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# Autocorrelation function for estimating static pressure fluctuation intensity in incompressible homogeneous turbulence under an intermediate Reynolds number

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**Abstract.** This study presents a simple form of an autocorrelation function that estimates the relative static pressure fluctuation intensity in the intermediate range of Reynolds numbers. When the Reynolds number is high or low, values of relative static pressure fluctuation intensity are each derived as simple values. This study attempts to estimate the intensity between high and low Reynolds numbers. Using the joint-normal approximation in the velocity field allows the relative static pressure fluctuation intensity to be given by integrating a function based on the autocorrelation function. For high and low Reynolds numbers, the autocorrelation function is given as exponential and Gaussian profiles, respectively. This study adds a higher-order term to these functions using Gram–Charlier and power series expansion. In the incompressible turbulence, a correlation function has to satisfy the continuity equation. Also, there is a defined relationship between the integral scale and the Taylor microscale derived from the autocorrelation function. This study investigates an autocorrelation function that satisfies these conditions. The value of one constant included in the present autocorrelation function shown in this study is derived as the appropriate value based on these conditions.

## 1. Introduction

Incompressible turbulence [1] is commonly found in the field of fluid engineering. Therefore, predicting and modelling this phenomenon has been important in this field. Incompressible turbulence in a region away from a wall can be considered as homogeneous turbulence without significant mean shear in the free stream. Many previous studies have examined the homogeneous turbulence, velocity, and vorticity fields [1], whereas few previous studies have addressed the static pressure fluctuation field. The static pressure fluctuation field of homogeneous turbulence has been studied mainly by direct numerical simulation (e.g., [2-7]), although experimental work is also known (e.g., [8]). In these studies, the relative static pressure fluctuation intensity was investigated as one of the most fundamental quantities that characterize the static pressure fluctuation field. The relative static pressure fluctuation intensity is given as  $(p/q^2)^2$ , where  $q = 2k/3$ , and  $k$  is the turbulent kinetic energy. Here, the relative static pressure fluctuation intensity is on the order of unity. This study considers that



predicting the relative static pressure fluctuation intensity is important in the field of incompressible homogeneous turbulence.

When the Reynolds number is sufficiently high or low, autocorrelation functions, with the application of the joint-normal approximation (e.g., [9-12]), derive the values of the relative static pressure fluctuation intensity simply as 0.5 and 1, respectively [9]. Here, autocorrelation  $f(r)$  is derived as  $f(r) = \langle u(x+r)u(x) \rangle$  as a longitudinal form, where  $\langle \rangle$  denotes ensemble average and  $u$  is velocity fluctuation for  $x$  direction. In contrast, in previous numerical analyses using direct numerical simulation, as the Reynolds number increased, the value of the relative static pressure fluctuation intensity decreased, and the values between the high and low Reynolds numbers were also investigated [2-6]. A form of the autocorrelation function that gives the relative static pressure fluctuation intensity in this intermediate Reynolds number range should be addressed. The continuity equation, which is one of the governing equations of incompressible flow, can also be given to correlation functions in the homogeneous turbulence. Also, for homogeneous turbulence, it is known that the integral scale needs to be larger than the Taylor microscale. An autocorrelation function that gives the relative static pressure fluctuation intensity has to meet all these conditions.

The purpose of this study is to investigate a simple autocorrelation function that calculates the relative static pressure fluctuation intensity in the field at an intermediate Reynolds number range of homogeneous turbulence. When the Reynolds number is sufficiently high or low, the values of the relative static pressure fluctuation intensity can be derived as shown in previous studies. In this study we develop an autocorrelation function for estimating the value that exists between these two values of Reynolds numbers. For this purpose, we add an additional term to the autocorrelation functions previously reported. In particular, in this study, Gram-Charlier expansion [13-16] and power series expansion are applied to obtain this higher-order term. Also, we investigate whether the autocorrelation function obtained in this study can satisfy the conditions described above. As described below, applying a higher-order term based on the power series expansion to the autocorrelation function can satisfy all of the setting conditions of this study. An appropriate value for the constant used in the proposed autocorrelation function can thus be given from the conditions set in this study.

## 2. Method

The intensity of relative static pressure fluctuations is considered a fundamental turbulence statistic in previous studies on incompressible turbulence. As a result, many of these studies, especially numerical analyses using the technique of direct numerical simulation, have shown values of relative static pressure fluctuation intensity. Although the relative static pressure fluctuation intensity is on the order of unity, the intensity value changes depending on the Reynolds number. Specifically, the value of the relative static pressure fluctuation intensity increases as the Reynolds number decreases. This study addresses the prediction of the value of relative static pressure fluctuation intensity.

Previous studies often assumed that the velocity field of incompressible homogeneous turbulence has a joint-normal nature (e.g., [9-12]). By assuming that the turbulence velocity field has this nature, the following simple relationship can be obtained between the relative static pressure fluctuation intensity and an autocorrelation function [9]:

$$\left(\frac{p}{q^2}\right)^2 = \int_0^\infty r \left(\frac{df(r)}{dr}\right)^2 dr, \quad (1)$$

where  $f(r)$  is an autocorrelation function,  $r$  is the longitudinal axis normalized by a turbulent characteristic length scale,  $p$  is the rms value of the static pressure fluctuation,  $q = (2/3)k$ , and  $k$  is the turbulent kinetic energy of the velocity field. As shown in this equation, the relative static pressure fluctuation intensity can be calculated by integrating the autocorrelation function. There is a simple form of autocorrelation function when the Reynolds number is sufficiently high or low. In previous

studies, the following autocorrelation functions were obtained under the conditions of low Reynolds number and high Reynolds number, respectively [9]:

$$f(r) = C \exp(-2r^2) \text{ and } f(r) = \exp(-r). \quad (2)$$

From these autocorrelation functions, the values of the relative static pressure fluctuation intensity under the conditions of both Reynolds numbers can be obtained as follows [9]:

$$\begin{aligned} (p/q^2)^2 &= 1 \quad \text{for } f(r) = \exp(-2r^2) \text{ and} \\ (p/q^2)^2 &= 0.5 \text{ for } f(r) = \exp(-r). \end{aligned} \quad (3)$$

However, as shown in the previous numerical analysis [2-6], the relative static pressure fluctuation intensity has a meaningful value between these two values in an actual turbulent flow field. This study elucidates a simple autocorrelation function that gives the value of the relative static pressure fluctuation intensity between the conditions of low and high Reynolds numbers. As shown in the previous work (e.g., [9]), the condition of the low Reynolds number corresponds to that of the final period of decay. On the other hand, the state of high Reynolds number corresponds to the case where the Reynolds number is sufficiently high. The intermediate Reynolds numbers in this study exist between low and high Reynolds numbers. We have added some descriptions to clarify this point to the revised manuscript.

This study focuses on the following three conditions to obtain an autocorrelation function with an intermediate Reynolds number:

$$0.5 < (p/q^2)^2 < 1, \quad (4)$$

$$\int_0^\infty r g(r) dr = 0, \text{ where } g(r) = f(r) + \frac{r}{2} \frac{df}{dr}, \quad (5)$$

$$L > \lambda. \quad (6)$$

Here  $g(r)$  is a horizontal correlation function. As shown in equation (4), the autocorrelation function given by this study is derived to give the relative static pressure fluctuation intensity in the range of 0.5–1 [9]. Also, as shown in equation (5), the continuity equation needs to hold in the correlation function because the flow is incompressible. In addition, the integral scale has to be larger than the Taylor microscale in the described turbulent flow field. Here, the integral scale and the Taylor microscale are defined from the autocorrelation function, respectively, as follows [1]:

$$L = \int_0^\infty f(r) dr \text{ and } \lambda = \frac{1}{\sqrt{-d^2 f(r)/dr^2|_{r=0}/2}}. \quad (7)$$

Here, the integral scale and Taylor microscale characterize large-scale and small-scale structures of turbulence, respectively. This study validates the four autocorrelation functions described below by determining whether the three conditions in equations (4)–(6) are satisfied.

### 3. Results and discussion

#### 3.1. Autocorrelation function using Gram–Charlier expansion

First, we focused on the Gram–Charlier expansion because it has been used in previous studies [13] to expand the Gaussian profile. Using this expansion and considering up to the second-order term, the following expansion can be derived:

$$f(r) = C_o (C \exp(-2r^2) - (C - 1)(r^2 - 1)\exp(-2r^2)). \quad (8)$$

Here, due to the definition of the autocorrelation function, the value of the function at  $r = 0$  is unity. From this condition, the two constants in expansion equation (8) are related as follows:

$$C_o = 1/(2C - 1). \quad (9)$$

Also, the value of the constant  $C$  determines the magnitude of the second-order term. For example, if the value is  $C = 1$  [1], the second-order term is zero. Note that the above autocorrelation function satisfies the continuity equation (Eq.(5)). Using the above form of an autocorrelation function, the relative static pressure fluctuation, the integral scale, and the Taylor microscale are derived, respectively, as follows:

$$(p/q^2)^2 = \frac{33C^2 - 34C + 9}{8(2C - 1)^2}, L = \frac{\sqrt{\pi}(7C - 3)}{2^{7/2}(2C - 1)}, \text{ and } \lambda = \frac{\sqrt{2C - 1}}{5C - 3}. \quad (10)$$

The form in equation (10) is an expansion that uses up to the second-order term. The prediction accuracy of the expansion may be improved by considering higher-order terms. The following expansion formula can be obtained using this expansion by considering higher-order terms up to the fourth-order term [13]:

$$f(r) = C_o(C \exp(-2r^2) - (C - 1)(r^2 - 1)\exp(-2r^2) + C_4(r^4 - 6r^2 + 3)\exp(-2r^2)). \quad (11)$$

Here, because the autocorrelation function is even, the first order and third-order terms are derived to be zero. In this expansion, because the value of the autocorrelation function is unity at  $r = 0$ , the relationship between the two constants can be obtained as follows:

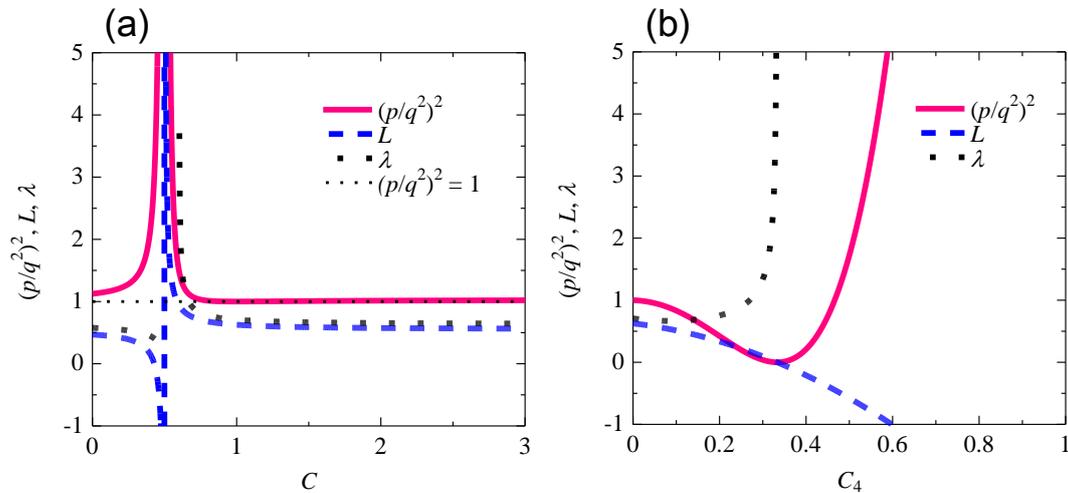
$$C_o = (1 - 3C_4)/(2C - 1). \quad (12)$$

When  $C = 1$ , the second-order term is zero, and  $C = 1$  was set in this form to investigate the effect of the fourth-order term. The above autocorrelation function also satisfies the continuity equation. This autocorrelation function gives the relative static pressure fluctuation, the integral scale, and the Taylor microscale, respectively, as follows:

$$(p/q^2)^2 = \frac{(1 - 3C_4)^2}{32}(387C_4^2 + 184C_4 + 32),$$

$$L = \frac{\sqrt{\pi}}{2^{11/2}}(1 - 3C_4)(27C_4 + 16), \text{ and} \quad (13)$$

$$\lambda = \frac{1}{\sqrt{2(1 - 3C_4)(6C_4 + 1)}}.$$



**Figure 1.** The relative intensity of static pressure fluctuation  $(p/q^2)^2$ , integral length scale  $L$ , and Taylor microscale  $\lambda$  depending on the model constant  $C$  or  $C_4$ . Here (a) and (b) are results for auto correlation function based on the second order and fourth-order forms, respectively.

Figure 1 shows the relative static pressure fluctuation, integral scale, and Taylor microscale calculated from the autocorrelation functions using the second-order and fourth-order terms and the Gram–Charlier expansion. Here, these values are shown as functions of each control parameter. As shown in the figure, when the second-order term is used, these quantities become unrealistic values when  $C$  is between 0 and 1. The relative static pressure fluctuation is obtained as unity approximately when  $C$  is larger than unity. As shown in Figure 1(b), when the autocorrelation function using the

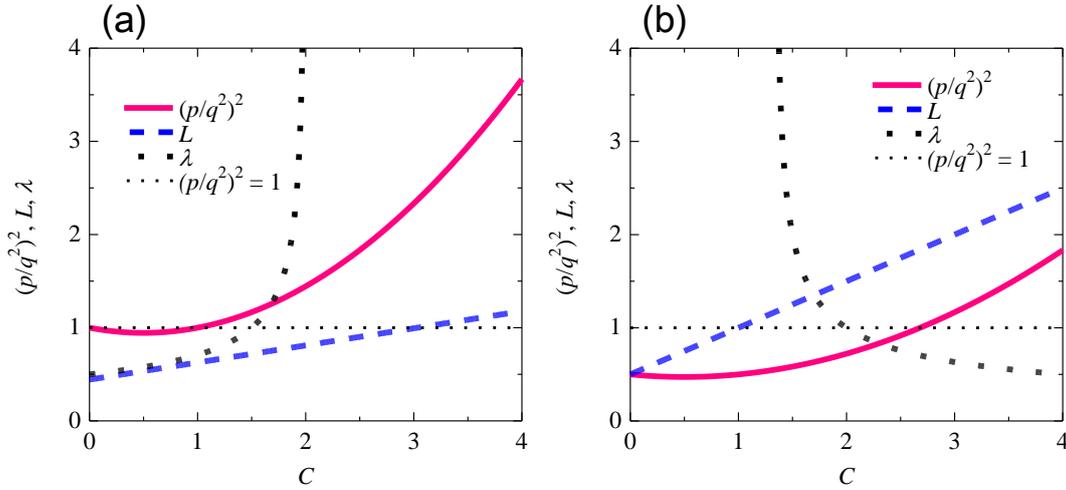
fourth-order term is used, the three quantities change depending on the constant. When  $C_4$  is in the range between 0 and 0.1, the relative static pressure fluctuation value is found to be 0.5–1. However, under this condition of the constant  $C_4$ , the Taylor microscale is larger than the integral scale, which is unrealistic. From these results, when the second-order and fourth order terms are used with the Gram–Charlier expansion, neither of the conditions shown in equation (6) are satisfied.

### 3.2. Autocorrelation function using power series expansion

Next, the form of the autocorrelation function using power series expansion was used. As shown in previous studies, autocorrelation functions based on a Gaussian profile and exponential function have been obtained for turbulent flows with low and high Reynolds numbers. These autocorrelation functions can theoretically derive the simple values of relative static pressure fluctuations. In this study, each second-order term was added to the function form of autocorrelations. First, a second-order term was added to the Gaussian profile used under the condition of low Reynolds number as follows based on power series expansion:

$$f(r) = C \exp(-2r^2) - (C - 1) \exp(-4r^2), \quad (14)$$

where the coefficient of the second-order term is given so that the value of the autocorrelation function is unity at  $r = 0$ . Also, this autocorrelation function with the second-order term satisfies the continuity equation. Using the autocorrelation function based on this Gaussian profile, the relative static pressure fluctuation, the integral scale, and the Taylor microscale are given, respectively, as follows:



**Figure 2.** Relative intensity of static pressure fluctuation  $(p/q^2)^2$ , integral length scale  $L$ , and Taylor microscale  $\lambda$  depending on the model constant  $C$ . Here (a) and (b) are results for the autocorrelation function based on Gaussian and exponential functions, respectively.

$$\begin{aligned} (p/q^2)^2 &= \frac{1}{9} (2C^2 - 2C + 9), \\ L &= \frac{\sqrt{\pi}}{4} ((1 - \sqrt{2})C - 1), \text{ and } \lambda = \frac{1}{\sqrt{2(2 - C)}}. \end{aligned} \quad (15)$$

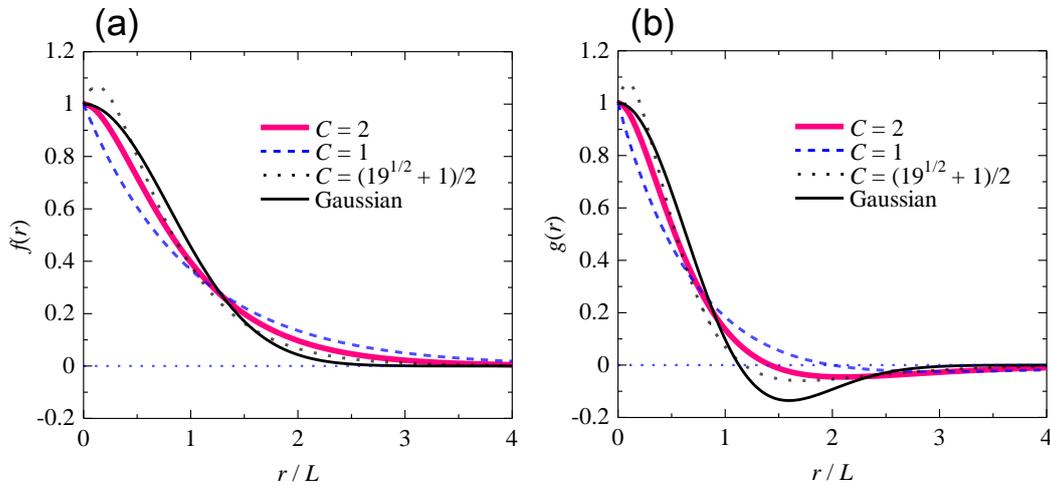
When the Reynolds number is sufficiently high, an exponential function is used to represent the autocorrelation function. By adding a second-order term to the autocorrelation function based on the exponential function using power series expansion, the following autocorrelation function can be given:

$$f(r) = C \exp(-r) - (C - 1) \exp(-2r), \quad (16)$$

where the second-order term's coefficient is given such that the value of the autocorrelation function is unity at  $r = 0$ . This autocorrelation function can also satisfy the continuity equation. Using the autocorrelation function and this second-order term, the values of relative static pressure fluctuation, integral scale, and Taylor microscale are respectively derived as follows:

$$(p/q^2)^2 = \frac{1}{18} (2C^2 - 2C + 9), L = \frac{C + 1}{2}, \text{ and } \lambda = \frac{\sqrt{2}}{\sqrt{3C - 4}}. \quad (17)$$

Figure 2 shows the relative static pressure fluctuation, integral scale, and Taylor microscale obtained from the autocorrelation functions with each second-order term added using power series expansion. In this figure, these quantities are shown as a function of the constant  $C$ . As shown in the figure, when the Gaussian profile and the exponential profile are used, these three quantities change depending on the control constant. When the Gaussian profile is used, the relative static pressure



**Figure 3.** Longitudinal and lateral autocorrelation functions, (a) and (b), depending on values of the model constant. Here the lateral autocorrelation function has been derived from the longitudinal function. Lateral axis is normalized by the integral scale.

variation is calculated to be less than unity in the range of  $0 < C < 1$ . In contrast, the Taylor microscale is found to be larger than the integral scale under this relative static pressure fluctuation. This derived relationship between the length scales is considered to be unrealistic. When an exponential profile is used, the relative static pressure variation is between 0.5 and 1 near the condition of  $C = 2$ . Under this given this relative static pressure fluctuation value, the integral scale is calculated to be larger than the Taylor microscale. These two conditions meet those shown in equations (4)–(6). Therefore, the autocorrelation function obtained by adding the second-order term to the exponential profile can meet the conditions of equations (4)–(6) by adjusting the range of the control constant.

### 3.3. Appropriate value of the model constant

The results shown in section 3.2 show that the form of the autocorrelation function obtained by adding the second-order term to the exponential profile, often used for turbulence at high Reynolds numbers, is as follows when the conditions satisfy equations (4)–(6). The specific form of this autocorrelation function can be obtained by determining the value of the constant  $C$  included in the form. In this study, three values were used for constant  $C$ . First, the condition  $C = 1$  was used, under which the second-order term is zero, as shown in Eq.(16). Next, we used the second condition of  $C = 2$ . Under this condition, as shown in Figure 3, the relative static pressure fluctuation can be set to a value of 0.5–1, and the integral scale is set to be larger than the Taylor microscale. For the third condition, the value of the constant that sets the value of the relative static pressure fluctuation to unity is used. This constant value is specifically derived as  $C = (19^{1/2}+1)/2$ . The autocorrelation function based on the Gaussian distribution without the second-order term gives a value of unity as the relative static pressure fluctuation. The shape of the autocorrelation function of the third condition is compared with that of this Gaussian profile. Gaussian and exponential functions are used to represent autocorrelation functions when the Reynolds numbers are low and high, respectively. In Figure 3, the result for  $C = 1$  corresponds to the autocorrelation function described as an exponential function. In this study, we focus on the intermediate Reynolds number range. Therefore, in addition to the exponential function, the Gaussian function is also shown in this figure for comparison with the autocorrelation function shown in this study.

Figure 3 shows longitudinal and lateral correlation functions. Here, the longitudinal correlation function is given by Eq.(16), as described above. The lateral correlation function is derived from the

vertical correlation function using Eq.(5). The horizontal variable is normalized by the integral scale that characterizes the large-scale turbulence structure. As shown in Figure 3, the profile of the autocorrelation function is changed by adding the second-order term. Specifically, the correlation value increases in the region where  $r$  is small and decreases in the region where  $r$  is large. This result is found in both the longitudinal and lateral autocorrelation functions. By adding the second-order term, the shape of the profiles that is approximately intermediate between the exponential and the Gaussian profiles is obtained. Under the third condition, the value of the relative static pressure fluctuations obtained from the autocorrelation function is equal to that of the Gaussian profile. However, the shapes of the profiles are significantly different from each other. Specifically, the autocorrelation function of the third condition, where  $C = (19^{1/2}+1)/2$ , has an unrealistic shape in the region where  $r$  is small. From these results,  $C = 2$  is appropriate as a constant value in the autocorrelation function obtained by adding the second-order term.

#### 4. Conclusions

The purpose of this study was to investigate various forms of the autocorrelation function for calculating the relative static pressure fluctuation intensity in a noncompressible fluid. A simple autocorrelation function that gives values of the relative intensity under the conditions of high or low Reynolds number has been previously reported. However, in real-world engineering, intermediate values between these conditions have been found. This study shows a simple autocorrelation function that gives the relative static pressure fluctuation intensity at intermediate Reynolds numbers.

The autocorrelation function for homogeneous turbulence must satisfy a few conditions. First, the autocorrelation function has to satisfy the continuity equation. Also, the integral scale and Taylor microscale can be derived from an autocorrelation function. In homogeneous turbulence, the integral scale needs to be larger than the Taylor microscale. This study constructed a simple autocorrelation function that satisfies all of these conditions by adding a higher-order term to the autocorrelation function given by previous studies.

This study focused on Gram–Charlier and power series expansions to add a higher-order term to the profile. The Gram–Charlier expansion, which has often been used in previous studies of turbulence, is used to add a higher-order term to the Gaussian profile used for conditions with a low Reynolds number. However, the autocorrelation function with the higher-order term added by the Gram–Charlier expansion gave unrealistic turbulence characteristics. Additionally, a higher-order term based on the power series expansion was added to the exponential and Gaussian profiles used for the high and low Reynolds number conditions, respectively. The autocorrelation function based on the Gaussian profile with the second-order term also gave unrealistic turbulence characteristics. In contrast, the autocorrelation function based on the exponential profile with the second-order term can satisfy all the conditions that were evaluated in this study. Therefore, the longitudinal and lateral correlation functions were used to obtain an appropriate value for the constant that determines the magnitude of the second-order term.

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