Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: Aqeel Falih Jaddoa 2021 *J. Phys.: Conf. Ser.* **1897** 012047

On Oscillatory to Nonlinear Impulsive Differential Equation of Second-Order with Damping Term

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Abstract

The paper focuses on the sufficient conditions for oscillatory property to the nonlinear impulsive differential equation(IDE) of the second order with damping term. We submitted a suitable impulsive conditions with the nonlinear equation. The obtained new results in this article generalize and extend some modification results to linear (IDEs) with damping term in the references. We have been given illustrative example to apply the conditions in the main results.

1. Introduction

Differential equations theory is considered effective tool in modeling many phenomena and continuous processes in optimal control, population dynamics, industrial robots [1-5] and so forth. Researchers have searched for many and various methods of finding solutions and stability to differential equations, whether with fractional order [6-11]. Moreover, the researchers studied the properties that characterize all solutions with their behavior, and delved into the extraction of those features of the solutions, analyzing them, giving the scientific meaning to them, and providing recommendations and observations.

We also know that many scientific phenomena are not always in the form of continuous processes, but there are moments of change in the behavior of the solutions for certain periods, as this phenomenon is called impulses. Thus, the (IDEs) are more realistic to represent the phenomenon or scientific issue as a mathematical formula [12].

Although these periods are in the form of a few moments, they are not neglected. Rather, conditions are set for them that are compatible with the issue. Therefore, we note that there is great importance in study of impulsive differential equations and extracting properties to solutions for this type of equations.

Many authors are investigated qualitative features to solutions of differential equations with impulses effect such as asymptotic behavior [13-15], oscillation criteria [16-19] and stability [20,21].

The importance of applications of impulsive differential equations with damping term, it paid attention the authors to published several papers such as Thandapani, Kannan and Pinelas [22] presented the sufficient conditions to oscillation to forced delay (IDE) with damping term. Zeng; Wen; Peng and Huang [23] investigated the oscillatory property to second order linear (IDE) with damping term.

Our research is to consider oscillation criteria to nonlinear (IDE) in form:

$$\left.\begin{array}{c} \Upsilon''(\varkappa)+\mathfrak{f}(\varkappa)\Upsilon'(\varkappa)+\mathfrak{g}(\varkappa)\hbar(\Upsilon(\varkappa))=0,\quad \varkappa\neq\varkappa_\iota \quad \iota=1,2,\dots \\ \mathfrak{a}_\iota\,\Delta\Upsilon'(\varkappa_\iota)+\mathfrak{b}_\iota\,\hbar\big(\Upsilon(\varkappa_\iota)\big)=0,\qquad \varkappa=\varkappa_\iota \end{array}\right\} \tag{1}$$

where $\mathfrak{a}_\iota, \mathfrak{b}_\iota$ are sequences of positive real numbers , $\varkappa_\iota$ are the moments of impulses effect.

Let $\mathfrak{f}(\varkappa), \mathfrak{g}(\varkappa) \in PC([0,\infty);\mathcal{R})$, $\hbar \in C(\mathcal{R},\mathcal{R})$, $\Upsilon\hbar(\Upsilon) > 0$ and $\hbar'(\Upsilon(\varkappa)) \geq \eta > 0$.

The $\mathfrak{f}(\varkappa), \mathfrak{g}(\varkappa)$ are piecewise continuous functions from the left and $\mathfrak{g}(\varkappa)$ is the damping function. We define $\mu(\varkappa)$ as the moments to impulses effect in $[\varkappa_0, \varkappa)$ with $\mu(\varkappa) = \begin{cases} \iota & ,\varkappa_\iota < \varkappa \leq \varkappa_{\iota+1} \\ 0 & ,\varkappa_0 \leq \varkappa \leq \varkappa_1 \end{cases}$.

Definition 1.1 [23]: Let $\Upsilon(\varkappa) = \Upsilon(\varkappa;\, \varkappa_0)$ be the solution of equation (1), then $\Upsilon(\varkappa)$ is characterized by the following:

(a) If $\varkappa \in (\varkappa_\iota, \varkappa_{\iota+1}]$, $\Upsilon(\varkappa)$ satisfies the first equation in (1).

(b) If $\varkappa = \varkappa_\iota$, $\Upsilon(\varkappa)$ satisfies the impulsive condition of (1).

(c) $\Upsilon'(\varkappa)$ has two side limits with left continuous in impulsive points, where $\Upsilon'(\varkappa_\iota)$ satisfies the impulsive condition of (1)

Definition 1.2 [23]: If $\Upsilon(\varkappa)$is a nontrivial solution to eq. (1), then $\Upsilon(\varkappa)$ achieves the non-oscillatory feature if it satisfies eventually positive feature or eventually negative. Otherwise, $\Upsilon(\varkappa)$ is oscillatory. The eq. (1) verifies oscillation feature if all nontrivial solutions are oscillatory.

2. Main Results

This section includes some new results to secure sufficient conditions for oscillatory feature for eq. (1).

Theorem 2.1: Let $\mathfrak{f}(\varkappa) > 0$ and $\phi: \mathcal{L} \to \mathcal{R}^+$ be continuous function, where $\mathcal{L} = \{(\varkappa, \xi): \varkappa \geq \xi \geq \varkappa_0\}$. Assume that $\phi(\varkappa, \xi)$ has continuous with non-positive partial derivative on ϕ for the variable ξ. Let $\varphi: \mathcal{L} \to \mathcal{R}^+$ be continuous function such that:

$$-\frac{\partial}{\partial\xi}\phi(\varkappa,\xi) = \varphi(\varkappa,\xi)\sqrt{\phi(\varkappa,\xi)},\quad \varkappa,\xi \in \mu \tag{2}$$

$$\text{And } \phi(\varkappa,\varkappa) = 0, \varkappa \geq \varkappa_0,\ \phi(\varkappa,\varkappa) > 0,\ \varkappa \geq \xi \geq \varkappa_0 \tag{3}$$

$$\lim_{\varkappa\to\infty}\sup\frac{1}{\phi(\varkappa,\varkappa_0)} \times \left\{\int_{\varkappa_0}^{\varkappa}\left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right)d\xi + \sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)\right\} = \infty \tag{4}$$

Then eq. (1) has oscillatory property.

Proof. Suppose that $\Upsilon(\varkappa)$ is nonoscillatory, so there is a $\mathcal{T} \geq \varkappa_0$ such that $\Upsilon(\varkappa) \neq 0, \varkappa \geq \mathcal{T}$.

We define the Riccati Transform as:

$$\mathcal{S}(\varkappa) = \frac{\Upsilon'(\varkappa)}{\hbar(\Upsilon(\varkappa))},\qquad \varkappa \geq \mathcal{T}$$

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$$\mathcal{S}'(\varkappa) = \frac{\Upsilon''(\varkappa)\,\hbar(\Upsilon(\varkappa)) - (\Upsilon'(\varkappa))^2 \hbar'(\Upsilon(\varkappa))}{\hbar^2(\Upsilon(\varkappa))}$$

$$\mathcal{S}'(\varkappa) = \frac{\Upsilon''(\varkappa)}{\hbar(\Upsilon(\varkappa))} - \left(\frac{\Upsilon'(\varkappa)}{\Upsilon(\varkappa)}\right)^2 \hbar'(\Upsilon(\varkappa))$$

So, by the first equation of (1):

$$\mathcal{S}'(\varkappa) = \frac{-\mathfrak{f}(\varkappa)\Upsilon'(\varkappa) - \mathfrak{g}(\varkappa)\hbar(\Upsilon(\varkappa))}{\hbar(\Upsilon(\varkappa))} - \left(\frac{\Upsilon'(\varkappa)}{\Upsilon(\varkappa)}\right)^2 \hbar'(\Upsilon(\varkappa))$$
$$\leq -\mathfrak{f}(\varkappa)\mathcal{S}(\varkappa) - \mathfrak{g}(\varkappa) - \eta\mathcal{S}^2(\varkappa)$$

$\mathcal{S}'(\varkappa) + \mathfrak{f}(\varkappa)\mathcal{S}(\varkappa) + \eta\mathcal{S}^2(\varkappa) \leq -\mathfrak{g}(\varkappa)$, for $\varkappa \neq \varkappa_\iota$.

When $\varkappa = \varkappa_\iota$ then:

$$\Delta\mathcal{S}(\varkappa_\iota) = \mathcal{S}(\varkappa_\iota^+) - \mathcal{S}(\varkappa_\iota^-)$$

$$= \frac{\Upsilon'(\varkappa_\iota^+)}{\hbar(\Upsilon(\varkappa_\iota^+))} - \frac{\Upsilon'(\varkappa_\iota^-)}{\hbar(\Upsilon(\varkappa_\iota^-))}$$

But the function $\hbar$ is continuous on $[\varkappa_0, \infty)$

$$= \frac{\Upsilon'(\varkappa_\iota^+) - \Upsilon'(\varkappa_\iota^-)}{\hbar(\Upsilon(\varkappa_\iota))}$$

$$\Delta\mathcal{S}(\varkappa_\iota) = \frac{\Delta\Upsilon'(\varkappa_\iota)}{\hbar(\Upsilon(\varkappa_\iota))} = \frac{-\mathfrak{b}_\iota}{\mathfrak{a}_\iota}$$

The transformation $\mathcal{S}(\varkappa)$ satisfies:

$$\left.\begin{array}{ll} \mathcal{S}'(\varkappa) + \mathfrak{f}(\varkappa)\mathcal{S}(\varkappa) + \eta\mathcal{S}^2(\varkappa) \leq -\mathfrak{g}(\varkappa), & \varkappa \neq \varkappa_\iota \\ \Delta\mathcal{S}(\varkappa_\iota) = \dfrac{-\mathfrak{b}_\iota}{\mathfrak{a}_\iota} & \end{array}\right\} \tag{5}$$

by choosing positive integer n with $\varkappa_{n-1} \leq \mathcal{T} < \varkappa_n$. Let $m = i(\varkappa)$ for large enough $\varkappa$ that means $\varkappa_m \leq \varkappa < \varkappa_{m+1}$. Assuming that $\mathcal{J}$ denotes to the set of interval $[\mathcal{T}, \varkappa]$ except the points $\varkappa_n, \varkappa_{n+1}, \dots, \varkappa_m$, so it can see of the first equation of (5):

$$-\int_{\mathcal{T}}^{\varkappa} \mathfrak{g}(\xi)\phi(\varkappa,\xi)d\xi = -\int_{\mathcal{J}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\,d\xi$$
$$\geq \int_{\mathcal{J}} \phi(\varkappa,\xi)\,\mathcal{S}'(\xi)d\xi + \eta\int_{\mathcal{J}} \phi(\varkappa,\xi)\,\mathcal{S}^2(\xi)d\xi$$
$$+ \int_{\mathcal{J}} \mathfrak{f}(\xi)\phi(\varkappa,\xi)\,\mathcal{S}(\xi)d\xi \tag{6}$$

Since $\mathcal{S}(\varkappa)$ is a piecewise continuous with continuity of ϕ, we see that:

$$\phi(\varkappa,\varkappa)\mathcal{S}(\varkappa) - \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) = \phi(\varkappa,\varkappa)\mathcal{S}(\varkappa) - \phi(\varkappa,\varkappa_\iota^+)\mathcal{S}(\varkappa_\iota^+) + \phi(\varkappa,\varkappa_\iota^+)\mathcal{S}(\varkappa_\iota^+) - \phi(\varkappa,\varkappa_\iota^-)\mathcal{S}(\varkappa_\iota^-)$$
$$+\phi(\varkappa,\varkappa_\iota^-)\mathcal{S}(\varkappa_\iota^-) - \phi(\varkappa,\varkappa_{\iota-1}^+)\mathcal{S}(\varkappa_{\iota-1}^+) + \phi(\varkappa,\varkappa_{\iota-1}^+)\mathcal{S}(\varkappa_{\iota-1}^+) - \phi(\varkappa,\varkappa_{\iota-1}^-)\mathcal{S}(\varkappa_{\iota-1}^-) + \phi(\varkappa,\varkappa_{\iota-1}^-)\mathcal{S}(\varkappa_{\iota-1}^-)$$
$$-\phi(\varkappa,\varkappa_{\iota-2}^+)\mathcal{S}(\varkappa_{\iota-2}^+) + \cdots + \phi(\varkappa,\varkappa_\iota^+)\mathcal{S}(\varkappa_\iota^+)$$

$$-\phi(\varkappa,\varkappa_\iota^-)\mathcal{S}(\varkappa_\iota^-)+\phi(\varkappa,\varkappa_\iota^-)\mathcal{S}(\varkappa_\iota^-)\ -\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})$$

$$=\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi+\sum_{\iota=n}^{\mu(\varkappa)}\big(\phi(\varkappa,\varkappa_\iota^+)\mathcal{S}(\varkappa_\iota^+)\ -\phi(\varkappa,\varkappa_\iota^-)\mathcal{S}(\varkappa_\iota^-)\big)$$

$$=\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi+\sum_{\iota=n}^{\mu(\varkappa)}\phi(\varkappa,\varkappa_\iota)\big(\mathcal{S}(\varkappa_\iota^+)\ -\mathcal{S}(\varkappa_\iota^-)\big)$$

$$=\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi+\sum_{\iota=n}^{\mu(\varkappa)}\phi(\varkappa,\varkappa_\iota)\Delta\mathcal{S}(\varkappa_\iota)$$

$$=\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi-\sum_{\iota=n}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)\tag{7}$$

$\phi(\varkappa,\varkappa)=0$, and by (7):

$$\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi=-\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})+\sum_{\iota=n}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)\tag{8}$$

by (6) and (8), and the relationship of $\phi(\varkappa,\xi)$ and $\varphi(\varkappa,\xi)$, we have

$$-\int_{\mathcal{J}}\mathfrak{g}(\xi)\phi(\varkappa,\xi)\,d\xi\geq\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\big(\phi(\varkappa,\xi)\mathcal{S}(\xi)\big)\,d\xi-\int_{\mathcal{J}}\frac{\partial}{\partial\xi}\phi(\varkappa,\xi)\mathcal{S}(\xi)\,d\xi$$

$$+\eta\int_{\mathcal{J}}\phi(\varkappa,\xi)\mathcal{S}^2(\xi)\,d\xi+\int_{\mathcal{J}}\mathfrak{f}(\xi)\phi(\varkappa,\xi)\mathcal{S}(\xi)\,d\xi$$

$$=-\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})+\sum_{\iota=n}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)+\int_{\mathcal{J}}\varphi(\varkappa,\xi)\sqrt{\phi(\varkappa,\xi)}\mathcal{S}(\xi)\,d\xi+\eta\int_{\mathcal{J}}\phi(\varkappa,\xi)\mathcal{S}^2(\xi)\,d\xi$$
$$+\int_{\mathcal{J}}\mathfrak{f}(\xi)\phi(\varkappa,\xi)\mathcal{S}(\xi)\,d\xi$$

$$=-\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})+\sum_{\iota=n}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)$$
$$+\int_{\mathcal{J}}\Big(\varphi(\varkappa,\xi)\sqrt{\phi(\varkappa,\xi)}\mathcal{S}(\xi)+\eta\phi(\varkappa,\xi)\mathcal{S}^2(\xi)+\mathfrak{f}(\xi)\phi(\varkappa,\xi)\mathcal{S}(\xi)\Big)\,d\xi$$

$$=-\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})+\sum_{\iota=n}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)$$
$$+\int_{\mathcal{J}}\Big(\Big(\varphi(\varkappa,\xi)+\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\Big)\sqrt{\phi(\varkappa,\xi)}\mathcal{S}(\xi)+\eta\phi(\varkappa,\xi)\mathcal{S}^2(\xi)\Big)\,d\xi$$

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$$
\begin{aligned}
&= -\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&\qquad + \int_{\mathcal{J}} \left(\frac{-1}{4\eta}\left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2\right. \\
&\qquad \left. + \left[\frac{1}{2\sqrt{\eta}}\left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right) + \sqrt{\eta}\sqrt{\phi(\varkappa,\xi)}\mathcal{S}(\xi)\right]^2\right) d\xi.
\end{aligned}
$$

So, we have:

$$
\begin{aligned}
&-\int_{\mathcal{J}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\, d\xi \\
&\qquad \geq -\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&\qquad + \int_{\mathcal{J}} \left(\frac{-1}{4\eta}\left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2\right. \\
&\qquad \left. + \left[\frac{1}{2\sqrt{\eta}}\left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right) + \sqrt{m}\sqrt{\phi(\varkappa,\xi)}\mathcal{S}(\xi)\right]^2\right) d\xi
\end{aligned}
$$

$$
\geq -\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) - \frac{1}{4\eta}\int_{\mathcal{J}} \left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2 d\xi.
$$

$$
\begin{aligned}
&-\int_{\mathcal{J}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\, d\xi + \frac{1}{4\eta}\int_{\mathcal{J}} \left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2 d\xi - \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&\qquad \geq -\phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) \qquad (9)
\end{aligned}
$$

$$
\int_{\mathcal{J}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\, d\xi - \frac{1}{4\eta}\int_{\mathcal{J}} \left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2 d\xi + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \leq \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})
$$

$$
\begin{aligned}
&\int_{\mathcal{J}} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&\qquad \leq \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}) \qquad (10)
\end{aligned}
$$

so, we have:

$$
\begin{aligned}
&\int_{\varkappa_0}^{\varkappa} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&= \int_{\varkappa_0}^{\mathcal{T}} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \\
&+ \int_{\mathcal{T}}^{\varkappa} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=n}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)
\end{aligned}
$$

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$$\leq \int_{\varkappa_0}^{\mathcal{T}} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) + \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})$$

$$= \int_{\varkappa_0}^{\mathcal{T}} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi(\varkappa,\xi) + \mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}\right)^2\right) d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) + \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T})$$

$$\leq \int_{\varkappa_0}^{\mathcal{T}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\, d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) + \phi(\varkappa,\mathcal{T})\mathcal{S}(\mathcal{T}), \text{since } \mathfrak{f}(\varkappa) > 0$$

$$\leq \int_{\varkappa_0}^{\mathcal{T}} \mathfrak{g}(\xi)\phi(\varkappa,\xi)\, d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) + \phi(\varkappa,\mathcal{T})|\mathcal{S}(\mathcal{T})|$$

$$\leq \phi(\varkappa,\varkappa_0)\left(\int_{\varkappa_0}^{\mathcal{T}} |\mathfrak{g}(\xi)|\, d\xi + \sum_{\iota=1}^{n-1} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota} + |\mathcal{S}(\mathcal{T})|\right)$$

the last inequality held by decreasing of $\phi(\varkappa,\xi)$ with respect to ξ.
So, we conclude that:

$$\lim_{\varkappa\to\infty} \sup \frac{1}{\phi(\varkappa,\varkappa_0)} \times \left\{ \int_{\varkappa_0}^{\varkappa} \left(\mathfrak{g}(\xi)\phi(\varkappa,\xi) - \frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi) + 2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)} + \mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right) d\xi + \sum_{\iota=1}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota) \right\} < \infty,$$

Which is a contradiction with condition (4). □

In the following result, we proposed two conditions and use the problem (5) to ensure the oscillatory property.

Theorem 2.2 Assume that, for $\varrho > 1, 0 \leq \vartheta < 1$,

$$\lim_{\varkappa\to\infty} \sup \frac{1}{\varkappa^{\varrho}} \left\{ \int_{\varkappa_0}^{\varkappa} \mathfrak{g}(\xi)(\varkappa-\xi)^{\varrho}\xi^{\vartheta} d\xi + \sum_{\iota=1}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}{\varkappa_\iota}^{\vartheta} \right\} = \infty \tag{11}$$

and

$$\lim_{\varkappa\to\infty} \sup \frac{1}{\varkappa^{\alpha}} \left\{ \int_{\varkappa_0}^{\varkappa} (\varrho\xi - \vartheta(\varkappa-\xi) + \mathfrak{f}(\xi)u(\varkappa-\xi))^2 (\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2} d\xi \right\} < \infty \tag{12}$$

Then eq. (1) has oscillation property.

Proof:

Let $\Upsilon(\varkappa)$ be a non-oscillatory, then there is a $\mathcal{T} \geq \varkappa_0$ with $\Upsilon(\varkappa) \neq 0$ for $\varkappa \geq \mathcal{T}$.

Therefore the Riccati transform $\mathcal{S}(\varkappa)$ satisfies (5):

$$\mathcal{S}'(\varkappa) + \mathfrak{f}(\varkappa)\mathcal{S}(\varkappa) + \eta\mathcal{S}^2(\varkappa) \le -\mathfrak{g}(\varkappa),\ \varkappa \neq \varkappa_\iota$$
$$\Delta\mathcal{S}(\varkappa_\iota) = \frac{-\mathfrak{b}_\iota}{\mathfrak{a}_\iota}$$

We multiply first inequality of (5) by $(\varkappa-\xi)^\varrho\xi^\vartheta$ with integrating it from $\varkappa_0$ to $\varkappa$:

$$-\int_{\varkappa_0}^{\varkappa} \mathfrak{g}(\xi)(\varkappa-\xi)^\varrho u^\vartheta d\xi$$
$$\ge \int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}'(\xi)d\xi + \int_{\varkappa_0}^{\varkappa} \mathfrak{f}(\xi)(\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}(\xi)d\xi$$
$$+\eta\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}^2(\xi)d\xi \qquad (13)$$

But

$$\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}'(\xi)d\xi$$
$$= (\varkappa-\varkappa_0)^\varrho\varkappa_0^\vartheta\mathcal{S}(\varkappa_0) + \sum_{\iota=1}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^\varrho\varkappa_\iota^\vartheta + \varrho\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}(\xi)d\xi$$
$$-\vartheta\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^{\vartheta-1}\mathcal{S}(\xi)d\xi \qquad (14)$$

Now put (14) in (13), we get:

$$-\int_{\varkappa_0}^{\varkappa} \mathfrak{g}(\xi)(\varkappa-\xi)^\varrho\xi^\vartheta d\xi$$
$$\ge (\varkappa-\varkappa_0)^\varrho\varkappa_0^\vartheta\mathcal{S}(\varkappa_0) + \sum_{\iota=1}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^\varrho\varkappa_\iota^\vartheta + \varrho\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}(\xi)d\xi$$
$$-\vartheta\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^{\vartheta-1}\mathcal{S}(\xi)d\xi + \int_{\varkappa_0}^{\varkappa} \mathfrak{f}(\xi)(\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}(\xi)d\xi$$
$$+\eta\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}^2(\xi)d\xi$$

$$= (\varkappa-\varkappa_0)^\varrho\varkappa_0^\vartheta\mathcal{S}(\varkappa_0) + \sum_{\iota=1}^{\mu(\varkappa)} \frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^\varrho\varkappa_\iota^\vartheta$$
$$+\int_{\varkappa_0}^{\varkappa} (\varrho\xi - \vartheta(\varkappa-\xi) + \mathfrak{f}(\xi)\xi(\varkappa-\xi))(\varkappa-\xi)^{\varrho-1}\xi^{\vartheta-1}\mathcal{S}(\xi)d\xi$$
$$+\eta\int_{\varkappa_0}^{\varkappa} (\varkappa-\xi)^\varrho\xi^\vartheta\mathcal{S}^2(\xi)d\xi$$

$$= (\varkappa-\varkappa_0)^{\varrho}\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}\varkappa_\iota^{\vartheta}$$
$$-\frac{1}{4\eta}\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi$$
$$+\int_{\varkappa_0}^{\varkappa}\left\{\sqrt{\eta}(\varkappa-\xi)^{\frac{\varrho}{2}}\xi^{\frac{\vartheta}{2}}\mathcal{S}(\xi)\right.$$
$$\left.+\frac{1}{2\sqrt{\eta}}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))(\varkappa-\xi)^{\frac{\varrho-2}{2}}\xi^{\frac{\vartheta-2}{2}}\right\}^2d\xi$$

From above, we get:

$$\int_{\varkappa_0}^{\varkappa}\mathfrak{g}(\xi)(\varkappa-\xi)^{\varrho}\xi^{\vartheta}d\xi$$
$$\leq-(\varkappa-\varkappa_0)^{\varrho}\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)-\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}\varkappa_\iota^{\vartheta}$$
$$+\frac{1}{4\eta}\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi$$
$$-\int_{t_0}^{t}\left\{\sqrt{\eta}(\varkappa-\xi)^{\frac{\varrho}{2}}\xi^{\frac{\vartheta}{2}}\mathcal{S}(\xi)\right.$$
$$\left.+\frac{1}{2\sqrt{\eta}}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))(\varkappa-\xi)^{\frac{\varrho-2}{2}}\xi^{\frac{\vartheta-2}{2}}\right\}^2d\xi$$
$$\leq-(\varkappa-\varkappa_0)^{\varrho}\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)-\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}\varkappa_\iota^{\vartheta}$$
$$+\frac{1}{4\eta}\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi.$$
$$\int_{\varkappa_0}^{\varkappa}\mathfrak{g}(\xi)(\varkappa-\xi)^{\varrho}\xi^{\vartheta}d\xi+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}\varkappa_\iota^{\vartheta}$$
$$\leq-(\varkappa-\varkappa_0)^{\varrho}\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)$$
$$+\frac{1}{4\eta}\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi.$$

By dividing the last inequality by $\varkappa^{\alpha}$ and take $\lim_{\varkappa\to\infty}$ sup for two sides:

$$\lim_{\varkappa\to\infty}\sup\frac{1}{\varkappa^{\varrho}}\left\{\int_{\varkappa_0}^{\varkappa}\mathfrak{g}(\xi)(\varkappa-\xi)^{\varrho}\xi^{\vartheta}d\xi+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(\varkappa-\varkappa_\iota)^{\varrho}{\varkappa_\iota}^{\vartheta}\right\}$$
$$\leq\lim_{\varkappa\to\infty}\sup\frac{-(\varkappa-\varkappa_0)^{\varrho}\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)}{\varkappa^{\varrho}}$$
$$+\frac{1}{4\eta}\lim_{\varkappa\to\infty}\sup\frac{1}{\varkappa^{\varrho}}\left\{\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi\right\}$$
$$=\lim_{\varkappa\to\infty}\sup\ -(1-\frac{\varkappa_0}{\varkappa})^{\varrho}\,\varkappa_0^{\vartheta}\mathcal{S}(\varkappa_0)$$
$$+\frac{1}{4\eta}\lim_{\varkappa\to\infty}\sup\frac{1}{\varkappa^{\varrho}}\left\{\int_{\varkappa_0}^{\varkappa}(\varrho\xi-\vartheta(\varkappa-\xi)+\mathfrak{f}(\xi)\xi(\varkappa-\xi))^2(\varkappa-\xi)^{\varrho-2}\xi^{\vartheta-2}d\xi\right\}<\infty$$

Which is a contradiction with condition (11). □

Corollary 2.1 Let $\phi(\varkappa,\xi)=(1-\frac{\xi}{\varkappa})^{2\varrho},\varrho>1$ and:

$$\limsup_{\varkappa\to\infty}\left\{\int_{\varkappa_0}^{\varkappa}\left((1-\frac{\xi}{\varkappa})^{\varrho}\mathfrak{g}(\xi)-\frac{1}{4\eta}\left(\frac{2\varrho}{\varkappa}(1-\frac{\xi}{\varkappa})^{\varrho-1}\mathfrak{f}(\xi)+(1-\frac{\xi}{\varkappa})^{\varrho}\mathfrak{f}^2(\xi)\right)\right)d\xi+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}=\infty \tag{15}$$

Then equation (1) is oscillatory.

Proof: Suppose that $\phi(\varkappa,\xi)=(1-\frac{\xi}{\varkappa})^{2\varrho},\varrho>1$. Then $\varphi(\varkappa,\xi)=\frac{2\varrho}{\varkappa}(1-\frac{\xi}{\varkappa})^{\varrho-1}$,
The functions ϕ and φ verify the conditions (2) and (3) in theorem (2.1).

Furthermore,

$$\limsup_{\varkappa\to\infty}\frac{1}{\phi(\varkappa,\varkappa_0)}\int_{\varkappa_0}^{\varkappa}\varphi^2(\varkappa,\xi)\,d\xi$$
$$=\limsup_{\varkappa\to\infty}\left\{\frac{1}{(1-\frac{\varkappa_0}{\varkappa})^{2\varrho}}\int_{\varkappa_0}^{\varkappa}\frac{4\varrho^2}{\varkappa^2}(1-\frac{\xi}{\varkappa})^{2\varrho-2}\,d\xi\right\}$$
$$=\limsup_{\varkappa\to\infty}\frac{4\varrho^2(1-\frac{\varkappa_0}{\varkappa})^{2\varrho-1}}{(2\varrho-1)\varkappa(1-\frac{\varkappa_0}{\varkappa})^{2\varrho}}$$
$$=\limsup_{\varkappa\to\infty}\frac{4\varrho^2}{(2\varrho-1)(\varkappa-\varkappa_0)}=0$$
$$\frac{1}{\phi(\varkappa,\varkappa_0)}=\frac{1}{(1-\frac{\varkappa_0}{\varkappa})^{2\varrho}}\to 1 \text{ when } \varkappa\to\infty$$

So, by applying the condition (4) of theorem (2.1):

$$\limsup_{\varkappa\to\infty}\frac{1}{\phi(\varkappa,\varkappa_0)}\times\left\{\int_{\varkappa_0}^{\varkappa}\left(\mathfrak{g}(\xi)\phi(\varkappa,\xi)-\frac{1}{4\eta}\left(\varphi^2(\varkappa,\xi)+2\varphi(\varkappa,\xi)\mathfrak{f}(\xi)\sqrt{\phi(\varkappa,\xi)}+\mathfrak{f}^2(\xi)\phi(\varkappa,\xi)\right)\right)d\xi+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}\phi(\varkappa,\varkappa_\iota)\right\}$$
$$=\limsup_{\varkappa\to\infty}\left\{\int_{\varkappa_0}^{\varkappa}\left((1-\frac{\xi}{\varkappa})^{2\varrho}\mathfrak{g}(\xi)-\frac{1}{4\eta}\left(\frac{4\varrho}{\varkappa}(1-\frac{\xi}{\varkappa})^{2\varrho-1}\mathfrak{f}(\xi)+(1-\frac{\xi}{\varkappa})^{2\varrho}\mathfrak{f}^2(\xi)\right)\right)d\xi+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}=\infty$$

By theorem (2.1), the eq. (1) is oscillatory. □

In the following example, we applied the conditions in theorem (2.1) to guarantee the oscillation property.

Example (2.1): Suppose that $\phi(\varkappa,\xi),\varphi(\varkappa,\xi)$ are defined as in corollary (2.1) and:

$$\Upsilon''(\varkappa)+\frac{1}{\varkappa^{\varrho+1}}\Upsilon'(\varkappa)+\varkappa^2\hbar(\Upsilon(\varkappa))=0,\quad \varkappa\neq\varkappa_\iota\quad \iota=1,2,\ldots$$
$$\iota\Delta\Upsilon'(\varkappa_\iota)+(\iota+1)\hbar\big(\Upsilon(\varkappa_\iota)\big)=0,\ \ \varkappa=\varkappa_\iota$$

Where $\mathfrak{f}(\varkappa)=\begin{cases}\varkappa, \varkappa\neq\varkappa_\iota\\ 2,\ \ \varkappa=\varkappa_\iota\end{cases}$, $\mathfrak{g}(\varkappa)=\begin{cases}\frac{2}{\eta}\varkappa^2,\ \ \varkappa\neq\varkappa_\iota\\ \frac{1}{3},\ \ \varkappa=\varkappa_\iota\end{cases}$, $\varrho>2,\ \varkappa\geq\xi\geq\varkappa_0\ \geq 1,$ and $\mathfrak{a}_\iota\ =\ \iota,$ $\mathfrak{b}_\iota\ =\iota+1$. By applying the condition (15):

$$\lim_{\varkappa\to\infty}\sup\left\{\int_{\varkappa_0}^{\varkappa}\left((1-\frac{\xi}{\varkappa})^{2\varrho}\mathfrak{g}(\xi)-\frac{1}{4\eta}\left(\frac{4\varrho}{\varkappa}(1-\frac{\xi}{\varkappa})^{2\varrho-1}\mathfrak{f}(\xi)+(1-\frac{\xi}{\varkappa})^{2\varrho}\,\mathfrak{f}^2(\xi)\right)\right)d\xi\right.$$
$$\left.+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\mathfrak{b}_\iota}{\mathfrak{a}_\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}$$
$$=\lim_{\varkappa\to\infty}\sup\left\{\int_{\varkappa_0}^{\varkappa}\left(\frac{2}{\eta}\xi^2(1-\frac{\xi}{\varkappa})^{2\varrho}-\frac{1}{4\eta}\left(\frac{4\varrho\xi}{\varkappa}(1-\frac{\xi}{\varkappa})^{2\varrho-1}+\xi^2(1-\frac{\xi}{\varkappa})^{2\varrho}\right)\right)d\xi\right.$$
$$\left.+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\iota+1}{\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}$$

$$=\lim_{\varkappa\to\infty}\sup\left\{\frac{2\varkappa_0^2(\varkappa-\varkappa_0)^{2\varrho+1}}{\eta(2\varrho+1)\varkappa^{2\varrho}}+\frac{\varkappa_0(\varkappa-\varkappa_0)^{2\varrho+2}}{\eta(\varrho+1)}+\frac{(\varkappa-\varkappa_0)^{2\varrho+3}}{\eta(\varrho+1)(2\varrho+3)}\right.$$
$$-\frac{1}{4\eta}\left(\frac{2\varkappa_0(\varkappa-\varkappa_0)^{2\varrho}}{\varkappa^{2\varrho}}+\frac{2(\varkappa-\varkappa_0)^{2\varrho+1}}{(2\varrho+1)\varkappa^{2\varrho}}+\frac{\varkappa_0^2(\varkappa-\varkappa_0)^{2\varrho+1}}{(2\varrho+1)\varkappa^{2\varrho}}+\frac{\varkappa_0(\varkappa-\varkappa_0)^{2\varrho+2}}{2\varrho+2}\right.$$
$$\left.\left.+\frac{(\varkappa-\varkappa_0)^{2\varrho+3}}{(2\varrho+2)(2\varrho+3)}\right)+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\iota+1}{\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}$$
$$=\lim_{\varkappa\to\infty}\sup\left\{\frac{(7\varkappa_0^2-2)(\varkappa-\varkappa_0)^{2\varrho+1}}{4\eta(2\varrho+1)\varkappa^{2\varrho}}+\frac{7\varkappa_0(\varkappa-\varkappa_0)^{2\varrho+2}}{8\eta(\varrho+1)}+\frac{7(\varkappa-\varkappa_0)^{2\varrho+3}}{8\eta(\varrho+1)(2\varrho+3)}\right.$$
$$\left.-\frac{1}{4\eta}\left(\frac{2\varkappa_0(\varkappa-\varkappa_0)^{2\varrho}}{\varkappa^{2\varrho}}\right)+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\iota+1}{\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}$$
$$=\lim_{\varkappa\to\infty}\sup\left\{\frac{(\varkappa-\varkappa_0)^{2\varrho}}{\varkappa^{2\varrho}}\left(\frac{(7\varkappa_0^2-2)(\varkappa-\varkappa_0)}{4\eta(2\varrho+1)}-\frac{2\varkappa_0}{4\eta}\right)+\frac{7\varkappa_0(\varkappa-\varkappa_0)^{2\varrho+2}}{8\eta(\varrho+1)}+\frac{7(\varkappa-\varkappa_0)^{2\varrho+3}}{8\eta(\varrho+1)(2\varrho+3)}\right.$$
$$\left.+\sum_{\iota=1}^{\mu(\varkappa)}\frac{\iota+1}{\iota}(1-\frac{\varkappa_\iota}{\varkappa})^{2\varrho}\right\}=\infty,$$

So, by corollary (2.1) the eq. (1) is oscillatory.

Conclusion

we obtained appropriate sufficient conditions to oscillation for equation (1) which generalize and extend some results in [23] to nonlinear case. By formulating appropriate impulsive conditions, we concluded that the impulses conditions play an important role to consider the qualitative features to solutions for differential equations.

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