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Research methods of the process of heat and mass transfer in different media with diffusion and subdiffusion

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Abstract. The problem of studying the laws governing the formation of the radon environment is not new. The development of the mining industry (to study the regularities of the formation of the radon environment in mine workings, it was necessary to simulate the flux of radon density, which led to the construction of various models of radon transfer), became the main catalyst for in-depth research in this direction. It should also be noted that according to the RF radiation safety standards (NRB-99), the average annual equivalent equilibrium volumetric activity (concentration) of radon in the air of residential and public buildings should not exceed the established limit. To implement this decree, various models of mass transfer (radon) were built. Most of these models are based on the advection-diffusion equation, which simulates the processes of mass transfer of matter or heat transfer in a medium with fractal geometry (in particular, in porous media). Moreover, the order of the fractional time derivative in this equation corresponds to the proportion of channels (the system described by this equation is open, that is, it is connected to the outside world either by a finite or infinite number of communication channels) open for flow in a fractal (porous) medium. This process is non-local in time. And the environment in which this process takes place will be an environment with memory. In this paper, we analyze boundary value problems for the considered equation. A method based on the separation of variables is presented, while the solution of the problems posed is written out in the form of an infinite series in the eigenfunctions of the operator generated by an ordinary differential expression of the fractional order (the order of the fractional derivative is greater than one but less than two) and boundary conditions of the Sturm-Liouville type.



1. Introduction

The problem of studying the patterns of formation of the radon environment is not new. The development of the mining industry (to study the regularities of the formation of the radon environment in mine workings, it was necessary to simulate the flux of radon density, which led to the construction of various models of radon transfer), became the main catalyst for in-depth research in this direction. Note also that according to the radiation safety standards of Russian Federation, the average annual equivalent equilibrium volumetric activity (concentration) of radon in the air of residential and public buildings should not exceed the established limit. To implement this decree, various models of mass transfer (radon) were built. Most of these models are based on the equation

$$d \frac{\partial^2 C(x)}{\partial x^2} - \lambda \cdot C(x) = 0 \quad (1)$$

where $C(x)$ – distribution of radon volumetric activity in the sample;

d – diffusion coefficient of radon;

λ – radon decay constant $2,09 \cdot 10^{-6}$ 1/s.

Equation (1) is obtained under the following assumptions:

- radon transfer occurs in one direction perpendicular to the sample cross section, while the influence of edge effects on its lateral surface is negligible;
- barometric pressures at the boundaries of the sample are the same during the experiment;
- the emission of radon in the sample material is negligible;
- there is no sorption of radon in the sample material.

2. Materials and methods

Model (1) describes a stationary regime of mass transfer. Let us present the formulation and solution of the nonstationary problem. For this, we present laboratory studies of the diffusion radon permeability of materials in a non-stationary mode given in [1]. There are two fundamentally different methods – “constant” and “instant” sources. In the well-known works of G. Zapalac presented the theoretical basis of the non-stationary method of “constant source”, as well as the scheme of the experimental setup and the results of determining the effective diffusion coefficient of radon in thin concrete samples. The mathematical formulation of the problem of radon mass transfer in the test sample corresponding to the experimental conditions is presented in the form of the equation

$$d \frac{\partial^2 C(x,t)}{\partial x^2} = \frac{\partial C(x,t)}{\partial t} \quad (2)$$

Equation (2) will be used by us in the future to model the process of radon transport in various environments. We also note the equation

$$d \frac{\partial^2 C(x,t)}{\partial x^2} - \lambda \cdot C(x,t) = \frac{\partial C(x,t)}{\partial t},$$

which is widely used in the theory of heat and mass transfer. When it comes to abnormal diffusion, there are two main methods

1) the first method is stochastic - in this case, diffusion is described using the process of a random walk of particles;

2) the second method is a method based on fractional calculus. Here we are talking about models based on nonstationary fractional differential equations of the form

$$d \frac{\partial^\alpha C(x,t)}{\partial x^\alpha} = \frac{\partial C(x,t)}{\partial t} \quad (3)$$

where

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^x \frac{u(\tau,t) d\tau}{(x-\tau)^{\alpha-1}}$$

- the derivative of the fractional (in the sense of Riemann-Liouville) order $1 < \alpha < 2$, which is widely used in modeling the process of radon transport [1].

In this paper we carry out a detailed discussion and development of the method of separation of variables (Fourier method) when solving boundary value problems for equation (3).

3. Results and discussions

Consider the first boundary value problem for the inhomogeneous fractional dispersion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D \cdot \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t), \end{cases} \quad (4)$$

$$\begin{cases} u(1,t) = u(0,t) = 0, \end{cases} \quad (5)$$

$$\begin{cases} u(x,0) = \varphi(x), \end{cases} \quad (6)$$

The following theorem holds

Theorem. The function

$$u(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n D t} \left[\int_0^t f_n(\tau) e^{-\lambda_n D \tau} d\tau + \varphi_n \right] x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha) \quad (7)$$

is a solution to the boundary value problem (4), (5), (6). Here

$$E_{\alpha,\alpha}(\lambda_n x^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda_n x^\alpha)^k}{\Gamma(\alpha + \alpha k)}$$

– is the well-known function of Mittag-Leffler type, φ_n – the decomposition coefficients of functions $u(x,t)$ and $f(x,t)$ in a basis of functions $\omega_n(\lambda_n, x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha)$.

The proof of formula (7) is based on the following lemma

Lemma. The function

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha) \quad (8)$$

is the solution of the following problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D_{0+}^\alpha u(x,t), \end{cases} \quad (9)$$

$$\begin{cases} u(0+,t) = u(1,t) = 0, \end{cases} \quad (10)$$

$$\begin{cases} u(x,0) = \phi(x), \end{cases} \quad (11)$$

here

$$E_{\alpha,\alpha}(\lambda_n x^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda_n x^\alpha)^k}{\Gamma(\alpha + \alpha k)}$$

– is the well-known function of Mittag-Leffler type, φ_n – the decomposition coefficients of function $\phi(x)$ in the non-orthogonal basis of system of the functions $X_n(\lambda_n, x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha)$.

Proof of the theorem. First consider the homogeneous fractional differential equation

$$\frac{\partial u(x,t)}{\partial t} = D \cdot \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \quad (12)$$

corresponding to fractional differential equation (1). In order to find a nontrivial solution to the homogeneous fractional differential equation (12), we use the Fourier method of separation of variables and represent the function $u(x, t)$ as a product

$$u(x, t) = \omega(x)\rho(t), \quad (13)$$

where $\omega(x)$ – is the function, depending only on x , $\rho(t)$ – is the function, depending only on t . Substituting the assumed form of solution (13) into equation (12), we obtain the following equality:

$$\omega(x)\rho'(t) = \rho(t)D_x^\alpha \omega(x). \quad (14)$$

Dividing both sides of equality (14) by the product $\omega(x)\rho(t)$, we obtain the equation

$$\frac{\rho'(t)}{\rho(t)} = \frac{D_x^\alpha \omega(x)}{\omega(x)}. \quad (15)$$

The left side of equation (15) depends only on the variable t , while the right side - only on the variable x . Therefore, equation (15) can be valid only in the case

$$\frac{\rho'(t)}{\rho(t)} = \frac{D_x^\alpha \omega(x)}{\omega(x)} = \lambda, \quad (16)$$

where λ – is a real number.

From the equation (16) follows

$$D_x^\alpha \omega(x) = \lambda \omega(x), \quad (17)$$

$$\rho'(t) = \lambda \rho(t). \quad (18)$$

Boundary conditions (2) in this case give

$$\omega(0) = 0, \omega(1) = 0.$$

So, to define the function $\omega(x)$ an eigenvalue problem is obtained (Sturm-Liouville problem):

$$\begin{cases} D_x^\alpha \omega(x) = \lambda \omega(x), \\ \omega(0) = 0, \\ \omega(1) = 0. \end{cases} \quad (19)$$

studied in [1]. In this paper was shown that eigenfunctions $\omega_n(\lambda_n, x)$

$$\omega_n(\lambda_n, x) = x^{\alpha-1} E_{\alpha, \alpha}(\lambda_n x^\alpha). \quad (20)$$

exists only for the eigenvalues $\{\lambda_k\}$, which are zeros of the function $E_{\alpha, \alpha}(\lambda)$.

It was also proved in [1] that the system of functions $\omega_n(\lambda_n, x)$ forms a basis in a Hilbert space $L_2(0,1)$. So, the functions $u(x, t)$ and $f(x, t)$ may be decompose in the Fourier series in a basis of system of the functions (20):

$$u(x, t) = \sum_{n=1}^{\infty} v_n(t) \omega_n(\lambda_n, x), \quad (21)$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \omega_n(\lambda_n, x). \quad (22)$$

In expressions (21) and (22), the functions $v_n(t)$ and $f_n(t)$ play the role of the Fourier coefficients of the function expansions $u(x, t)$ and $f(x, t)$ in a basis of system (20).

Now, substituting (21) and (22) into equation (1), we obtain (7).

4. Summary

Thus, in this paper we consider a general method for solving a one-dimensional boundary value problem for a one-dimensional advection-diffusion equation, based on the separation of variables.

This problem was solved in two stages. First, we find the exact solution in the form of an infinite series in the eigenfunctions of the operator generated by an ordinary differential expression of fractional order (the order of the fractional order derivative is greater than one, but less than two) and boundary conditions of the Sturm-Liouville type.

Since this series converges very quickly, it is natural to replace it with the sum of the first few terms. The approximate solution obtained in this way is quite suitable for practical numerical calculations. The paper shows the use of the fractional diffusion equation for modelling the process of radon transport in various media. This is a very actual problem nowadays.

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