFF}, \hat{\mathcal{B}}_{\alpha,m}] = -\omega(2m-1)\hat{\mathcal{B}}_{\alpha,m}. \quad (74)$$

With the help of the CBT, one learns that the operator  $\hat{\mathcal{A}}_{\alpha,m}$  annihilates the states  $\phi_{j,m}$  with  $j = 0, 1, \dots, m$ , while the operator  $\hat{\mathcal{B}}_{\alpha,m}$  annihilates the states  $\phi_{l,m}$  with  $l = 0, 1, \dots, m-1$ .

Among these states we have all the eigenfunctions corresponding to the part separated from the infinite equidistant part of the spectrum. One finds also that they effectively relate the states of one tower with the states of another,

$$\hat{A}_{\alpha,m}\phi_{n,m}^{\alpha} \propto \psi_{n-2m-1,m}^{\alpha}, \quad \hat{B}_{\alpha,m}\phi_{n,m}^{\alpha} \propto \psi_{n-2m,m}^{\alpha}, \quad (75)$$

$$\hat{A}_{\alpha,m}\psi_{n,m}^{\alpha} \propto \phi_{n,m}^{\alpha}, \quad \hat{B}_{\alpha,m}\psi_{n,m}^{\alpha} \propto \phi_{n,m}^{\alpha}. \quad (76)$$

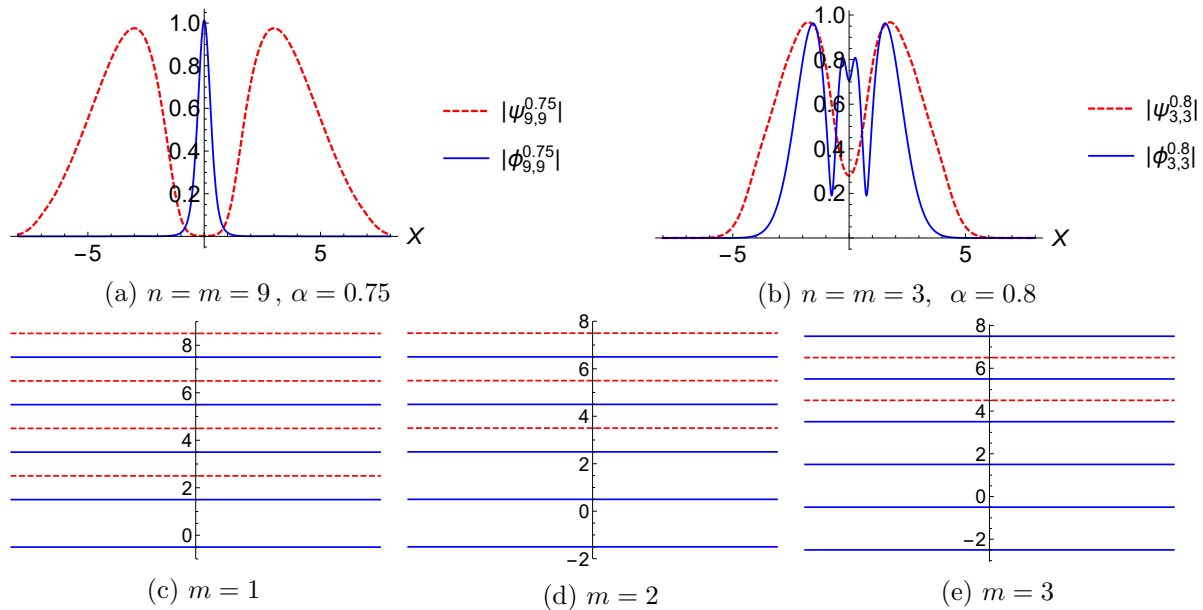


Figure 1: In Figs. 1a and 1b, the modules of the states  $\phi_{n,m}^{\alpha}$  and  $\psi_{n,m}^{\alpha}$  (normalized by their respective maxima) with some values of  $n$ ,  $m$  and  $\alpha$  are compared. On Figs. 1c, 1d and 1e, the energy levels  $\mathcal{E}_{n,m}$  (continuous blue lines) and  $E_{n,m}$  (dashed red lines) are shown for different values of the parameter  $m$ .

Finally, via application of  $\hat{\mathcal{S}}_{\alpha}$  to eigenstates of  $\hat{H}_{\alpha,\nu}$  we get the overcomplete set of coherent states by allowing  $\kappa$  to take complex values,

$$\hat{\mathcal{S}}_{\alpha}\psi_{\kappa,\nu}^{\alpha}(\frac{1}{\sqrt{2}}x) = 2^{\frac{1}{4}}e^{-\frac{1}{2}(x+i\alpha)^2+\frac{1}{4}\kappa^2}\sqrt{x+i\alpha}J_{\nu+1/2}(\kappa(x+i\alpha)) := \Psi_{\nu}^{\alpha}(x,\kappa), \quad (77)$$

$$\hat{\mathcal{S}}_{\alpha}\phi_{\kappa,\nu}^{\alpha}(\frac{1}{\sqrt{2}}x) = 2^{\frac{1}{4}}e^{-\frac{1}{2}(x+i\alpha)^2+\frac{1}{4}\kappa^2}\sqrt{x+i\alpha}Y_{\nu+1/2}(\kappa(x+i\alpha)) := \Phi_{\nu}^{\alpha}(x,\kappa). \quad (78)$$

Considering the issue of the inner product, one could try to take it in the form  $\langle\chi_1|\hat{I}_{-2\alpha}|\chi_2\rangle$ , and then we obtain  $\langle\psi_{n,\nu}^{\alpha}|\hat{I}_{-2\alpha}|\psi_{n,\nu}^{\alpha}\rangle \propto \langle\psi_{n,\nu}^0|\psi_{n,\nu}^0\rangle$ . However the quantity  $\langle\phi_{n,\nu}^{\alpha}|\hat{I}_{-2\alpha}|\phi_{n,\nu}^{\alpha}\rangle$  diverges when  $\nu \neq 0$ , since functions  $\phi_{n,\nu}^0$  are singular in the origin, see Eq. (68). Note, however, here that in the case of  $\mathcal{PT}$ -symmetric harmonic oscillator ( $\nu = 0$ ) we do not have these problems since the states  $\phi_{n,0}^0$  can be written in terms of even Hermite polynomials, which do not have singularities in the real line. So, the interesting open problem is to find a Hermitian system  $\hat{\mathcal{H}}_{\nu}$ , defined on the entire real line, with the same spectrum of  $\hat{H}_{\alpha,\nu}$ , and an operator  $\hat{\mathcal{O}}_{\alpha}$  such that  $\hat{\mathcal{O}}_{\alpha}\hat{\mathcal{H}}_{\nu}\hat{\mathcal{O}}_{\alpha}^{-1} = \hat{H}_{\alpha,\nu}$ . One can expect that such a Hamiltonian will be a non-local operator of the nature similar to that considered in [58], see also [59].

### 3.4. A $\mathcal{PT}$ -symmetric reinterpretation of CBT

According to [37, 39], the  $\mathcal{PT}$  symmetric Hamiltonian operator

$$\mathcal{H}_{\alpha,\beta,\gamma} = \alpha\hat{a}^{+}\hat{a}^{-} + \beta(\hat{a}^{-})^2 + \gamma(\hat{a}^{+})^2, \quad x \in \mathbb{R}, \quad (79)$$

where  $\hat{a}^+$  and  $\hat{a}^-$  are the Hermitian conjugate raising and lowering ladder operators (28), is characterized by a purely real spectrum if the real parameters  $\alpha$ ,  $\beta$  and  $\gamma$  satisfy the relation  $\alpha^2 - 4\beta\gamma \geq 0$ . In the particular case in which  $\alpha = 0$ ,  $\beta = \omega$  and  $\gamma = -\omega$ , this operator takes the form

$$\hat{\mathcal{H}}_{0,\omega,-\omega} = 2i\omega\hat{D}, \quad (80)$$

where  $\hat{D}$  corresponds to the dilatation operator appearing in (21). The eigenstates of this Hamiltonian are the functions  $\Omega_n$  presented in Eq. (24), and the eigenvalue problem corresponds to the third equation in (25). From Eq. (30) one deduces that the CBT generator  $\hat{\mathcal{S}}$  works as the operator that relates the Hamiltonian (80) with the harmonic oscillator system described by the Hermitian Hamiltonian. Indeed, from equations (16) we obtain the  $\mathcal{PT}$ -symmetric conjugation

$$\hat{\mathcal{H}}_{0,\omega,-\omega}\hat{\Theta} = \hat{\mathcal{H}}_{0,\omega,-\omega}^\dagger\hat{\Theta}, \quad \hat{\Theta} = (\hat{\mathcal{S}})^2, \quad (81)$$

implying that the  $\mathcal{PT}$ -symmetric normalization of the eigenstates  $\Omega_n$  is just equivalent to normalization of eigenstates of the quantum harmonic oscillator under the usual inner product in  $\mathbb{R}$ . In the same vein, we note that at the classical level, the time evolution produced by the Hamiltonian  $\mathcal{H}_{0,\omega,-\omega} = 2i\omega D$  in the variables  $x$  and  $p$  is governed by the equations  $\dot{x} = \{x, \mathcal{H}_{0,\omega,-\omega}\} = i\omega x$  and  $\dot{p} = \{p, \mathcal{H}_{0,\omega,-\omega}\} = -i\omega p$ , which resemble the equations of motion of the classical analog  $a^+$  and  $a^-$  of the first order ladder operators of the harmonic oscillator system. Off course, both systems are related to each other by the classical version of the conformal bridge transformation reviewed in Section 2.1.

The model (79) can be generalised up to a concrete realization of the  $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$  generators, by changing the harmonic oscillator operators by

$$\hat{\mathcal{H}}_{\alpha,\beta,\gamma} = \alpha\hat{\mathcal{J}}_0 + \beta\hat{\mathcal{J}}_- + \gamma\hat{\mathcal{J}}_+ = \frac{\alpha-\beta-\gamma}{2\omega}\hat{H} + \frac{\omega(\alpha+\beta+\gamma)}{2}\hat{K} + i(\beta-\gamma)\hat{D}. \quad (82)$$

Relation (81) holds since it is true by the conformal algebraic arguments. This also means that the  $\mathcal{PT}$ -symmetric normalization of the considered physical states of the system, which are also the rank  $[n/2] + 1$  Jordan states of zero energy of  $\hat{H}$ , correspond to the normalization of the eigenstates of the system given by the Hamiltonian  $\hat{\mathcal{J}}_0$ . In this way, if we select the realization (36), where the generators are defined on  $\mathbb{R}^+$ , the physical eigenstates of (82) with  $\alpha = 0$ ,  $\beta = -\gamma$  correspond to (38), and the corresponding eigenvalue equation is given by Eq. (40).

#### 4. CBT for higher-dimensional Euclidean systems

Let us start with realization of the operators  $\hat{H}$ ,  $\hat{D}$  and  $\hat{K}$  in higher-dimensional systems. In the simplest case of a free particle in  $\mathbb{R}^d$ , the generators of its  $\mathfrak{so}(2,1)$  conformal symmetry are given by (in this section we restore the dimensional constants):

$$\hat{H} = \sum_{i=1}^d \hat{H}_i, \quad \hat{D} = \sum_{i=1}^d \hat{D}_i, \quad \hat{K} = \sum_{i=1}^d \hat{K}_i, \quad (83)$$

$$\hat{H}_i = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2}, \quad \hat{D}_i = -i\frac{\hbar}{2} \left( x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right), \quad \hat{K}_i = \frac{m}{2} x_i^2. \quad (84)$$

These symmetry generators are complemented by the Heisenberg algebra generators

$$\hat{p}_j = -i\hbar \frac{\partial}{\partial x_j}, \quad \hat{\xi}_j = mx_j, \quad [\hat{\xi}_j, \hat{\xi}_k] = [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{\xi}_j, \hat{p}_k] = i\hbar m \delta_{jk}, \quad (85)$$

and the angular momentum tensor  $\hat{M}_{ij} = \frac{1}{m}(\hat{\xi}_i \hat{p}_j - \hat{\xi}_j \hat{p}_i)$ . Together, all these generators produce a  $d$ -dimensional Schrödinger symmetry of a free particle [52].

The commutation relations in (85) imply that different conformal bridge transformations can be applied for each spatial direction, and each of them works in the same way as in the one-dimensional case considered in Sec. 3.1.



First, let us consider the isotropic CBT produced by the operator

$$\hat{\mathfrak{S}} = \prod_{i=1}^d \hat{\mathfrak{S}}_i, \quad \hat{\mathfrak{S}}_i = e^{-\frac{\omega}{\hbar} \hat{K}_i} e^{\frac{\hat{H}_i}{2\hbar\omega}} e^{\frac{i}{\hbar} \ln(2) \hat{D}_i}, \quad [\hat{\mathfrak{S}}_i, \hat{\mathfrak{S}}_j] = 0. \quad (86)$$

This operator generates a composed CBT with equal frequencies in each direction. For this reason, it commutes with the angular momentum tensor, and so, is rotationally invariant. Applying the similarity transformation given by (86) to the free particle, we obtain, in accordance with Eqs. (16) and (14), the  $d$ -dimensional isotropic harmonic oscillator, with the  $\mathfrak{sl}(2, \mathbb{R})$  generators  $\hat{\mathcal{J}}_{\pm}$  to be quadratic radial ladder operators. Also one gets

$$\hat{\mathfrak{S}}(\hat{p}_j) \hat{\mathfrak{S}}^{-1} = -i\sqrt{m\hbar\omega} \hat{a}_j^-, \quad \hat{\mathfrak{S}}(\hat{\xi}_j) \hat{\mathfrak{S}}^{-1} = \sqrt{\frac{m\hbar}{\omega}} \hat{a}_j^+, \quad (87)$$

where  $\hat{a}_i^{\pm}$  are the first order ladder operators for each direction,

$$\hat{a}_i^{\pm} = \sqrt{\frac{m\omega}{2\hbar}} \left( x_i \mp \frac{\hbar}{m\omega} \frac{\partial}{\partial x_i} \right), \quad [\hat{a}_i^{\pm}, \hat{a}_j^{\pm}] = 0, \quad [\hat{a}_i^-, \hat{a}_j^+] = \delta_{ij}. \quad (88)$$

The second option is the anisotropic CBT composed from generators with different values of frequencies  $\omega_i > 0$ ,

$$\hat{\mathfrak{S}}_{\omega_1, \dots, \omega_d} = \prod_{i=1}^d \hat{\mathfrak{S}}_{\omega_i}, \quad \hat{\mathfrak{S}}_{\omega_i} = e^{-\frac{\omega_i}{\hbar} \hat{K}_i} e^{\frac{\hat{H}_i}{2\hbar\omega_i}} e^{\frac{i}{\hbar} \ln(2) \hat{D}_i}, \quad [\hat{\mathfrak{S}}_{\omega_i}, \hat{\mathfrak{S}}_{\omega_j}] = 0. \quad (89)$$

Via the similarity transformation, this operator and its inverse transform the linear combination  $2i\hat{\mathcal{D}}_{\omega_1, \dots, \omega_d}^{\epsilon_1, \dots, \epsilon_d} = 2i \sum_{i=1}^d \omega_i \epsilon_i \hat{D}_i$  into the  $d$ -dimensional anisotropic oscillator Hamiltonian  $\hat{H}_{\omega_1, \dots, \omega_d}^{\epsilon_1, \dots, \epsilon_d} = \sum_i^d \epsilon_i \hat{H}_{\text{os}}^{\omega_i}$  with  $\hat{H}_{\text{os}}^{\omega_i} = \hbar\omega_i (\hat{a}_{\omega_i}^+ \hat{a}_{\omega_i}^- + \frac{1}{2})$ , and  $\hat{a}_{\omega_i}^{\pm} = \sqrt{\frac{m\omega_i}{2\hbar}} \left( x_i \mp \frac{\hbar}{m\omega_i} \frac{\partial}{\partial x_i} \right)$ , where each  $\epsilon_i$  can be chosen as 1 or  $-1$ . On the other hand, up to multiplicative constants, one also gets  $\hat{\mathfrak{S}}_{\omega_1, \dots, \omega_d} : (\hat{\xi}_i, \hat{p}_i) \rightarrow (\hat{a}_{\omega_i}^+, \hat{a}_{\omega_i}^-)$ .

Systems in  $d$  Euclidean dimensions, such as the free particle and isotropic or anisotropic harmonic oscillators, can have hidden symmetries generated by higher order integrals of motion [60]. Since these integrals are always written in terms of  $\hat{x}$  and  $\hat{p}$  (or in terms of (88)), the CBT maps the symmetry generators of one system into those of another. In practical terms, application of the CBT to the operators that commute with  $2i\hat{D}$  ( $2i\hat{\mathcal{D}}$ ) produces the Hamiltonian symmetries of the isotropic (anisotropic) case. This scheme allows a reinterpretation from the perspective of  $\mathcal{PT}$  symmetry, in the spirit of Sec. 3.4, since the operators  $2i\hat{D}$  and  $2i\hat{\mathcal{D}}$  are a generalization of the Hamiltonian (80) to  $d$ -dimensions, in the isotropic and anisotropic case, respectively.

To see some concrete applications of the higher dimensional CBT in detail, we will consider only the case  $d = 2$ . In particular, we focus our attention on the exotic rotationally invariant harmonic oscillator (ERIHO) system [30], which corresponds to the planar isotropic harmonic oscillator extended by a Zeeman type term. This model is generated by the isotropic CBT by generalizing the already considered constructions. It represents a one parametric family of systems revealing different phases, two of which correspond to the Landau problem. Additionally, in [30] it also was shown that in spite of the explicit rotationally invariant nature, the model is unitary equivalent to the planar anisotropic harmonic oscillator (AHO) via the application of a certain  $\mathfrak{su}(2)$  rotation accompanied by an anisotropic  $\mathfrak{so}(1, 1)$  Bogolyubov transformation [61].

#### 4.1. The ERIHO system: classical case

Starting from the two-dimensional free particles system, let us consider the following complex combination of its symmetry generators

$$2iD_0 + g p_{\varphi} = x_j \Delta_{jk} p_k \quad g \in \mathbb{R}, \quad (90)$$

$$\Delta_{jk} = i\delta_{jk} + g\epsilon_{jk}, \quad \Delta_{jk} \Delta_{jl} = (g^2 - 1)\delta_{jl}, \quad \det \Delta = g^2 - 1. \quad (91)$$

The generator  $p_\varphi = M_{12} = \epsilon_{ij}x_i p_j$  of  $\mathfrak{so}(2)$  rotations is invariant under the classical isotropic CBT. As a result, (90), multiplied by  $\omega$ , is transformed into the classical Hamiltonian of the ERIHO system,

$$H_g = H_{\text{osc}} + g\omega p_\varphi, \quad H_{\text{osc}} = \frac{1}{2m}p_i p_i + \frac{1}{2}m\omega^2 x_i x_i. \quad (92)$$

System (92) admits the following three different physical interpretations [30].

First,  $H_g$  corresponds to the Hamiltonian of a planar particle in a non-inertial frame rotating with angular velocity  $\Omega = g\omega$  and subjected to the action of the isotropic harmonic trap  $U = \frac{1}{2}kx_i x_i$ . The cases  $k > m\Omega^2$ ,  $k = m\Omega^2$  and  $0 < k < m\Omega^2$  correspond, respectively, to the phases  $0 < g^2 < 1$ ,  $g^2 = 1$  and  $g^2 > 1$  of the system (92), while the inertial case  $\Omega = 0$ ,  $k = m\omega^2$  corresponds to the phase of the isotropic oscillator of (92) with  $g = 0$ .

Second, in the cases  $g = +1$  and  $g = -1$ , (92) takes the form of the Hamiltonian of Landau problem in symmetric gauge with different orientation (sign) of the magnetic field  $B$  and  $\omega = g\omega_B$ ,  $\omega_B = qB/2mc$ , where  $q$  is the charge of a particle. Then the phases with  $0 \leq g^2 < 1$  and  $g^2 > 1$  of (92) correspond to the extended Landau problem in the presence of the additional harmonic potential term  $\frac{1}{2}m\Lambda x_i x_i$  with  $\Lambda > 0$  and  $0 > \Lambda > -\omega_B^2$ , respectively, where  $\omega = \sqrt{\Lambda + \omega_B^2}$ , and  $g = \omega_B/\omega$ . The repulsive critical,  $\Lambda = -m\omega_B^2$ , and supercritical,  $\Lambda < -m\omega^2$ , cases of the extended Landau problem have no analogs in the system (92).

Finally, in terms of the classical analogues of the circular ladder operators,

$$b_1^- = \frac{1}{\sqrt{2}}(a_1^- - ia_2^-), \quad b_1^+ = (b_1^-)^*, \quad b_2^- = \frac{1}{\sqrt{2}}(a_1^- + ia_2^-), \quad b_2^+ = (b_2^-)^*, \quad (93)$$

$$a_i^\pm = \sqrt{\frac{m\omega}{2}} \left( x_i \mp \frac{i}{m\omega} p_i \right), \quad (94)$$

Hamiltonian (92) takes the form

$$H_g = \omega (\ell_1 b_1^+ b_1^- + \ell_2 b_2^+ b_2^-), \quad \ell_1 = 1 + g, \quad \ell_2 = 1 - g. \quad (95)$$

It looks like the anisotropic harmonic oscillator Hamiltonian, but system (92) is manifestly rotational invariant.

In correspondence with relations (91) and the comments on different interpretations, it is expected that the system (92) should have essentially different physical properties and symmetries in the cases  $g^2 < 1$  and  $g^2 > 1$ , as well as when  $g = \pm 1$ . Indeed, the system corresponds to the planar isotropic harmonic oscillator when  $g = 0$ , meanwhile, as it was already mentioned, the model at  $g = \pm 1$  represents the Landau problem in the symmetric gauge. In the case  $|g| < 1$ , the Hamiltonian (92) formally looks like the Euclidean AHO with different frequencies  $\omega_1 \neq \omega_2$ ,  $\omega_i = \ell_i \omega$ , contrary to the case of  $|g| > 1$ , when (92) has instead the form of a Hamiltonian of the Minkowskian AHO with frequencies of two different signs. This last family of systems resembles the Pais-Uhlenbeck oscillator, which recently attracted a considerable attention in relation to the  $\mathcal{PT}$ -symmetry, see Refs. [62, 63, 64]. Finally, in the limit  $g \rightarrow \infty$ , one has

$$g^{-1}H_g \rightarrow \omega p_\varphi = \omega(b_1^+ b_1^- - b_2^+ b_2^-) = b_i^+ \eta_{ij} b_j^-, \quad \eta = \text{diag}(1, -1), \quad (96)$$

which can be interpreted as the isotropic Minkowskian oscillator.

From the point of view of the  $\mathcal{PT}$  symmetry, the generator (90) is a generalization of the classical analogue of the  $\mathcal{PT}$  invariant Hamiltonian of the form (80), extended now by the angular momentum taken with arbitrary coupling constant. Here, the isotropic CBT provides us the transformation that connects this system with its real (Hermitian in the quantum case) counterpart  $H_g$ .

By solving the equations of motion for  $b_j^\pm$ ,  $\dot{b}_j^\pm = \pm i\omega \ell_j b_j^\pm$ ,  $j = 1, 2$ , and using the relation  $\sqrt{m\omega}(x_1 + ix_2) = b_1^+ + b_2^-$ , we get the trajectories of the system,

$$z(t) = x_1(t) + ix_2(t) = R_1 e^{i\gamma_1} e^{i\omega \ell_1 t} + R_2 e^{-i\gamma_2} e^{-i\omega \ell_2 t}, \quad (97)$$

where  $R_i \geq 0$  and  $\gamma_i \in \mathbb{R}$  are the integration constants. The energy and angular momentum of the system are given by  $E_g = m\omega^2(\ell_1 R_1^2 + \ell_2 R_2^2)$ ,  $p_\varphi = m\omega(R_1^2 - R_2^2)$ . Notice that for  $g^2 < 1$  the exponents in (97) evolve in opposite directions, while in the case of  $g^2 > 1$  they change in the same direction that depends on the sign of  $g$ . On the other hand, at  $g = +1$  ( $g = -1$ ), one gets  $\omega_2 = 0$  ( $\omega_1 = 0$ ), and the orbit is a circumference of radius  $R_1$  ( $R_2$ ) centered at  $(X_1, X_2)$  with  $Z = X_1 + iX_2 = R_2 e^{-i\gamma_2}$  ( $Z = R_1 e^{i\gamma_1}$ ).

In general case, the trajectory is closed for arbitrary choice of the integration constants iff the condition  $\ell_1/\ell_2 = q_2/q_1$  with  $q_1, q_2 \in \mathbb{Z}$  is fulfilled. This implies rational values for the parameter  $g = (q_2 - q_1)/(q_1 + q_2)$ . Some trajectories for rational values of  $g$  are shown in Figs. 2 and 3. In the case of Minkowskian isotropic oscillator (96), the trajectories are obtained by applying the transformation  $\omega \rightarrow \omega/|g|$ , and taking the limit  $|g| \rightarrow \infty$  in (97). As a result one gets a circle centered in the origin of the coordinate system,  $z(t) = e^{i\epsilon\omega t}(R_1 e^{i\gamma_1} + R_2 e^{i\gamma_2})$ , where  $\epsilon = \pm 1$  for  $g \rightarrow \pm\infty$  [30].

The closed character of the trajectories for rational values of the parameter  $g$  indicates that some additional true integrals of motion have to appear in the corresponding systems. To obtain them, let us employ the classical CBT.

In the case in which we select  $g$  as the irreducible rational number

$$g_{<}^{s_1, s_2} = (s_2 - s_1)/(s_1 + s_2), \quad s_1, s_2 = 1, 2, \dots, \quad |g_{<}^{s_1, s_2}| < 1, \quad (98)$$

it is easy to see that the phase space functions of the classical free particle

$$S_{s_1, s_2}^+ = (\xi_+)^{s_1} (p_+)^{s_2}, \quad \hat{S}_{s_1, s_2}^- = (p_-)^{s_1} (\xi_-)^{s_2}, \quad (99)$$

where  $p_\pm = p_1 \pm ip_2$  and  $\xi_\pm = \xi_1 \pm i\xi_2$ , Poisson commute with (90). After the application of the classical CBT we get (up to certain constant multiplicative factors)

$$\mathcal{L}_{s_1, s_2}^+ = (b_1^+)^{s_1} (b_2^-)^{s_2}, \quad \mathcal{L}_{s_1, s_2}^- = (\mathcal{L}_{s_1, s_2}^+)^*. \quad (100)$$

These new generators are the true integrals of motion for the system (92). Together with  $p_\varphi$  and  $H_g$ , they generate a non-linear deformation of the  $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  algebra [30], which in the case of  $s_1 = s_2 = 1$ ,  $g = 0$ , reduces to the  $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  Lie algebraic symmetry of the isotropic oscillator. As  $\{\mathcal{L}_{s_1, s_2}^+, \mathcal{L}_{s_1, s_2}^-\}$  is a polynomial of  $H_g$  and  $p_\varphi$  (of order  $s_1 + s_2$ ), effectively (100) provides us with only one new integral independent from  $H_g$  and  $p_\varphi$ .

If instead of (98) we chose now the irreducible fraction

$$g_{>}^{s_1, s_2} = (s_2 + s_1)/(s_2 - s_1), \quad |g_{>}^{s_1, s_2}| > 1, \quad (101)$$

one can show that the polynomials of the free particle Heisenberg generators

$$\Xi_{s_1, s_2}^+ = (\xi_+)^{s_1} (\xi_-)^{s_2}, \quad \Xi_{s_1, s_2}^- = (p_+)^{s_1} (p_-)^{s_2} \quad (102)$$

Poisson commute with (90), and after the application of the conformal bridge transformation, we obtain (up to certain multiplicative constants) the true integrals of motion for our system,

$$\mathcal{J}_{s_1, s_2}^+ = (b_1^+)^{s_1} (b_2^+)^{s_2}, \quad \mathcal{J}_{s_1, s_2}^- = (\mathcal{J}_{s_1, s_2}^+)^*. \quad (103)$$

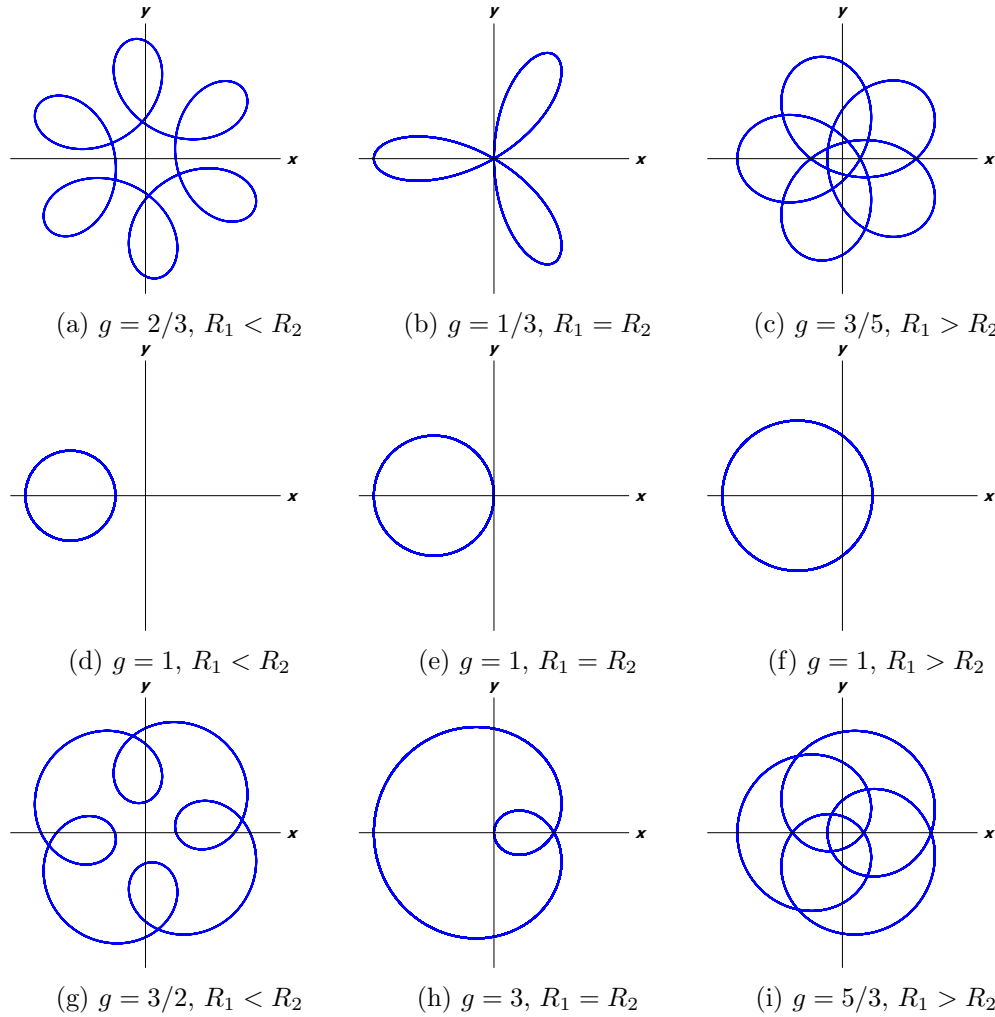


Figure 2: Trajectories for some rational values of  $g$ . In cases b), e) and h),  $p_\varphi = 0$  and trajectories pass through the origin. For  $R_1 \neq R_2$ ,  $\text{sign}(p_\varphi) = \text{sign}(R_1 - R_2)$ .

As in the previous case, these integrals generate a non-linear algebra, which corresponds here to a deformation of the  $\mathfrak{gl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$  algebra [30].

The symmetries of the phases  $g = 1$  and  $g = -1$  of the Landau problem as well as of the isotropic Minkowskian oscillator,  $|g| = \infty$ , can also be reproduced by the described CBT, see ref. [30].

We do not discuss here the complete algebraic structure generated by computing the Poisson brackets between the true integrals and the rest of dynamical symmetries of the system. Nevertheless we notice, that in the case  $g = g_{<}^{s_1, s_2}$  ( $g = g_{>}^{s_1, s_2}$ ) the integrals  $\mathcal{J}_{s_1, s_2}^\pm$  ( $\mathcal{L}_{s_1, s_2}^\pm$ ) are dynamical, and can be generated via the Poisson brackets between the integrals  $\mathcal{L}_{s_1, s_2}^\pm$  ( $\mathcal{J}_{s_1, s_2}^\pm$ ) and the quantities  $\mathcal{L}_{1,1}^\pm := \mathcal{L}_\pm$  ( $\mathcal{J}_{1,1}^\pm := \mathcal{J}_\pm$ ), which also are dynamical integrals of the system. In this process we generate a large but still finite number (depending on the values of the integer parameters  $s_1$  and  $s_2$ ) of dynamical integrals, which together with four true integrals generate a finite non-linear algebra of the  $W$  type. From this point of view we also have a kind of transmutation of symmetries for the “dual” pairs of the systems with  $g = g_{<}^{s_1, s_2}$  and  $g = g_{>}^{s_1, s_2} = 1/g_{<}^{s_1, s_2}$ , where the non-linearly deformed  $\mathfrak{u}(2)$  and  $\mathfrak{gl}(2, \mathbb{R})$  subalgebras generated

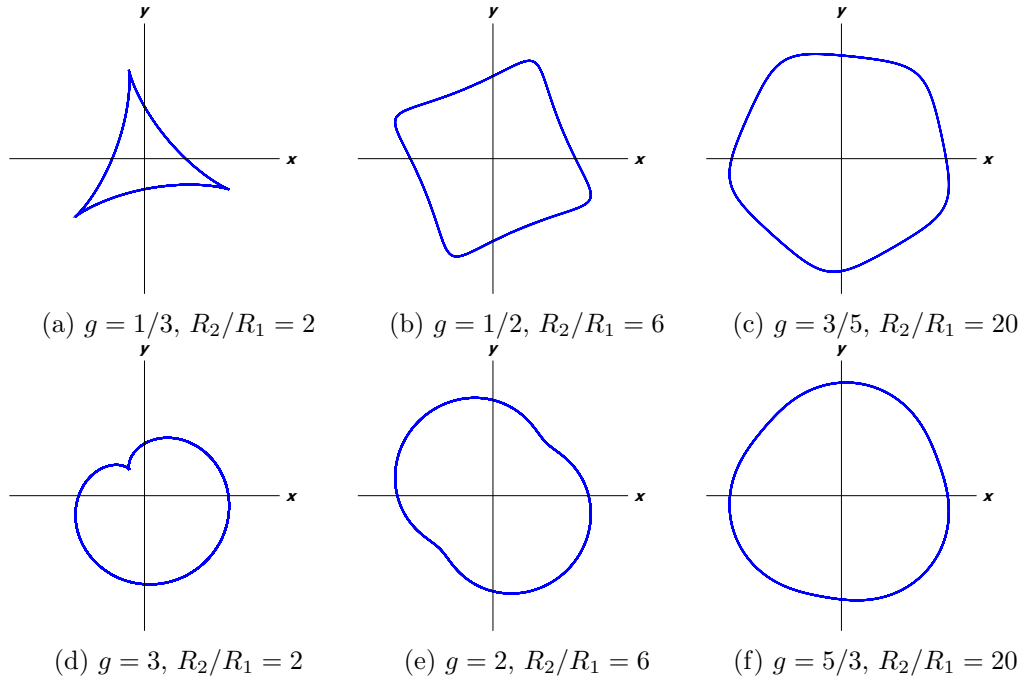


Figure 3: Trajectories for some rational values of  $g$  and  $R_1/R_2$ . The “dual” figures (a) and (d), see below, correspond to a general case  $R_1|\ell_1| = R_2|\ell_2|$  of the trajectories with cusps, in which velocity turns into zero.

by the sets  $(H_g, \mathcal{L}_2, \mathcal{L}_{s_1, s_2}^\pm)$  and  $(H_g, \mathcal{J}_0, \mathcal{J}_{s_1, s_2}^\pm)$  change their role in the sense of the true and dynamical sub-symmetries. For more details see ref. [30].

#### 4.2. Quantum case of the ERIHO system

At the quantum level we have

$$\hat{H}_g = \hat{H}_{\text{osc}} + \omega \hat{p}_\varphi = \hat{\mathfrak{S}} \omega (2i\hat{D} + g\hat{p}_\varphi) \hat{\mathfrak{S}}^{-1} = \hbar \omega (\ell_1 \hat{b}_1^+ \hat{b}_1^- + \ell_2 \hat{b}_2^+ \hat{b}_2^- + 1), \quad (104)$$

$$\ell_1 = 1 + g, \quad \ell_2 = 1 - g, \quad (105)$$

$$\hat{b}_1^- = \frac{1}{\sqrt{2}}(\hat{a}_1^- - i\hat{a}_2^-), \quad \hat{b}_1^+ = (\hat{b}_1^-)^\dagger, \quad \hat{b}_2^- = \frac{1}{\sqrt{2}}(\hat{a}_1^- + i\hat{a}_2^-), \quad \hat{b}_2^+ = (\hat{b}_2^-)^\dagger. \quad (106)$$

Notice here that the operator  $2i\hat{D} + g\hat{p}_\varphi$ , to which we apply the CBT, is  $\mathcal{PT}$  symmetric if  $\mathcal{P}$  is identified as a spatial reflection operator in two dimensions,  $\mathcal{P}x_1 = -x_1\mathcal{P}$ ,  $\mathcal{P}x_2 = x_2\mathcal{P}$ . To obtain the eigenstates and the spectrum of this system, analogously to the procedure described in Sec. 3, we first have to solve the eigenvalue equation

$$(2i\hat{D} + g\hat{p}_\varphi)\phi_\lambda = \hbar \left( (1+g)z \frac{\partial}{\partial z} + (1-g)z^* \frac{\partial}{\partial z^*} \right) \phi_\lambda = \lambda \phi_\lambda, \quad z = x_1 + ix_2. \quad (107)$$

The well defined in  $\mathbb{R}^2$  solutions of this equation correspond to  $\phi_{n_1, n_2} = z^{n_1} (z^*)^{n_2}$ , where  $n_1$  and  $n_2$  are non-negative integers. These are the Jordan states of the two-dimensional free particle that satisfy the equations  $\hat{H}\phi_{n_1, n_2} = -\frac{2\hbar}{m}n_1n_2\phi_{n_1-1, n_2-1}$ ,  $2i\hat{D}\phi_{n_1, n_2} = \hbar(n_1 + n_2 + 1)\phi_{n_1, n_2}$ ,  $\hat{p}_\varphi\phi_{n_1, n_2} = \hbar(n_1 - n_2)\phi_{n_1, n_2}$ , which imply that  $(2i\hat{D} + g\hat{p}_\varphi)\phi_{n_1, n_2} = \hbar(n_1\ell_1 + n_2\ell_2 + 1)\phi_{n_1, n_2}$ . The isotropic two-dimensional CBT produces (up to multiplicative constants) a map

$$\hat{\mathfrak{S}} : (\hat{\xi}_+, \hat{\xi}_-, \hat{p}_+, \hat{p}_-) \rightarrow (\hat{b}_1^+, \hat{b}_2^+, \hat{b}_2^-, \hat{b}_1^-), \quad (108)$$

where  $\hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2$  and  $\hat{\xi}_\pm = \hat{\xi}_1 \pm i\hat{\xi}_2$ , as well the map (16) with  $\hat{H}$ ,  $\hat{D}$  and  $\hat{K}$  given by (83) with  $d = 2$ , while  $\hat{p}_\varphi$  is left invariant. By computing the action of the generators  $(\hat{H}, \hat{D}, \hat{K}, \hat{p}_\varphi, \hat{\xi}_+, \hat{\xi}_-, \hat{p}_+, \hat{p}_-)$  on the states  $\phi_{n_1, n_2}$ , and with the subsequent application of the CBT generator  $\hat{\mathcal{S}}$  from the left, one obtains the equations

$$\hat{b}_1^\pm \Psi_{n_1, n_2} = \sqrt{n_1 + \beta_\pm} \Psi_{n_1 \pm 1, n_2}, \quad \hat{b}_2^\pm \Psi_{n_1, n_2} = \sqrt{n_2 + \beta_\pm} \Psi_{n_1, n_2 \pm 1}, \quad (109)$$

$$\hat{J}_\pm \Psi_{n_1, n_2} \sqrt{(n_1 + \beta_\pm)(n_2 + \beta_\pm)} \Psi_{n_1 \pm 1, n_2 \pm 1}, \quad (110)$$

$$\hat{H}_g \Psi_{n_1, n_2} = E_{n_1, n_2} \Psi_{n_1, n_2}, \quad \hat{p}_\varphi \Psi_{n_1, n_2} = \hbar(n_1 - n_2) \Psi_{n_1, n_2}, \quad (111)$$

$$E_{n_1, n_2} = \hbar\omega(\ell_1 n_1 + \ell_2 n_2 + 1), \quad \beta_\pm = \frac{1 \pm 1}{2}. \quad (112)$$

Here the physical eigenstates  $\Psi_{n_1, n_2}(x_1, x_2)$  are given by

$$\hat{\mathcal{S}}\phi_{n_1, n_2} = \mathcal{N}_{n_1, n_2} \Psi_{n_1, n_2}, \quad \mathcal{N}_{n_1, n_2} = \left(\frac{2\hbar}{m\omega}\right)^{\frac{n_1+n_2}{2}} \sqrt{n_1! n_2! \pi}, \quad (113)$$

$$\Psi_{n_1, n_2} = \sqrt{\frac{m\omega}{\hbar\pi n_1! n_2!}} H_{n_1, n_2} \left(\sqrt{\frac{m\omega}{\hbar}} x_1, \sqrt{\frac{m\omega}{\hbar}} x_2\right) e^{-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)}, \quad (114)$$

where the functions are the generalized Hermite polynomials of two indexes [65].

From equations (109) one deduces that the operators  $\hat{b}_i^\pm$  are the spectrum generating ladder operators of the system for arbitrary values of  $g$ . Eqs. (111) and (112) yield the energy spectrum of the system and the angular momentum value of each stationary state. In dependence on the value of  $g$ , the spectrum has the following properties. It is degenerate iff  $g$  is a rational number, that we assume from now on. The spectrum is positive, has a finite degeneracy, and the ground state is not degenerate when  $|g| < 1$ . In the case  $|g| > 1$ , it is not bounded from below, and has infinite degeneracy in each energy level. Finally, we have the spectrum of the Landau problem when  $|g| = 1$ , see [27].

In the case in which  $g$  is equal to (98) one gets that the integrals

$$\hat{\mathcal{L}}_{s_1, s_2}^+ = (\hat{b}_1^+)^{s_1} (\hat{b}_2^-)^{s_2}, \quad \hat{\mathcal{L}}_{s_1, s_2}^- = (\hat{\mathcal{L}}_{s_1, s_2}^-)^\dagger, \quad (115)$$

which are the direct quantum analogs of  $\mathcal{L}_{s_1, s_2}^\pm$ , act as follows,

$$\hat{\mathcal{L}}_{s_1, s_2}^\pm \Psi_{n_1, n_2} = \sqrt{\frac{\Gamma(n_1 + \beta_\pm s_1 + 1) \Gamma(n_2 + \beta_\mp s_2 + 1)}{\Gamma(n_1 - \beta_\mp s_1 + 1) \Gamma(n_2 - \beta_\pm s_2 + 1)}} \Psi_{n_1 \pm s_1, n_2 \mp s_2}. \quad (116)$$

Besides, when  $g$  corresponds to the case (101), the action of the quantum analogs of the integrals  $\hat{\mathcal{J}}_{s_1, s_2}^\pm$ ,

$$\hat{\mathcal{J}}_{s_1, s_2}^+ = (\hat{b}_1^+)^{s_1} (\hat{b}_2^+)^{s_2}, \quad \hat{\mathcal{J}}_{s_1, s_2}^- = (\hat{\mathcal{J}}_{s_1, s_2}^-)^\dagger, \quad (117)$$

yields

$$\hat{\mathcal{J}}_{s_1, s_2}^\pm \Psi_{n_1, n_2} = \sqrt{\frac{\Gamma(n_1 + \beta_\pm s_1 + 1) \Gamma(n_2 + \beta_\pm s_2 + 1)}{\Gamma(n_1 - \beta_\mp s_1 + 1) \Gamma(n_2 - \beta_\mp s_2 + 1)}} \Psi_{n_1 \pm s_1, n_2 \pm s_2}. \quad (118)$$

All the normalizable eigenfunctions with the same energy can be obtained by repeated application of these operators to some fixed state  $\Psi_{n_1, n_2}$ . When considering the case (98), the action of the integrals  $\hat{\mathcal{L}}_{s_1, s_2}^\pm$  with both upper signs produces a finite list of states. This happens due to obligatorily appearance of the poles in the Gamma function in a denominator of some coefficients. In contrast, when  $g$  is given by Eq. (101), equations (118) imply that the repeated action of  $\hat{\mathcal{J}}_{s_1, s_2}^-$  at some step annihilates a state, but the repeated application of  $\hat{\mathcal{J}}_{s_1, s_2}^+$  will never produces zero. The described properties of the quantum integrals reflect the properties of the spectrum in dependence on the corresponding rational value of  $g$ .



As in the previous section, we can construct the coherent states of the system. The way to obtain them is to apply the CBT operator to the eigenstates of the free particle Hamiltonian. For this, we consider the plane wave  $e^{\frac{1}{\sqrt{2}}(\alpha_1 z + \alpha_2 z^*)}$ , which, in dependence on the values of the parameters  $\alpha_1, \alpha_2 \in \mathbb{C}$  can be a physical or non-physical, formal eigenstate of  $\hat{H}$ . The resulting  $L^2(\mathbb{R}^2)$  integrable functions are eigenstates of operators  $\hat{b}_i^-$  with eigenvalues  $\sqrt{\frac{\hbar}{m\omega}}\alpha_i$ ,  $i = 1, 2$ , and they hold their shape under the time translations and rotations [30].

## 5. CBT in cosmic strings and Dirac monopole backgrounds

Here we discuss applications of CBT with non-trivial realizations of conformal generators in more than one dimension. In the first subsection we consider the relationship between the free particle and the harmonic oscillator on a cosmic string background [29]. In the second subsection we comment on the three-dimensional example in the Dirac monopole background [28]. This second example corresponds to a direct generalization of the relationship between the one-dimensional Calogero type model and the AFF conformal mechanics studied in Sec. 3.2 to the case of three-dimensional spaces.

### 5.1. CBT in a cosmic string background

The metric corresponding to the  $(2+1)$  cosmic string space-time is given by [66, 67]

$$dS^2 = -c^2 dt^2 + ds^2, \quad ds^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) (dr^2 + r^2 d\varphi^2), \quad (119)$$

where  $G$  is Newton constant,  $c$  is the speed of light,  $\mu$  is the linear mass density of the cosmic string and  $r_0$  corresponds to the cosmic string radius. By introducing the new coordinate

$$r'^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) r^2, \quad \alpha^2 dr'^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right) dr^2, \quad \alpha = \frac{1}{1 - \frac{4\mu G}{c^2}} > 0, \quad (120)$$

one gets (renaming  $r' \rightarrow r$ )

$$ds^2 = \alpha^2 dr^2 + r^2 d\varphi^2. \quad (121)$$

When  $\alpha > 1$ , which implies  $\mu > 0$ , metric (121) is obtained from the three-dimensional Euclidean metric reduced to the conic surface  $z = \lambda_E r$ . On the other hand, when  $0 \leq \alpha < 1$ , that means  $\mu < 0$ , metric (121) is obtained by reducing a  $(2+1)$  dimensional Minkowski space metric  $ds^2 = -c^2 d\tau^2 + dr^2 + r^2 d\varphi^2$  to the surface  $c\tau = \lambda r$ ,  $0 < \lambda < 1$ . Such metric also appears in condensed matter systems [68, 69, 70, 71, 72].

The non-relativistic action of a free particle in this space is  $I = \int L dt$ ,  $L = \frac{m}{2} g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} = \frac{m}{2} (\alpha^2 \dot{r}^2 + r^2 \dot{\varphi}^2)$ , and the classical Hamiltonian corresponds to

$$H^{(\alpha)} = \frac{1}{2m} \left( \frac{p_r^2}{\alpha^2} + \frac{p_\varphi^2}{r^2} \right). \quad (122)$$

The formal analogs of the momenta integrals and the Galilean boosts generators are given by

$$\Pi_\pm = \Pi_1 \pm i\Pi_2 = \left( \frac{p_r}{\alpha} \pm i \frac{p_\varphi}{r} \right) e^{\pm i \frac{\varphi}{\alpha}}, \quad (123)$$

$$\Xi_\pm = \Xi_1 \pm i\Xi_2 = \left[ \alpha m r - t \left( \frac{p_r}{\alpha} \pm i \frac{p_\varphi}{r} \right) \right] e^{\pm i \frac{\varphi}{\alpha}}. \quad (124)$$

These are well defined phase space functions only when  $\alpha^{-1}$  is an integer, while in the general case they are multi-valued. Despite this obstacle, we can use these formal conserved quantities to construct the well defined integrals for the system. In the general case of  $\alpha$ , we have the

$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$  generators which are the Hamiltonian  $H^{(\alpha)}$ , the dilatations generator  $D$ , the generator of the special conformal transformations  $K$ , and the generator of rotations  $J_0$ ,

$$H^{(\alpha)} = \frac{1}{2m} \Pi_+ \Pi_- , \quad D = \frac{1}{4m} (\Xi_+ \Pi_- + \Pi_+ \Xi_-) , \quad (125)$$

$$K = \frac{1}{2m} \Xi_+ \Xi_- , \quad J_0 = \frac{i}{4m} (\Xi_+ \Pi_- - \Pi_+ \Xi_-) = \frac{\alpha}{2} p_\varphi . \quad (126)$$

For the case of rational values of  $\alpha = q/k$ , with  $q, k = 1, 2, \dots$ , one can construct

$$\mathcal{O}_{\mu, \nu}^\pm = (\Xi_\pm)^\mu (\Pi_\pm)^\nu , \quad \mu = 0, 1, \dots, q, \quad \nu = q - \mu , \quad (127)$$

$$\mathcal{S}_{\mu', \nu'}^\pm = (\Xi_\pm)^{\mu'} (\Pi_\pm)^{\nu'} , \quad \mu' = 0, 1, \dots, 2q, \quad \nu' = 2q - \mu' . \quad (128)$$

Here, the generators  $\mathcal{O}_{\mu, \nu}^\pm$  ( $\mathcal{S}_{\mu', \nu'}^\pm$ ) have the angular dependence  $e^{\pm i k \varphi}$  ( $e^{\pm i 2 k \varphi}$ ), and therefore, they are well defined phase space functions. The finite sets of generators (127) and (128) are obtained by taking repeated Poisson brackets between  $K$  (or  $H^{(\alpha)}$ ) with  $\mathcal{O}_{0, q}^\pm$  ( $\mathcal{O}_{q, 0}^\pm$ ) and  $\mathcal{S}_{0, 2q}^\pm$  ( $\mathcal{S}_{2q, 0}^\pm$ ) respectively. On the other hand, the brackets  $\{\mathcal{O}_{\mu, \nu}^\pm, \mathcal{O}_{\lambda, \sigma}^\pm\}$  and  $\{\mathcal{S}_{\mu', \nu'}^\pm, \mathcal{S}_{\lambda', \sigma'}^\pm\}$  are polynomial functions of  $m$ ,  $D$ ,  $J_0$ , and  $H^{(\alpha)}$  only. These properties imply that the sets  $\mathcal{U}_1 = \{H^{(\alpha)}, K, D, J_0, \mathcal{O}_{\mu, \nu}^\pm\}$  and  $\mathcal{U}_2 = \{H^{(\alpha)}, K, D, J_0, \mathcal{S}_{\mu', \nu'}^\pm\}$  generate independent non-linear subalgebras. The complete symmetry algebra of the system corresponds to  $\mathcal{U}_1 \cup \mathcal{U}_2$  and also one can show that  $\mathcal{U}_1$  is an ideal subalgebra [29]. For subsequent application of the conformal bridge transformation, it is useful to write down explicitly the brackets

$$\{D, \mathcal{O}_{\mu, \nu}^\pm\} = \frac{\nu - \mu}{2} \mathcal{O}_{\mu, \nu}^\pm , \quad \{D, \mathcal{S}_{\mu', \nu'}^\pm\} = \frac{\nu' - \mu'}{2} \mathcal{S}_{\mu', \nu'}^\pm . \quad (129)$$

From them one sees that in the case  $q = 2n$ , the integrals that Poisson commute with  $D$  correspond to  $(\mathcal{O}_{n, n}^\pm, \mathcal{S}_{2n, 2n}^\pm = (\mathcal{O}_{n, n}^\pm)^2)$ , while in the case  $q = 2n + 1$ , only the integral  $\mathcal{S}_{2n+1, 2n+1}^\pm$  are dilatation invariant.

These properties associated with the parameter  $\alpha$  can be predicted by analyzing the classical trajectories

$$r(\varphi) = \frac{r_*}{\cos((\varphi - \varphi_*)/\alpha)} , \quad r_* = \frac{p_\varphi}{\sqrt{2mH^{(\alpha)}}} , \quad -\frac{\pi}{2}\alpha \leq \varphi - \varphi_* \leq \frac{\pi}{2}\alpha , \quad (130)$$

from where we learn that the scattering angle is  $\varphi_{\text{scat}} = \alpha\pi$ . Some examples of the trajectories are shown on Fig 4.

Though in a free case special values of the parameter  $\alpha$  associated with existence of additional non-trivial integrals of motion reveal themselves in dynamics only in rational values of the scattering angle in comparison with a flat case where  $\varphi_{\text{scat}} = \pi$ , they will explicitly be detected in the dynamics after applying the conformal bridge transformation.

After quantization, the Hamiltonian operator, its eigenstates and its spectrum are given by

$$\hat{H}^{(\alpha)} = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} = -\frac{\hbar^2}{2m} \left( \frac{1}{\alpha^2 r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) , \quad (131)$$

$$\psi_{\kappa, l}^\pm(r, \varphi) = \sqrt{\frac{\kappa}{2\pi\alpha}} J_{\alpha l}(\kappa r) e^{\pm i l \varphi} , \quad E = \frac{\hbar^2 \kappa^2}{2m\alpha^2} , \quad \kappa \geq 0 , \quad l = 0, 1, \dots . \quad (132)$$

The eigenfunctions satisfy  $\langle \psi_{\kappa, l}^\pm | \psi_{\kappa', l'}^\mp \rangle = \delta_{ll'} \delta(\kappa - \kappa')$ , where  $\langle \Psi_1 | \Psi_2 \rangle = \int_V \Psi_1^* \Psi_2 \sqrt{g} dV = \int_0^\infty \alpha r dr \int_0^{2\pi} d\varphi \Psi_1^* \Psi_2$ . The quantum versions of the formal integrals  $\Pi_\pm$  are given by

$$\hat{\Pi}_\pm = e^{\pm \frac{i\varphi}{2\alpha}} \left( \frac{1}{\alpha} \hat{p}_r \pm \frac{i}{r} \hat{p}_\varphi \right) e^{\pm \frac{i\varphi}{2\alpha}} = -i\hbar \frac{1}{\alpha} e^{\pm \frac{i\varphi}{2\alpha}} \left( \frac{\partial}{\partial r} \pm i \frac{\alpha}{r} \frac{\partial}{\partial \varphi} \right) , \quad (133)$$

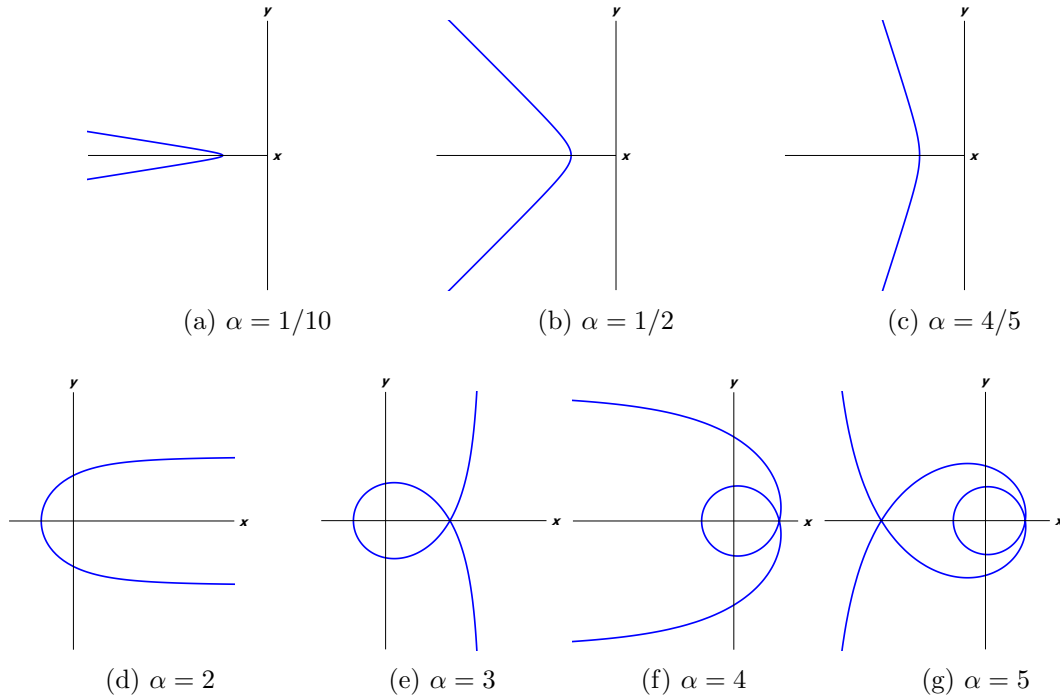


Figure 4: Some examples of the geodesic motion in the conical geometry in coordinates  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ . From the first three figures one sees that for  $0 < \alpha < 1$ , the dynamics resembles that of the repulsive Kepler-Coulomb problem. When  $\alpha > 1$  and is even,  $\alpha = 2n$ , the particle experiences a backscattering. When  $\alpha$  is odd,  $\alpha = 2n + 1$ , the particle approaches the initial direction asymptotically after  $n$  times circling the vertex of the cone.

and from the exponential factors one deduces that the action of these operators on eigenstates produce non-physical solutions in the general case. Explicitly we have

$$\hat{\Pi}_{\pm} \psi_{\kappa, l}^{\pm}(r, \varphi) = i \frac{\hbar \kappa}{\alpha} \sqrt{\frac{\kappa}{2\pi\alpha}} J_{\alpha l+1}(\kappa r) e^{\pm i(l+\frac{1}{\alpha})\varphi}, \quad (134)$$

$$\hat{\Pi}_{\pm} \psi_{\kappa, l}^{\mp}(r, \varphi) = -i \frac{\hbar \kappa}{\alpha} \sqrt{\frac{\kappa}{2\pi\alpha}} J_{\alpha l-1}(\kappa r) e^{\pm i(l-\frac{1}{\alpha})\varphi}. \quad (135)$$

Quantum analogs of the generators (125) and (126) can be constructed straightforward for arbitrary values of  $\alpha$ . But this is not the case for the integrals corresponding to rational values of this geometrical parameter. In fact, with the help of expressions (134) and (135) one can show that the well defined symmetry operators that are the quantum analogs of the integrals (127), (128) can only be constructed for the special case of integer values of  $\alpha = q$ . This reveals a kind of the quantum anomaly in the system, since this is the only case in which the action of the corresponding operators do not produce functions outside the Hilbert space constructed from the eigenstates (132), see [29] for more details.

By applying the classical conformal bridge transformation to generators (125) and (126) we obtain the  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$  generators of the harmonic oscillator in the geometry defined by (121),

$$\mathcal{J}_0 = \frac{1}{2} \mathfrak{b}_a^+ \mathfrak{b}_a^- = \frac{1}{2\omega} H_{\text{os}}^{(\alpha)}, \quad \mathcal{J}_{\pm} = \mathfrak{b}_1^{\pm} \mathfrak{b}_2^{\pm}, \quad \mathcal{L}_2 = \frac{1}{2} (\mathfrak{b}_1^+ \mathfrak{b}_1^- - \mathfrak{b}_2^+ \mathfrak{b}_2^-) = \frac{1}{2} \alpha p_{\varphi}, \quad (136)$$

$$\mathfrak{b}_1^- = \frac{1}{2} e^{i(\omega t - \frac{\varphi}{\alpha})} \left( \alpha \sqrt{m\omega} r + \frac{p_{\varphi}}{\sqrt{m\omega} r} + \frac{ip_r}{\alpha \sqrt{m\omega}} \right), \quad \mathfrak{b}_1^+ = (\mathfrak{b}_1^-)^*, \quad (137)$$

$$\mathfrak{b}_2^- = \frac{1}{2} e^{i(\omega t + \frac{\varphi}{\alpha})} \left( \alpha \sqrt{m\omega} r - \frac{p_{\varphi}}{\sqrt{m\omega} r} + \frac{ip_r}{\alpha \sqrt{m\omega}} \right), \quad \mathfrak{b}_2^+ = (\mathfrak{b}_2^-)^*, \quad (138)$$

where

$$H_{\text{os}}^{(\alpha)} = \frac{1}{2m} \left( \frac{p_r^2}{\alpha^2} + \frac{p_\varphi^2}{r^2} \right) + \frac{m\omega^2 \alpha^2}{2} r^2, \quad (139)$$

is the Hamiltonian of the isotropic harmonic oscillator in a cosmic string background, and the formal dynamical integrals  $\mathbf{b}_i^\pm$  correspond to the mapping

$$\hat{\mathfrak{S}} : (\hat{\Xi}_+, \hat{\Xi}_-, \hat{\Pi}_+, \hat{\Pi}_-) \rightarrow \left( \sqrt{\frac{2m\hbar}{\omega}} \hat{\mathbf{b}}_1^+, \sqrt{\frac{2m\hbar}{\omega}} \hat{\mathbf{b}}_2^+, -i\sqrt{2m\omega\hbar} \hat{\mathbf{b}}_2^-, -i\sqrt{2m\omega\hbar} \hat{\mathbf{b}}_1^- \right). \quad (140)$$

By solving the associated equations of motion of the system one gets

$$r^2(\varphi) = \frac{p_\varphi^2}{mH_{\text{os}}^{(\alpha)}} \left( 1 + \delta \cos\left(\frac{2}{\alpha}(\varphi - \varphi_*)\right) \right)^{-1}, \quad \delta = \sqrt{1 - \left( \frac{\omega \alpha p_\varphi}{H_{\text{os}}^{(\alpha)}} \right)^2}, \quad (141)$$

from where one finds that the closed trajectories are possible only in the rational case  $\alpha = q/k$ , see Fig. 5.

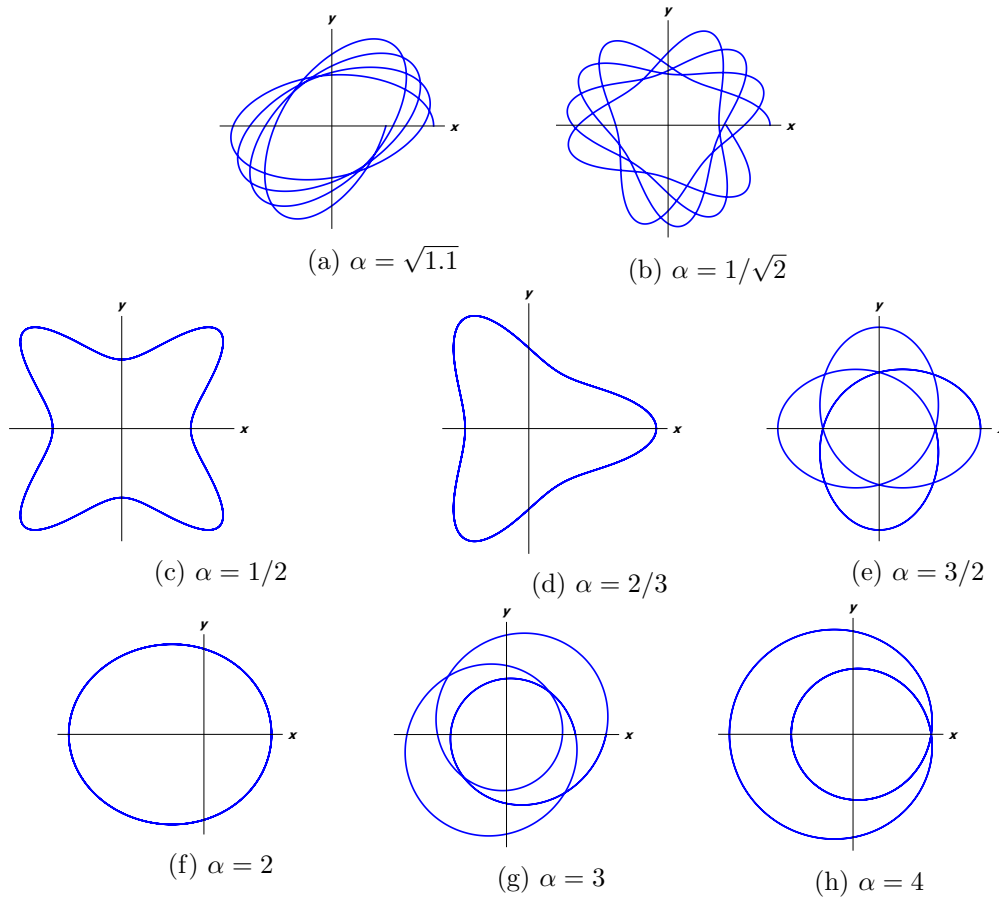


Figure 5: Images of the trajectory for some irrational and rational values of  $\alpha$ . One can show in particular that  $r(\varphi) = r(\varphi + \alpha l\pi)$ ,  $l = 1, 2, \dots$ . From here one deduces that in the case  $\alpha = q/k$  the number of maxima/minima of  $r$  on the orbit is  $\mathcal{N}_{\text{max/min}} = k(q \bmod 2 + 1)$ .

In correspondence with (141), there are globally well defined in the phase space integrals of motion that control the periodic behaviour of the trajectory iff  $\alpha$  is rational. To find these

integrals we use the relations (140) to transform the quantities (127) and (128) (up to inessential multiplicative constant factors) into

$$\mathcal{G}_{\mu,\nu}^+ = (\mathbf{b}_1^+)^{\mu} (\mathbf{b}_2^-)^{\nu}, \quad (\mathcal{G}_{\mu,\nu}^+)^* = \mathcal{G}_{\nu,\mu}^-, \quad \mathcal{F}_{\mu',\nu'}^+ = (\mathbf{b}_1^+)^{\mu'} (\mathbf{b}_2^-)^{\nu'}, \quad (\mathcal{F}_{\mu',\nu'}^+)^* = \mathcal{F}_{\nu',\mu'}^-. \quad (142)$$

Since the classical CBT is a canonical transformation, the algebraic properties of the free particle algebra are inherited by the harmonic oscillator algebra (with  $2i\omega D$  as a pre-image of the harmonic oscillator Hamiltonian). This implies that in the case  $q = 2n$  with  $n = 1, 2, \dots$ , the true integrals of the harmonic oscillator system in the cosmic string background are  $(\mathcal{G}_{n,n}^{\pm}, \mathcal{F}_{2n,2n}^{\pm} = (\mathcal{G}_{n,n}^{\pm})^2)$  and in the case  $q = 2n + 1$ , the true integrals are  $\mathcal{F}_{2n+1,2n+1}^{\pm}$ . This is due to the dilatation invariance of their corresponding pre-images, see (129) and comments below.

At the quantum level, the corresponding Hamiltonian operator, the eigenstates and the spectrum are given by

$$\hat{H}_{\text{os}}^{(\alpha)} = -\frac{\hbar^2}{2m} \left( \frac{1}{\alpha^2 r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi} \right) + \frac{\alpha^2 m \omega^2}{2} r^2, \quad (143)$$

$$\psi_{n_r,l}^{\pm}(r, \varphi) = \left( \frac{m \omega \alpha^2}{\hbar} \right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{2\pi \alpha \Gamma(n_r + \alpha l + 1)}} \zeta^{\alpha l} L_{n_r}^{(\alpha l)}(\zeta^2) e^{-\frac{\zeta^2}{2} \pm i l \varphi}, \quad \zeta = \sqrt{\frac{m \alpha^2 \omega}{\hbar}} r, \quad (144)$$

$$E_{n,l} = \hbar \omega (2n_r + \alpha l + 1), \quad n_r, l = 0, 1, \dots \quad (145)$$

Eigenstates (144) and the spectrum can be obtained by applying the corresponding realization of the operator  $\hat{\mathcal{S}}$  on the zero energy Jordan states of the free particle Hamiltonian (131), which are simultaneously eigenstates of  $2i\omega \hat{D}$ . In this case these Jordan states are given by  $\Omega_{n_r,l}^{\pm}(r, \varphi) = r^{2n_r + \alpha l} e^{\pm i l \varphi}$ . In the same vein, the application of  $\hat{\mathcal{S}}$  to functions (132) gives us the coherent states of the system. Due to this connection one deduces that the quantum anomaly mentioned above for the free system is also present in the harmonically confined one. We refer for the details to [29].

In correspondence with the spectral properties of the confined system, one notes that it acquires a special degeneracy when  $\alpha = q/k$ , however, due to the presence of the quantum anomaly, only in the case  $\alpha = q$  one can construct well defined operators that correctly reflect the degeneracy in the spectrum [29]. When  $\alpha = 2n$  ( $\alpha = 2n + 1$ ), these operators are  $\hat{\mathcal{G}}_{n,n}^{\pm}$  ( $\hat{\mathcal{F}}_{2n+1,2n+1}^{\pm}$ ).

### 5.2. CBT in a Dirac monopole background

The non-trivial three dimensional example we present here corresponds to the dynamics of a particle with electric charge  $e$ , which is coupled to a Dirac magnetic monopole with magnetic charge  $g$  and is subjected to the central potential  $V(r) = \frac{m \omega^2 r^2}{2} + \frac{\alpha}{2mr^2}$ . Parameter  $\alpha$  is a real numerical constant and  $\omega > 0$  is a frequency associated to the harmonic trap. The model and its supersymmetric extensions were extensively studied in [28] and here we just consider the system in relation to CBT.

The Hamiltonian of the system is

$$H_{\omega} = \frac{\pi_r^2}{2m} + \frac{m \omega^2 r^2}{2} + \frac{\alpha}{2mr^2}, \quad \boldsymbol{\pi} = \mathbf{p} - e \mathbf{A}, \quad \nabla \times \mathbf{A} = \mathbf{B} = g \frac{\mathbf{r}}{r^3}. \quad (146)$$

By considering the Poincaré integral of the system

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} - \nu \mathbf{n}, \quad J^2 = \mathbf{J}^2, \quad \mathbf{n} \cdot \mathbf{J} = -\nu, \quad eg = \nu, \quad (147)$$

we note that the Hamiltonian (146) in spherical coordinates admits an ‘AFF model representation’,

$$H_{\omega} = \frac{\pi_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2} + \frac{m \omega^2 r^2}{2}, \quad \mathcal{L}^2 := \mathbf{J}^2 - \nu^2 + \alpha, \quad (148)$$

and by using the fact that  $\{r, \pi_r\} = 1$ , it is deduced that the generators

$$\mathcal{J}_0 = \frac{1}{2\omega^2} H_\omega, \quad \mathcal{J}_\pm = -\frac{1}{2\omega} (H_0 - \omega^2 K_0 \pm i2\omega D_0), \quad (149)$$

produce the  $\mathfrak{sl}(2, \mathbb{R})$  algebra. In these last relations we have introduced the  $\mathfrak{so}(2, 1)$  symmetry generators of the system without the harmonic trap

$$H_0 = \frac{\pi_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2}, \quad D_0 = \frac{1}{2} r \pi_r - H_0 t, \quad K_0 = \frac{mr^2}{2} - Dt - H_0 t^2. \quad (150)$$

Both forms of dynamics are connected by the classical conformal bridge transformation (6). The model  $\hat{H}_0$  and its supersymmetric extensions were studied in details in [73]. It is worth to mention that the Poincaré integral  $\mathbf{J}$  Poisson commutes with all generators (149) and (150), and plays the role of angular momentum of the system.

After solving the trajectory equation for the system (146), one gets  $\mathbf{r} = r(\varphi)\mathbf{n}$ , where

$$r^2(\varphi) = \frac{\mathcal{L}^2}{mH_\omega} [1 - \rho \cos(\frac{2\mathcal{L}}{J} \varphi)]^{-1}, \quad \rho = \sqrt{1 - \frac{\omega^2 \mathcal{L}^2}{H_\omega^2}}, \quad (151)$$

$$\mathbf{n} = \frac{-\nu \mathbf{J}}{J} + \mathbf{n}_\perp(\varphi), \quad \mathbf{n}_\perp(\varphi) = \mathbf{n}_\perp(0) \cos(\varphi) + J^{-1} \mathbf{J} \times \mathbf{n}_\perp \sin(\varphi), \quad (152)$$

and  $\mathbf{n}_\perp \cdot \mathbf{J} = 0$ . After that, the angle  $\varphi = \varphi(t)$  is obtained by substitution of  $r^2(\varphi)$  into the  $\mathfrak{sl}(2, \mathbb{R})$  generators (149). From (152) one concludes that the dynamics occurs on the surface of a dynamical cone defined by the equation  $\mathbf{r} \cdot \mathbf{J} = -\nu r$ . Also from these solutions, with taking into account the definition (148) of  $\mathcal{L}$ , we find that the trajectories are closed for arbitrary values of angular momentum  $J$  only when  $\alpha = \nu^2$ . On the other hand, for  $\alpha \neq \nu^2$ , the trajectories are closed only for special values of  $J$  given by the equation

$$\alpha = \nu^2 + \left( \frac{1}{4} \frac{l_a^2}{l_r^2} - 1 \right) J^2, \quad l_a, l_r = 1, 2, \dots \quad (153)$$

The special properties at  $\alpha = \nu^2$  are expected to be reflected in the presence of the hidden symmetry associated with additional non-trivial integrals of motion in the system [28]. In Fig. 6 examples of the trajectories are shown for some irrational and rational values of the parameter  $\alpha$ .

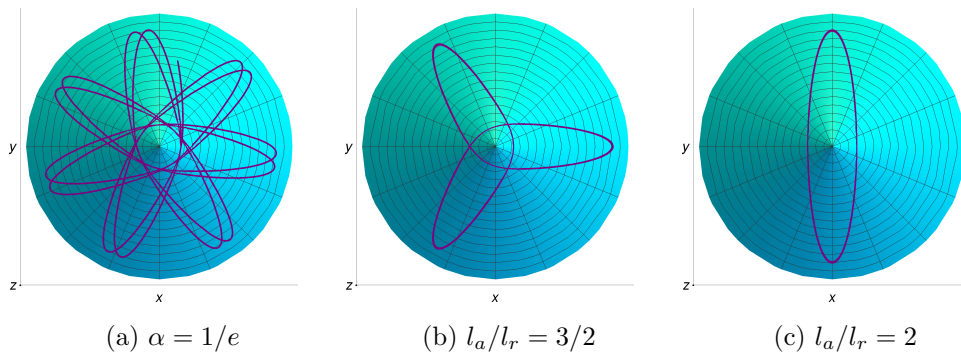


Figure 6: Examples of some non-closed and closed trajectories are shown. The last relation  $l_a/l_r = 2$  corresponds to the special case  $\alpha = \nu^2$ .

To find the already anticipated hidden integrals of the system we employ the classical CBT. For this aim, let us introduce the analogs of the Laplace-Runge-Lenz vector and generator



of the Galilean boost transformations for the asymptotically free system governed by the conformal generators (150), that are only available when  $\alpha = \nu^2$  [73],

$$\mathbf{V} = \boldsymbol{\pi} \times \mathbf{J}, \quad \mathbf{G} = (m\mathbf{r} - \boldsymbol{\pi}t) \times \mathbf{J}. \quad (154)$$

The components of these vector quantities satisfy the relations

$$\{H_0, G_i\} = -V_i, \quad \{K_0, V_i\} = G_i, \quad \{H_0, V_i\} = \{K_0, G_i\} = 0 \quad (155)$$

$$\{D_0, V_i\} = \frac{1}{2}V_i, \quad \{D_0, G_i\} = -\frac{1}{2}G_i, \quad \{D_0, V_i G_j\} = 0, \quad (156)$$

and the classical conformal bridge transformation corresponds to the mapping

$$\mathfrak{S} : (\mathbf{V}, \mathbf{G}) \rightarrow (-i\sqrt{m\omega}\mathbf{a}, \sqrt{\frac{m}{\omega}}\mathbf{a}^*), \quad \mathbf{a} = \sqrt{\frac{m\omega}{2}}(\mathbf{r} + \frac{i}{m\omega}\boldsymbol{\pi}) \times \mathbf{J} e^{i\omega t}, \quad (157)$$

$$\mathfrak{S} : (-i\omega(G_i V_j + G_j V_i), \omega(G_i V_j - G_j V_i)) \rightarrow (T_{(ij)}, T_{[ij]}), \quad (158)$$

$$T_{(ij)} = m\omega(a_i^* a_j + a_j^* a_i), \quad T_{[ij]} = -im\omega(a_i^* a_j - a_j^* a_i). \quad (159)$$

Here,  $T_{(ij)}$  is the symmetric tensor integral of the system (146), being the analog of the Fradkin tensor integral of the three-dimensional isotropic harmonic oscillator [74], while  $T_{[ij]}$  is the anti-symmetric tensor proportional to the Poincaré integral. In terms of  $\mathbf{r}$  and  $\boldsymbol{\pi}$ , the explicit form of the components of these tensors are

$$2T_{(ij)} = (\boldsymbol{\pi} \times \mathbf{J})^i (\boldsymbol{\pi} \times \mathbf{J})^j + m^2 \omega^2 (\mathbf{r} \times \mathbf{J})^i (\mathbf{r} \times \mathbf{J})^j, \quad 2T_{[ij]} = \epsilon_{ijk} m\omega (J^2 - \nu^2) J_k, \quad (160)$$

from where we explicitly see that these are the higher order integrals of motion of the hidden symmetries. It turns out that the complete geometric information on the trajectory that appears in Fig 6c is encoded in the symmetric tensor, see [28].

As in the previous examples, here we can also obtain all the information for the quantum version of the model by applying the quantum CBT to the asymptotically free version without harmonic potential. This time it is necessary to take into account the quantization prescription for the Dirac monopole, where the parameter  $\nu$  has to be an integer or half integer number [75, 76, 77], and the eigenstates are given in terms of the monopole harmonics [78, 79]. Due to the presence of the integral (160), the spectrum, which is discrete and bounded from below, has a special degeneracy depending on the choice of the quantized parameter  $\nu$ . Furthermore, the corresponding spectrum generation operators can be constructed from the quantum version of the complex vectors  $\mathbf{a}$  and  $\mathbf{a}^*$  introduced in (159). For more details, see ref. [28].

## 6. $\mathcal{PT}$ -symmetric systems and extreme waves

The conformal bridge transformation presented in all the previous sections shows how to derive the properties of harmonically confined systems from the associated model whose dynamics is asymptotically free. In this section, with the help of the generalized Darboux transformations [1, 80] we construct reflectionless  $\mathcal{PT}$ -symmetric systems with rational potentials of the type we already considered in section 3.3. Then we promote the obtained stationary potentials to the complex  $\mathcal{PT}$ -symmetric solutions to the KdV equation and higher equations of its hierarchy, whose peculiar evolution reveals the properties typical for extreme waves. We follow here refs. [8, 9], and work in the units  $\hbar = 1$ ,  $m = 1/2$ .

First, we remind that the usual Darboux transformation and its generalizations allow ones to generate from a given one-dimensional quantum system  $\hat{H}_0 = -\frac{d^2}{dx^2} + V(x)$  another system described by the Hamiltonian

$$\hat{H}_{[n]} = -\frac{d^2}{dx^2} + V(x) - 2\frac{d^2}{dx^2} \ln(W(\phi_1, \dots, \phi_n)). \quad (161)$$

Here functions  $(\phi_1, \dots, \phi_n)$  are the so-called seed states, which are physical or formal, non-physical eigenfunctions of  $\hat{H}_0$  with different eigenvalues  $\lambda_j$ , and  $W(\dots)$  is the Wronskian. If the seed states are chosen so that  $W(\phi_1, \dots, \phi_n) \neq 0$  in the domain where the potential  $V(x)$  of the initial system  $\hat{H}_0$  is regular, then the potential of the generated system  $\hat{H}_{[n]}$  will be nonsingular in the same domain. The Darboux transformation ensures that any, physical or non-physical, eigenfunction  $\psi$  of  $\hat{H}_0$  of eigenvalue  $E$  not included in the set of the seed states can be mapped into the corresponding eigenfunction  $\Psi_{[n]}$  of  $\hat{H}_{[n]}$ ,

$$\Psi_{[n]} = \frac{W(\phi_1, \dots, \phi_n, \psi)}{W(\phi_1, \dots, \phi_n)}, \quad \hat{H}_{[n]} \Psi_{[n]} = E \Psi_{[n]}. \quad (162)$$

These relations can be verified by employing the intertwining relations  $\hat{\mathbb{A}}_n^- \hat{H}_0 = \hat{H}_{[n]} \hat{\mathbb{A}}_n^-$  and  $\hat{\mathbb{A}}_n^\dagger \hat{H}_{[n]} = \hat{H}_0 \hat{\mathbb{A}}_n^\dagger$ , where the operator

$$\hat{\mathbb{A}}_n^- = \hat{A}_n^- \dots \hat{A}_1^-, \quad \hat{A}_k^- = \hat{\mathbb{A}}_{k-1} \phi_i \frac{d}{dx} \left( \frac{1}{\hat{\mathbb{A}}_{k-1} \phi_k} \right), \quad \hat{\mathbb{A}}_0^- = 1, \quad (163)$$

is constructed iteratively. With the help of this operator, eigenstate (162) can be presented in the form  $\Psi_{[n]} = \hat{\mathbb{A}}_n^- \psi$ . In particular case in which  $n = 1$ , the function  $W(x) = -(\ln(\phi_1(x)))'$  is called super-potential, and the systems  $\hat{H}_0$  and  $\hat{H}_{[n]}$  can also be presented in the equivalent, up to an additive common shift, form

$$\hat{H}_\pm = -\frac{d^2}{dx^2} + V_\pm, \quad V_\pm = W^2 \pm W'. \quad (164)$$

The confluent Darboux transformation follows the same rules but now Jordan states of  $\hat{H}_0$  can appear in the set of the seed states, see [8, 9, 48, 49, 50]. In Sec. 3.3 we showed how the  $\mathcal{PT}$ -regularized Calogero system with integer coupling constant  $\nu = m$  can be related with the free particle by means of an appropriate Darboux transformation.

It worth to note here a similarity of the non-unitary CBT we described in the previous sections with the Darboux transformations. As in the CBT construction, the system  $\hat{H}_{[n]}$  produced from the initial system  $\hat{H}_0$  in general case is not completely isospectral to it. However, the generated system  $\hat{H}_{[n]}$  inherits some important properties of  $\hat{H}_0$ . This happens, for instance, in the case when  $\hat{H}_0$  corresponds to the free particle, from which reflectionless quantum systems are produced. They represent snapshots of the soliton solutions to the KdV equation. This can be compared with the CBT that transforms an asymptotically free system possessing conformal symmetry into the harmonically trapped system with the same conformal symmetry but realized in another form. The essential difference between the two transformations is that the Darboux transformation, being generated by an operator of finite differential order, is local. The generator of the CBT  $\hat{\mathbb{S}}$  is, however, essentially non-local since it includes in its structure the exponent of the second order differential Hamiltonian operator of the initial conformally invariant system. In the next, concluding section, we will return to some aspect of similarity between the CBT, based on the conformal symmetry, and supersymmetry generated by the Darboux transformations of the second order with  $n = 2$ .

Let us start with the  $\mathcal{PT}$ -regularized Calogero type Hamiltonian

$$2\hat{H}_{\alpha,1} = -\frac{d^2}{dx^2} + \frac{2}{\xi^2}, \quad \xi = x + i\alpha. \quad (165)$$

Putting  $\nu = 1$  in Eqs. (56) and (57), one deduces that the system (165) can be obtained from the free particle via the first order Darboux transformation by selecting its formal zero energy

eigenfunction  $\Omega_{0,0}^\alpha = \xi = x + i\alpha x$  as the seed state. Eq. (55) yields us then the Lax-Novikov integral of the system (165),

$$2\hat{\mathcal{P}}_{\alpha,1} = -\left(\frac{d}{dx} - \frac{1}{\xi}\right)\hat{p}\left(\frac{d}{dx} + \frac{1}{\xi}\right) = -i\frac{1}{4}\hat{M}, \quad \hat{M} = -4\frac{d^3}{dx^3} + 6u\frac{d}{dx} + 3u, \quad (166)$$

where  $u(\xi) = \frac{2}{\xi^2}$ . The condition of commutativity of the third order operator (166) with Hamiltonian (165) means that the potential  $u(\xi) = \frac{2}{\xi^2}$  satisfies the stationary KdV equation  $-6uu_x + u_{xxx} = 0$ . Using the Galilean invariance of the KdV equation, one finds then that the function

$$U(x, \tau) = -\frac{1}{6}c + \frac{2}{(x+i\alpha-c\tau)^2}, \quad c \in \mathbb{R}, \quad (167)$$

will satisfy the dynamical KdV equation, which can be presented equivalently in the Lax form

$$\partial_\tau \hat{L} = [\hat{L}, \hat{M}] \Leftrightarrow U_\tau - 6UU_x + U_{xxx} = 0, \quad (168)$$

where  $L = -\frac{d^2}{dx^2} + U(x, \tau)$  and  $\hat{M}$  is given by Eq. (166) with  $u$  there changed for  $U(x, \tau)$ . Notice that if we extend the definition of the time reflection operator  $\mathcal{T}$  by requiring additionally  $\mathcal{T}\tau = -\tau\mathcal{T}$ , the time-dependent KdV equation will be invariant under the  $\mathcal{PT}$  transformation if  $U(x, \tau)$  is  $\mathcal{PT}$ -symmetric:  $[U(x, \tau), \mathcal{PT}] = 0$ . The real and imaginary part of such a field  $U(x, \tau) = v(x, \tau) + iw(x, \tau)$ , like this happens in the particular simplest case (167), will satisfy the system of coupled non-linear dynamical equations

$$v_\tau - 3(v^2 - w^2)_x + v_{xxx} = 0, \quad w_\tau - 6(vw)_x + w_{xxx} = 0. \quad (169)$$

To construct a more interesting  $\mathcal{PT}$ -symmetric solution to the KdV equation, let us use system (165) as a starting point for a new Darboux transformation. To this aim we select as the seed state the function  $\psi_{\alpha,\gamma}^1 = \gamma\Xi_{0,1}^\alpha + \Omega_{0,1}^\alpha = \gamma\xi^{-1} + \xi^2$ . This is a zero energy eigenfunction of (165). With pure imaginary parameter  $\gamma$ , function  $\psi_{\alpha,\gamma}^1$  is  $\mathcal{PT}$ -invariant. A further restriction  $\gamma = i\rho\alpha^3$  with  $\rho \in \mathbb{R}$ ,  $\rho \neq -8, 1$ , guarantees that  $\mathcal{PT}$ -odd superpotential  $W = -\frac{d}{dx} \ln(\psi_{\alpha,\gamma}^1) = \frac{1}{\xi} - \frac{3\xi^2}{\xi^3 + \gamma}$  does not take zero value. The generated potentials of the corresponding supersymmetric partner systems (164),

$$V_+ = \frac{6}{\xi^2} - \frac{6\gamma(4\xi^3 + \gamma)}{\xi^2(\xi^3 + \gamma)^2} := V_+^{(1)}(x; \alpha, \gamma), \quad V_- = \frac{2}{\xi^2}, \quad (170)$$

are  $\mathcal{PT}$ -symmetric non-singular functions. Potential  $V_+$  is a stationary solution to the higher order equation of the KdV hierarchy

$$U_\tau + 30U^2U_x - 20U_xU_{xx} - 10UU_{xxx} + U_{xxxxx} = 0. \quad (171)$$

The substitution  $\gamma \rightarrow \gamma(\tau) = 12\tau + i\rho\alpha^2$  with  $\rho > 1$  transforms function  $V_+$  into the dynamical field  $V_+^{(1)}(x, \tau; \alpha, \rho) = V_+^{(1)}(x; \alpha, \gamma(\tau))$  which is a  $\mathcal{PT}$ -symmetric solution of the KdV equation to be non-singular for  $\tau \in (-\infty, \infty)$  [8]. Some plots of the real and imaginary parts of the inverted field  $-V_+^{(1)}(x, \tau; \alpha, \rho)$  are shown on Fig. 7.

In the same way we provide a second example based on the  $\mathcal{PT}$ -symmetric system

$$2\hat{H}_{\alpha,2} = -\frac{d^2}{dx^2} + \frac{6}{\xi^2}. \quad (172)$$

This system can be obtained via a second order confluent Darboux transformation from the free particle system, where the corresponding seed states are  $\Omega_{0,0} = \xi$  and  $\Omega_{1,0} = \xi^3$  [8]. The Lax-Novikov integral of this model is given by the five order differential operator

$$4\hat{\mathcal{P}}_{\alpha,2} = \left(\frac{d}{dx} - \frac{2}{\xi}\right)\left(\frac{d}{dx} - \frac{1}{\xi}\right)\hat{p}\left(\frac{d}{dx} + \frac{1}{\xi}\right)\left(\frac{d}{dx} + \frac{2}{\xi}\right). \quad (173)$$

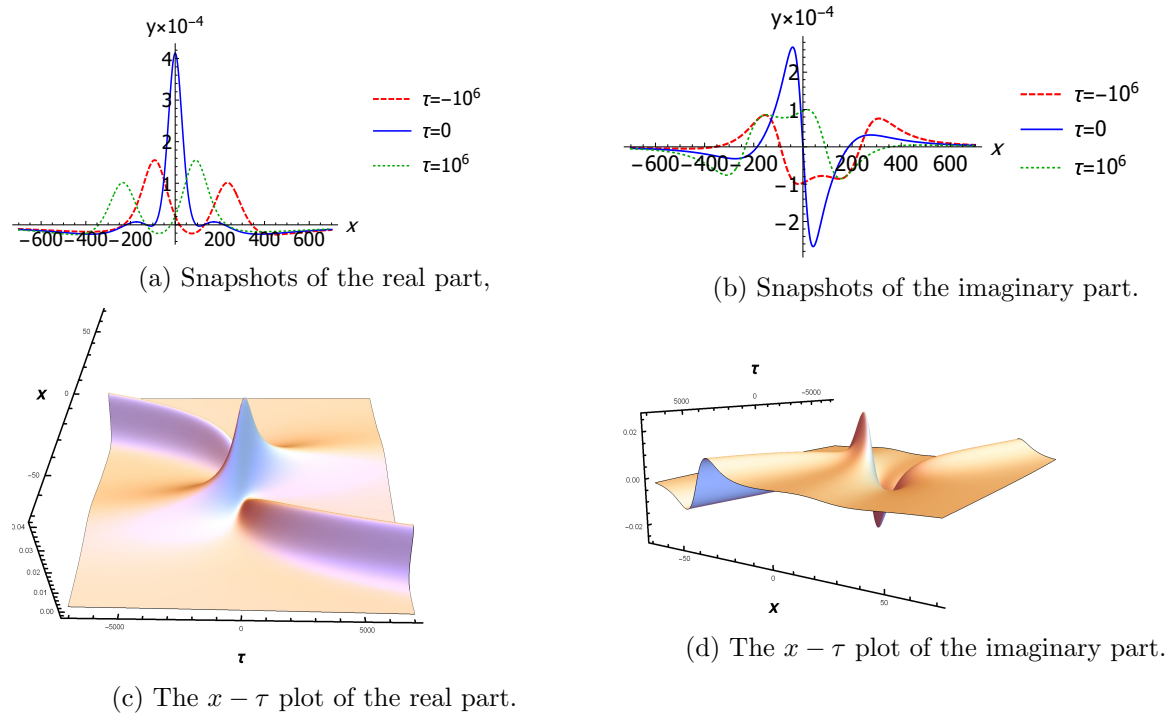


Figure 7: The plots of the real and imaginary parts of  $-V_+^{(1)}(x, \tau; \alpha, \varrho)$  with  $\alpha = 10$  and  $\varrho = 5$ .

As in the previous example, one can show that this operator is written in terms of derivatives and the potential  $u = \frac{6}{x^2}$  only, which is a stationary solution to the higher order equation of the KdV hierarchy (171).

As a seed state we choose the following nodeless zero energy eigenfunction of (172),  $\psi_{\alpha, \gamma}^2 = \gamma \Xi_{0,2}^\alpha + \Omega_{0,2}^\alpha = \gamma \xi^{-2} + \xi^3$ , with  $\gamma = i\varrho\alpha^5$ , and now  $\varrho$  is a real numerical parameter different from  $-1$  and  $-4$ . With this choice one gets the  $\mathcal{PT}$ -odd superpotential  $W = -\frac{d}{dx} \ln(\psi_{\alpha, \gamma}^2) = \frac{2}{\xi} - \frac{5\xi^4}{\xi^5 + \gamma}$ , and the  $\mathcal{PT}$ -symmetric potentials of the super-partner systems are given by

$$V_+ = \frac{12}{\xi^2} - \frac{10\gamma(6\xi^5 + \gamma)}{\xi^2(\xi^5 + \gamma)^2} := V_+^{(2)}(x; \alpha, \gamma), \quad V_- = \frac{6}{\xi^2}. \quad (174)$$

Similarly to the previous example, after the substitution  $\gamma \rightarrow \gamma(\tau) = -720\tau + i\varrho\alpha^5$  with  $\varrho > 24$ , potential  $V_+^{(2)}(x; \alpha, \gamma(\tau))$  satisfies the higher order non-linear field equation (171). The real and imaginary parts of the inverted function  $-V_+^{(2)}(x; \alpha, \gamma(\tau))$  are shown in Fig 8.

Finally, we note that near a critical value of the parameter  $\varrho$ , which in the first example is  $\varrho = 1$  and in the second case it is  $\varrho = 24$ , the real part of the potentials, defined respectively in (170) and (174), have a  $\delta$ -function like behaviour, while the corresponding imaginary parts have a form similar to  $\delta'$ -function [8, 9]. This behaviour is typical for the extreme (or, the so-called rogue) waves, that corresponds to soliton type waves with extreme values of the amplitude emerging in the process of their evolution.

## 7. Discussion and outlook

In conclusion, we indicate some open questions and problems related to the considered topics that deserve a further attention.

1. The two systems presented in Sec. 3.3 have interesting spectral properties. The first one is asymptotically free, and corresponds to a  $\mathcal{PT}$ -regularized Calogero type model. It is a perfectly

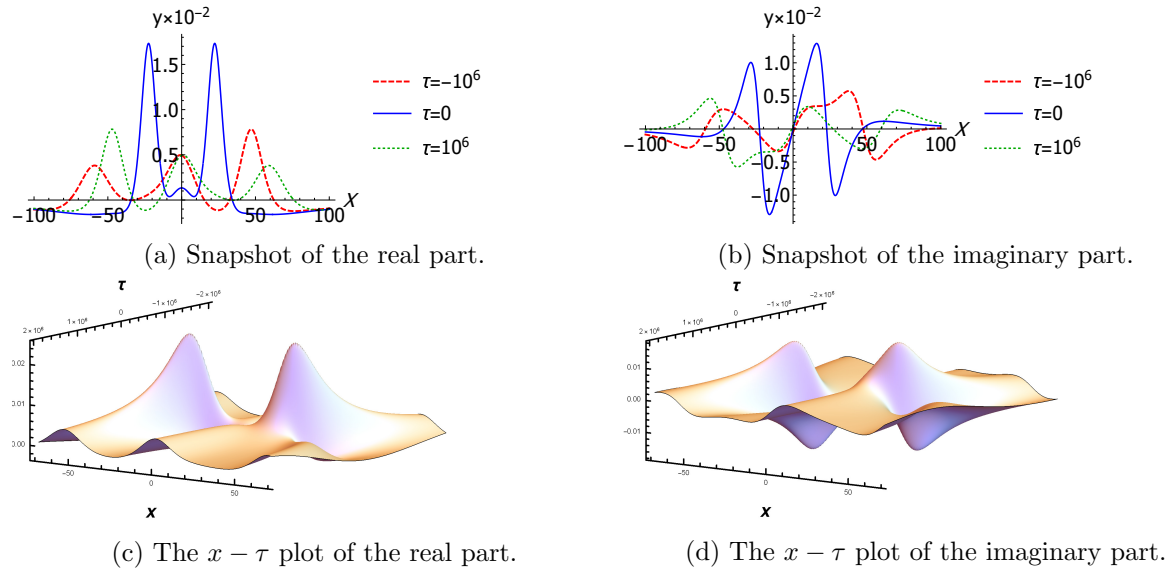


Figure 8: The plots of the real and imaginary parts of  $-V_+^{(2)}(x; \alpha, \gamma(\tau))$  with  $\alpha = 20$  and  $\varrho = 25$ .

invisible zero-gap system with the unique  $L^2(\mathbb{R})$  integrable eigenfunction of zero energy when the parameter  $\nu$  in the potential term  $\nu(\nu + 1)(x + i\alpha)^{-2}$  is an integer number,  $\nu = m$ , and the system can be related to a free particle by the Darboux transformation. These spectral properties are coherently reflected by the presence (available only at  $\nu = m$ ) of a well defined on all the real line Lax-Novikov integral [8, 9]. On the other hand, the harmonically confined  $\mathcal{PT}$ -regularized AFF model has two spectral towers that do not touch each other for any value of the parameter  $\nu$ , and for which the complete set of the spectrum generating ladder operators can only be constructed when, again, the parameter  $\nu$  is integer. As it was indicated at the end of that section, the well defined indefinite scalar product for those  $\mathcal{PT}$ -symmetric systems is unknown for us. We also do not know the equivalent Hermitian systems into which they can be transformed. However, due to the similarity of the spectrum of the  $\mathcal{PT}$ -regularized AFF model to the spectrum of a non-local model presented in ref. [58], one can expect that those two families of the  $\mathcal{PT}$ -regularized models can be related somehow to the non-local models with Hamiltonians that include in the structure the spatial reflection operator  $\mathcal{P}$ .

**2.** The higher order confluent Darboux transformations, which appeared in the  $\mathcal{PT}$ -regularized Calogero type systems with  $\nu = m$ , are directly related to the construction of higher order quantum supersymmetry [2, 6, 50, 80, 81] since the higher order intertwining operators can be promoted to higher order supercharges. In general this kind of systems, including the non-Hermitian ones, are described by non-linear superalgebras [2, 6, 8, 9, 81]. The generators of the higher order Darboux transformations can be factorized into generators of Darboux transformations of the corresponding lower orders, and such factorization is non-unique [82, 83]. In dependence on the choice of the factorization, the initial and final Darboux-related Hermitian systems can be related via “virtual” systems which can be non-Hermitian. This happens, in particular, in the case of the second order supersymmetry with Darboux generators to be differential operators of the second order [83]. In some examples considered there, the  $\mathcal{PT}$ -symmetric systems like (165) do appear in the form of non-Hermitian virtual systems.. An interesting question is whether other non-Hermitian systems obtained in this way can fit into the scheme of the  $\mathcal{PT}$ -symmetric models and CBT.

3. Consider the following similarity transformation of the operator  $i\hat{D}$ ,

$$\hat{\mathfrak{S}}_{a,b}(i\hat{D})\hat{\mathfrak{S}}_{a,b}^{-1} = i(1 + 2ab)\hat{D} + b\hat{H} - a(1 + ab)\hat{K} := i\hat{D}_{a,b}, \quad \hat{\mathfrak{S}}_{a,b} = e^{\frac{a}{\hbar}\hat{K}}e^{\frac{b}{\hbar}\hat{H}}, \quad (175)$$

where  $a = \alpha\omega$ ,  $b = \beta\omega^{-1}$ , and  $\alpha$  and  $\beta$  can be complex in general case dimensionless parameters. Our CBT generator (17) corresponds here to the particular choice  $\alpha = -1$ ,  $\beta = 1/2$ . The operator  $\hat{\mathfrak{S}}_{a,b}$  is a generator of the internal automorphism of  $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2, \mathbb{R})$  of the most general form since the inclusion of the operator  $e^{\frac{a}{\hbar}\hat{K}}$  in its structure reduces just to the change of the parameters  $\alpha$  and  $\beta$ . The operator  $i\hat{D}_{a,b}$  has the structure of the operator (82) with corresponding identification of the parameters. From (175) we obtain the relation

$$\hat{\mathfrak{S}}\hat{\mathfrak{S}}_{a,b}^{-1}(i\hat{D}_{a,b})\hat{\mathfrak{S}}_{a,b}\hat{\mathfrak{S}}^{-1} = \hbar\hat{\mathcal{J}}_0, \quad (176)$$

where  $\hat{\mathfrak{S}}$  is the  $\mathcal{PT}$ -symmetric generator (17) of our CBT. In our CBT scheme, the eigenvectors  $|\lambda\rangle$  that are transformed into the physical eigenstates of  $\hbar\hat{\mathcal{J}}_0$  correspond to the zero energy Jordan states of  $\hat{H}$ , which, in turn, are also eigenstates of  $2i\hat{D}$  with real eigenvalues. Then, in this extended scheme, the eigenstates of the operator  $i\hat{D}_{a,b}$ , that have to be transformed into the physical eigenstates of  $\hbar\hat{\mathcal{J}}_0$ , are  $e^{\frac{a}{\hbar}\hat{K}}e^{\frac{b}{\hbar}\hat{H}}|\lambda\rangle$ . However, in the general case we cannot say too much about the behaviour of the resulting functions in the coordinate representation for particular choice of the initial asymptotically free system described by the Hamiltonian  $\hat{H}$ . One can expect that the detailed analysis of this aspect should restrict the choice of the parameters  $\alpha$  and  $\beta$ . We just note that in our CBT with  $\mathcal{PT}$ -symmetric generator  $\hat{\mathfrak{S}}$ , the Jordan states of  $\hat{H}$ , which are transformed into eigenstates of  $\hat{\mathcal{J}}_0$ , satisfy the equation  $\hat{H}^n|\lambda\rangle = 0$  with some integer  $n$ . This is a scale-invariant equation for asymptotically free examples of the systems considered by us here. It is related with a unique peculiarity of our CBT: its generator  $\hat{\mathfrak{S}}$ , having the property  $\hat{\mathfrak{S}}^4 = \mathcal{P}$ , and so, being internal automorphism of the  $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2, \mathbb{R})$  algebra, maps the first order scale-invariant differential operator  $2i\hat{D}$  into the second order differential operator  $\hbar\hat{\mathcal{J}}_0$ .

4. For our  $\mathcal{PT}$ -symmetric CBT it does not matter if the number of degrees of freedom is greater than two, or if we are working in some exotic geometry. The only decisive factor is to have the generators of the  $\mathfrak{so}(2,1)$  conformal symmetry of the initial asymptotically free system to be able to construct the CBT operator. In this way, the operator  $i\hat{D}$ , to which we apply the CBT to get the Hamiltonian operator of the associated harmonically trapped system, can be extended by a  $\mathcal{PT}$ -symmetric ‘Zeeman type’ term  $\hat{\mathcal{Z}}$  that commutes with the  $\mathfrak{so}(2,1)$  generators. In this case the operator  $(2i\hat{D} + g\hat{\mathcal{Z}})$  will be mapped into  $2\hbar\hat{\mathcal{J}}_0 + g\hat{\mathcal{Z}}$ . For example if we consider a free particle in the cosmic string background, and take the combination  $2\omega(i\hat{D} + g\hat{\mathcal{J}}_0)$ , with  $\hat{D}$  and  $\hat{\mathcal{J}}_0$  to be the corresponding quantum analogues of the quantities (125), (126), the application of the corresponding CBT yields us

$$\hat{H}_g^{(\alpha)} = -\frac{\hbar^2}{2m} \left( \frac{1}{\alpha^2 r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) + \frac{\alpha^2 m \omega^2}{2} r^2 - i\hbar \omega g \alpha \frac{\partial}{\partial \varphi}. \quad (177)$$

This is a direct analog of the ERIHO quantum Hamiltonian (see Sec. 4) in the cosmic string metric (conical background from the viewpoint of condensed matter physics [69, 71]). Results related to this particular system will be presented by us soon.

In the case of two dimensions we also can add fermionic degrees of freedom by taking a term of the form  $\mathcal{Z} = \omega g \sigma_3 p_\varphi$ . In particular, the application of the isotropic CBT to the generator  $\omega(2i\hat{D} + \sigma_3 \hat{p}_\varphi)$  produces Hamiltonian of the supersymmetric Landau problem [80]. This is an important indication how the CBT can be generalized for supersymmetric case. In the same vein, the scheme of the Swanson model can be extended. The indicated generalizations can be interesting, particularly, from the point of view of physics of Bose-Einstein condensates [84, 85, 86, 87] and physics of anyons [88, 89, 90].



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