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Wyle’s Vanishing Tensor of Nearly Cosymplectic Manifold

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Abstract. The present paper focuses on the geometry of Wyle’s tensor of nearly cosymplectic manifold. In particular, the flatness properties of Wyle’s tensor were traded. These properties facilitated us to identify the necessary and sufficient condition for nearly cosymplectic manifold to be an Einstein manifold.

1. Introduction

In previous years, many scientific studies appeared on the subject of nearly cosymplectic manifold, which received the attention of many scholars. We will mention some of those. Blair [1], Blair and Showers [2] studied some properties of nearly cosymplectic structure and they are considered analog the concept nearly Kahler structure in Hermition geometry. There are many researchers studied this class, for example, Banaru [3], Endo [4], [5], [6]. Kirichenko and Kusova [7] studied the geometry of nearly cosymplectic manifold in the G- adjoined structure space. In particular, they found its structure equations and components of Riemannian curvature tensor. Abood and Nawaf [8][9][10][11], are studied some properties of nearly cosymplectic structure. By using the adjoined G-structure space method, we investigated the geometrical properties of one tensor on some types of almost contact manifold. In particular, we studied the Wyle's tensor and nearly cosymplectic manifold. In each Riemannian manifold, a Wyle's tensor (conformal curvature tensor) is a tensor of type $(3,1)$, which is invariant with respect to the metric transformation. This tensor has been studied on some classes of almost Hermitian manifold, see [12], [13]. In this paper, we have got some outcomes on Wyle's tensor in nearly cosymplectic manifold. In detail, we have found the necessary and sufficient conditions for nearly cosymplectic manifold to be an Einstein manifold.

Preliminaries

Suppose that M is smooth manifold of odd-dimensional greater than 3, X(M) is the modules of smooth vector field on M, $X^c(M)$ is complexification of module X(M) and $T_p^c(M)$ is the complexification of tangent space $T_p(M)$ at the point p in M .

The set $(M, \eta, \xi, \Phi,$ g) is called an almost contact manifold (*AC*-manifold), where η is differential 1-form called contact form, ξ is a vector field called a characteristic, Φ is endomorphism of *X(M)* called a structure endomorphism and g is the Riemannian metric on *M*. Moreover, the following conditions are implemented:

(1) $\eta(\xi) = 1;$ (2) $\Phi(\xi) = 0;$ (3)$\eta \circ \Phi = 0;$ (4)$\Phi^2 - id + \eta \otimes \xi;$ (5)$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), X, Y \in X(M)$.[14]

The matrices of the structure endomorphism $\Phi(p)$ and Riemannian metric g_p in A-frame are respectively given by the following forms:

$$\left(\Phi_j^i\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \tag{2.1}$$

An almost contact manifold is called a nearly cosymplectic manifold (*NC*-manifold) if the equality $\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0$, holds for X and Y in *X(M)* supports [15].

The following theorem explains the structure equations of *NC*-manifold in the G - adjoined structure space.

Theorem 2.1 [7].

In the G-adjoined structure space, the structure equations of *NC*-manifold are given by the following forms:

1) $d\omega^a = -\omega_b^a \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^a{}_b\omega \wedge \omega^b,$

2) $d\omega_a = \omega_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_a{}^b\omega \wedge \omega_b,$

3) $d\omega = C_c^b\omega^c \wedge \omega_b$,

where $\omega = \omega^0 = \omega_0 = \pi^*(\eta)$, π is a natural projection of the associated G-structure space onto the manifold M,

$B_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{a}_{[b,c]}$, $B_{ab}{}^{c} = \frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,\hat{c}}$,

$B^{abc} = \frac{\sqrt{-1}}{2}\Phi^{a}_{[\hat{b},\hat{c}]}$, $B^{ab}{}_{c} = -\frac{\sqrt{-1}}{2}\Phi^{a}_{\hat{b},c}$,

$B^{a}{}_{b} = \sqrt{-1}\Phi^{a}_{0,b}$, $C^{ab} = \sqrt{-1}\Phi 0^{0}_{[\hat{a},\hat{b}]}$,

$B_{a}{}^{b} = -\sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}}$, $C_{ab} = -\sqrt{-1}\Phi^{0}_{[a,b]}$,

$C_{a} = \sqrt{-1}\Phi^{0}_{a,0}$, $C^{a} = -\sqrt{-1}\Phi^{0}_{\hat{a},0}$

$C^{a}_{b} = -\sqrt{-1}\left(\Phi^{0}_{b,\hat{a}} + \Phi^{0}_{\hat{a},b}\right) = B^{a}{}_{b} - B_{b}{}^{a}$,

$B_{ab} = \sqrt{-1}\left(\frac{1}{2}\Phi^{\hat{a}}_{\hat{b},0} - \Phi^{\hat{a}}_{0,b}\right)$,

$B^{ab} = -\sqrt{-1}\left(\frac{1}{2}\Phi^{a}_{\hat{b},0} - \Phi^{a}_{0,\hat{b}}\right)$.

The tensors B, C and A are respectively called the first, second and third structure tensors.

Definition 2.1 [16]: A Riemann Christoffel tensor R of a smooth manifold M is a tensor of type (4,0) which is defined as:

$$R(X,Y,Z,W) = g\,(R(Z,W)Y,X),$$

where $R\,(X,Y)Z = \left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\right)Z$; $X,Y,Z,W \in T_p(M)$ and satisfies the next properties:

1)$R(X,Y,Z,W) = -R(Y,X,Z,W)$;

2)$R(X,Y,Z,W) = -R(X,Y,W,Z)$;

3)$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$;

4)$R(X,Y,Z,W) = R(Z,W,X,Y)$.

The components of Rieman-Christoffel tensor of NC-manifold are given in Lemma below.

Lemma 2.1 [7]: The components of the Rieman-Christoffel tensor of NC- manifold are given by the following forms:

1) $R_{abcd} = -2B_{ab[cd]}$;

2) $R_{\hat{a}bcd} = 0$;

3)$R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;

4) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d}$;

5) $R_{\hat{a}0b0} = 2C^{ac}C_{bc}$,

The other components of Rieman-Christoffel tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.2 [17]. A Ricci tensor is a tensor of type (2, 0) which is defined by

$$r_{ij=} R^{k}_{ijk} = g^{kl}R_{kijl} \ .$$

Definition 2.3 [8]: An NC-manifold has Φ- invariant Ricci tensor when

$$\Phi\mathrm{a}r = \mathrm{ra}\Phi\mathrm{a}$$

Lemma 2.2 [8]: An NC-manifold has Φ-invariant Ricci tensor if, and only if, in the G-adjoined structure space the following condition $r^{\hat{a}}_{b} = 0$ holds.

Definition 2.4 [18]: A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where e is cosmological constant.

The main results

Definition 3.1[19]: The Wyle's tensor W is a tensor of rank (4.0) which is defined on Riemannian manifold by the form:

$$W_{ijkl} = R_{ijkl} + \frac{1}{2(n-1)}\left(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il}\right) - \frac{\mathcal{K}}{2n(2n-1)}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right)$$

Now, we can redefine the Wyle's tensor on AC-manifold of odd-dimensional by the components form as follows:

$$W_{ijkl} = R_{ijkl} + \frac{1}{2n-1}\left(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il}\right) - \frac{\mathcal{K}}{2n(2n-1)}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) \quad (3.1)$$

Theorem 3.1: In the G adjoined structure space, the components of the Wyle tensor of NC-manifold are given by the following forms:

1) $W_{abcd} = -2B_{ab[cd];}$

2) $W_{\hat{a}bcd} = 0;$

3) $W_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd} + \frac{1}{2n-1}\left(r_{\hat{a}c}\delta_d^b + r_{\hat{b}d}\delta_c^a - r_{\hat{a}d}\delta_c^b - r_{\hat{b}c}\delta_d^a\right) + \frac{K}{2n(2n-1)}\left(\delta_d^a\delta_c^b - \delta_c^a\delta_d^b\right);$

4) $W_{\hat{a}bc\hat{d}} = A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} + \frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d + r_{b\hat{d}}\delta_c^a\right) - \frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right);$

5) $W_{\hat{a}0b0} = C^{ac}C_{bc} + \frac{1}{2n-1}(r_{\hat{a}b} - 2C^{cd}C_{cd}\delta_b^a.$

And the others are conjugate to the above components or equal to zero.

Proof:

By using the equations (3.1) and Lemma 2.1, directly we obtain the above components.

Remark 3.1[20]: On the space of the G-structure, the identities $CR_1 - CR_3$ are equivalent to the relations:

$$CR_1 \leftrightarrow R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0$$

$$CR_2 \leftrightarrow R_{\hat{a}bcd} = R_{abcd} = 0$$

$$CR_3 \leftrightarrow R_{\hat{a}bcd} = 0$$

$$CR_1 \subset CR_2 \subset CR_3$$

Definition 3.1: In the adjoined G-structure space, an AC −manifold is a manifold of class:

$$WR_1 \leftrightarrow W_{\hat{a}bcd} = W_{abcd} = W_{\hat{a}\hat{b}cd} = 0$$

$$WR_2 \leftrightarrow W_{\hat{a}bcd} = W_{abcd} = 0$$

$$WR_3 \leftrightarrow W_{\hat{a}bcd} = 0$$

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$$WR_1 \subset WR_2 \subset WR_3$$

Theorem 3.2: Any weakly cosymplectic manifold is a sub manifold of class WR_3.

Proof:

Let M be NC-manifold,

Since $W_{\hat{a}bcd} = 0$, this implies that M is a structure of class WR_3.

Theorem 3.3: Let *M* be AC-manifold withϕ-invariant Ricci tensor and vanishing wyle tensor then:

1) If M has vanishing Ricci tensor , then M A^{ad}_{ad} is constant.

2) If M has vanishing A^{ad}_{bc} them M is Einstein manifold.

Proof : Since M have vanishing Wyle tensor,

$$A^{ad}_{bc} - B^{adh}B_{hbc} - \tfrac{5}{3}C^{ad}C_{bc} + \tfrac{1}{2n-1}\left(r_{\hat{a}c}\delta^d_b + r_{b\hat{d}}\delta^a_c\right) - \tfrac{K}{2n(2n-1)}\left(\delta^a_c\delta^d_b\right) = 0 \tag{3.2}$$

1) Let M has vanishing Ricci tensor then the equation (3.2) becomes

$$A^{ad}_{bc} - B^{adh}B_{hbc} - \tfrac{5}{3}C^{ad}C_{bc} - \tfrac{K}{2n(2n-1)}\left(\delta^a_c\delta^d_b\right) = 0 \tag{3.3}$$

By symmetrizing(a, b), then we get

$$A^{ad}_{bc} = \frac{K}{2n(2n-1)}\left(\delta^a_c\delta^d_b\right)$$

By Contracting(a, b), and (d, c), we have

$$A^{ad}_{ad} = \frac{K}{2(2n-1)}$$

2) Let M has vanishing A^{ad}_{bc} , then the equation (3.2) becomes

$$-B^{adh}B_{hbc}-\frac{5}{3}C^{ad}C_{bc}+\frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d+r_{b\hat{d}}\delta_c^a\right)-\frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right)=0$$

By symmetric and anti-symmetric(a,d), we have

$$\frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d+r_{b\hat{d}}\delta_c^a\right)-\frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right)=0$$

By contracting (a,b) , we get

$$r_c^d=e\delta_c^d, \text{where } e=\frac{K}{4n}$$

Since M with ϕ-invariant Ricci tensor and by using The definition 2.4, then M is Einstein manifold.

Theorem 3.4: Suppose that M is a vanishing Wyle nearly cosymplectic manifold with ϕ invariant Ricci tensor, then the necessary and sufficient condition in which M is an Einstein manifold if

$$A=\frac{5}{3}C^{ad}C_{bd}+C_0\delta_b^a.$$

Proof: Since M have vanishing Wyle tensor,

$$A_{bc}^{ad}-B^{adh}B_{hbc}-\frac{5}{3}C^{ad}C_{bc}+\frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d+r_{b\hat{d}}\delta_c^a\right)-\frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right)=0$$

By symmetric and anti-symmetric(h,d), we have

$$A_{bc}^{ad}-\frac{5}{3}C^{ad}C_{bc}+\frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d+r_{b\hat{d}}\delta_c^a\right)-\frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right)=0 \qquad (3.4)$$

Let M be Einstein manifold, then we get

$$A_{bc}^{ad}-B^{adh}B_{hbc}-\frac{5}{3}C^{ad}C_{bc}+\frac{e}{2n-1}\left(\delta_c^a\delta_b^d+\delta_b^d\delta_c^a\right)-\frac{K}{2n(2n-1)}\left(\delta_c^a\delta_b^d\right)=0 \qquad (3.5)$$

By contracting the equation (3.4) by the indices (d,c), we obtain

$$A=\frac{5}{3}C^{ad}C_{bd}+C_0\delta_b^a \qquad (3.6)$$

Conversely

Let $A=\frac{5}{3}C^{ad}C_{bd}+C_0\delta_b^a$

By contracting the equation (3.4) by the indices (d,c), we obtain

$$A_{bd}^{ad}-\frac{5}{3}C^{ad}C_{bd}+\frac{2}{2n-1}(r_{\hat{a}b})-\frac{K}{2n(2n-1)}(\delta_b^a)=0 \qquad (3.7)$$

From the equations 3.6 and 3.7 we get

$$r_{\hat{a}b} = e\delta_b^a$$

Since M has ϕ-invariant Ricci tensor, then M is Einstein manifold.

Theorem 3.5: If M is a manifold of class WR_2, then the first structure tensor is parallel in the first canonical connection.

Proof : Suppose that M is a manifold of class WR_2, then from the definition 1 we have

$$W_{\hat{a}bcd} = W_{abcd} = 0$$

From The Theorem 3.1, we obtain

$$B_{abcd} = 0$$

According the fundamental theorem of analysis we have :

$$\nabla B_{abc} = dB_{abc} + B_{dbc}W_a^d + B_{adc}W_b^d + B_{abd}W_c^d = B_{abcd}W^d$$

So we obtain

$$\nabla B_{abc} = B_{abcd}W^d$$

Then we get

$$\nabla B_{abc} = 0$$

Which means the tensor B_{abc} is parallel in the connection if its covariant derivative is equal to zero.

Theorem 3.6: Let M be NC - manifold.. Then the classes CR_3 and WR_3 are coincide if and only if, M has ϕ-invariant Ricci tensor.

Proof : Suppose that CR_3 and WR_3 are coincide

Taking into account the lemmas 1 and 2 , we have

$$\frac{1}{2n-1}\left(r_{\hat{a}c}\delta_b^d + r_{b\hat{a}}\delta_c^a\right) = 0$$

By contracting the above equation by the indices (a,d), we get

$$r_{bc} = 0$$

Conversely

Suppose that M ϕ-invariant Ricci tensor

Making use of Theorem 3.1and Lemma 2.1 , it follows that

$$W_{\hat{a}bcd} = R_{\hat{a}bcd}$$

Therefore, the classes CR_3 and WR_3 are coincide.

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