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The Stability Analysis of the diseased predator-prey model incorporating migration in the contaminated environment

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Abstract. We proposed and analyzed a predator–prey model. The disease effects in predator due to pollution in environment, as well the immigration factor effected is discussed. We assumed that, the population are divided into three parts prey, susceptible predator and infected predator. Firstly, the existence, uniqueness and bounded-ness of the solution of the model are discussed. Secondly, we studied the existence and local stability of all equilibrium points. Furthermore, some of the Sufficient conditions of the global stability of the positive equilibrium are established using suitable Lyapunov functions. Finally, those theoretical results are demonstrated with numerical simulations.

Key words: Prey-Predator model, Immigrants, pollution environment, Stability.

1. Introduction

To study the dynamical behavior of a phenomenon, the mathematical modeling is used as an effective tool to describe and analyze this phenomenon. Around 1800, the British Economist Malthus formulated a single species model and subsequently modified by Verhulst [1].

In the beginning of the twentieth century several attempts have been made to predict the evolution and existence of species mathematically. Indeed, the first major attempt in this direction was due to the well-known classical Lotka and Volterra [2, 3]. They proposed the prey-predator model in 1927. They also describe the continuous Lotka-Volterra model by ordinary differential equations. Further the delay differential equations is widely used to characterize the dynamics of biological systems.

During the last three decades, the relationship between the predator and their prey is studied and its very crucial component of study in ecology. The prey-predator interaction is prominent and significant area of research in applied mathematical modeling and population dynamics. Venturino [4] investigated the long-term behavior in predator-prey model which assuming the epidemics occurred in prey population and can be transmitted by the contact of predators. Mathematical ecology and mathematical epidemiology are two different fields in the study of biology and applied mathematics. The combination of these fields are studied which termed as an eco-epidemiology. Many authors have studied eco-epidemiological models and considered infection in prey population only. Later, other authors such as Kant and Kumar [5] formulated and studied a predator-prey model with migrating prey and disease infection in both species. In [6], Haque and Venturino analyzed the prey-predator model by considering a Holling-Tanner functional response. They also investigated some bifurcations around the disease-free equilibrium. Recently, many authors have proposed and discussed eco-epidemiological models with some assumptions (for instance,[7-10]). They considered prey-predator model with infection in prey population only. Naji and Mustafa [9] discussed the dynamics of an eco-epidemiological model with nonlinear incidence rate.

On the other hand, there are another category papers in literature, in which the authors consider the eco-epidemiological models where the disease spreads in predator population [11-14].

Also, a migration can be described as an important demographic process that occurs in all living beings. The physical movement from one place to another is termed as a migration. For example, the bird migration is the regular seasonal movement, often north and south along a flyway, between breeding and wintering grounds and the timing of migration seems to be controlled primarily by changes in day length, and the reason behind the migration varies in organisms such as climatic changes, to seek refuge, for food, shelter etc., [15].

Because we have taken prey-predator model so scientifically, the effect of migration can be considered as a significant aspect in the formatting of prey predator mathematical models.

Dingle and Drake explained the term migration for different species. They recognized a migration as an adaptation to resources that fluctuate spatiotemporally either seasonally or less predictably. Some authors have studied predator-prey model by taking migration in prey species. In [16], the migration and diffusion are important in the dynamics of spatial prey–predator interaction which is confirmed by both theoretical and numerical analysis. Keeping the above in view, we combine a prey-predator model with an epidemiological model.system more scientifically, the effect of migration must be taken into consideration while formulating of mathematical model of the prey-predator systems.

In our work, we proposed a prey-predator system with disease in prey population. The interactions between the healthy predator, susceptible prey, and infected prey and vice versa. Furthermore, the combine of two cases are studied using four-dimensional model along with the migration in both prey and predator population. The mortality rate of infected prey population is possible due to natural death and the death due to disease, along with the incorporation of migration factor. The rest of the paper is structured as follows. In section 2 we present the mathematical model with basic considerations. Boundedness and positivity of the solutions of the model are established in section 3. Section 4 deals with all the possible equilibrium points of the model and their feasibility conditions. Stability of the model at various equilibrium points is discussed in section 5. Computer simulations are carried out to illustrate our analytical findings numerically in section 6. Section 7 contains the general discussion and biological significance of our analytical findings.

2. Mathematical Model

Our model consists of two populations, namely, the prey, whose population density is denoted by $X(t)$ and the predator, whose population density is denoted by $Y_1(t)$ and $Y_2(t)$,where Y_1 denotes to the healthy predator, while Y_2 represents the infected predator and t is the time variable. We make the following assumptions to formulate our model:

H1. In the absence of disease and predation, the prey population grows according to logistic law with growth rate $r > 0$ and carrying capacity $k_1 > 0$.

H2. It is assumed that the disease spreads among the predator population only and the transmission of disease between healthy and infected predator follow the simple law of mass action and environment $\beta Y_1 Y_2$ and $\gamma Y_1 E$, where β and γ are the forces of infection and $E \in [0,1]$.

H3. Predators get the same reward from the predating on the prey with different search efficiencies. Also, infected predator becomes less active and therefore they could not catch

easily the prey compared to the healthy predator. Thus, we assume that searching coefficient of the healthy predator for prey is greater than that of infected predator.

H4. The functional response of the predator to the prey is assumed to be of Lotka-Volterra type.

H5. Prey population has migration rates as m_1. It is a natural factor that healthy predator is stronger as compared with the infected predator and therefore we neglected the probability of migration of infected predator.

H6. It has been assumed that infected predator recovers with rate α_2.

Based on the above assumptions, the mathematical model takes the following form as:

Table 1 **Details of the parameters in the system (1):** Parameters and their definition

Symbol	Definition
r	Intrinsic growth rate
n	measure of the quality of the prey or food for the predator
nX	interpreted as a prey dependent carrying capacity for the predator
c	environmental carrying capacity for the predator
k_1	prey environment carrying capacity
k_2	half saturation constants
m_1	Migration rate of prey
m_2	Migration rate of healthy predator
α_1	Healthy prey-predation coefficient
α_2	Conversion coefficient from infected to healthy predator
β	Infection coefficient of healthy predator
γ	Conversion coefficient from healthy to infected predator by contaminated environment
d_1	Natural death rate of healthy predator
d_2	Natural death rate of infected predator
μ	Death rate of infected predator due to infection

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k_1}\right) - \frac{\alpha_1 XY_1}{k_2 + X} - m_1 X$$
$$\frac{dY_1}{dt} = sY_1\left(1 - \frac{Y_1 + Y_2}{nX + c}\right) - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - m_2 Y_1 - d_1 Y_1$$
$$\frac{dY_2}{dt} = \gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu)Y_2$$
$$\frac{dE}{dt} = \theta Y_2(1 - E) - \gamma_1 E$$

(1)

Subject to the initial conditions with $X(0) \geq 0$, $Y_1(0) \geq 0$, $Y_2(0) \geq 0$, $E(0) \geq 0$, while all the parameters are describe in the table (1).

3. Mathematical analysis

3.1. Boundedness of the solution.

Since all the parameters are non-negative and the interaction functions are continuously differentiable the right hand side of system (1) is a smooth function of variables (X, Y_1, Y_2, E) in the positive octant,

$$\Omega = \{((X, Y_1, Y_2, E)) \big| X \geq 0, Y_1 \geq 0, Y_2 \geq 0, E \geq 0\}.$$

Furthermore, it is easy to prove that Ω is an invariant set. In addition, it is easy to verify that, all the interaction functions are globally Lipschitz and then the system (1) has a unique solution. Now we will prove the boundedness of the system (1).

3.2 Theorem: All solutions of system (1) which initiate in $\mathfrak{R}_+^4$ are uniformly bounded.

Proof: Let $\left(X(t), Y_1(t), Y_2(t), E(t)\right)$ be any solution of the system (1) with non-negative initial condition $\left(X(0), Y_1(0), Y_2(0), E(0)\right)$, From the first equation, we get as $t \to \infty$

$$sup\left[rX\left(1 - \frac{X}{k_1}\right)\right] \leq \frac{rk_1}{4} \tag{2}$$

And from the second equation, we get

$$sup\left[sY_1\left(1 - \frac{Y_1}{\frac{nrk_1+4c}{4}}\right)\right] \leq \frac{s(nrk_1+4c)}{16} \tag{3}$$

Let $N(t) = X(t) + Y_1(t) + Y_2(t) + E(t)$, then from the model we get

$$\frac{dN}{dt} = rX\left(1 - \frac{X}{k_1}\right) + sY_1\left(1 - \frac{Y_1}{nX+c}\right) - m_1X - (m_2 + d_1)Y_1 - (d_2 + \mu)Y_2$$

Assuming a positive constant $q > 0$ and $q = min\{m_1, m_2 + d_1, d_2 + \mu\}$, we get

$$\frac{dN}{dt} + qN \leq H \left(= \frac{4rk_1+s(nrk_1+4c)}{16}\right) \tag{4}$$

Now by using Gronweall Lemma it obtain that

$$N(t) \leq \frac{H}{q} + \left(N_0 - \frac{H}{q}\right)e^{-qt} \tag{5}$$

Therefore, $N(t) \leq \frac{H}{q}$, as $t \to \infty$. Now from the last equation of system (1) we have

$$\frac{dE}{dt} = \theta Y_2(1 - E) - \gamma_1 E$$

Then $\frac{dE}{dt} \leq \theta Y_2 - \gamma_1 E \leq \theta \frac{H}{q} - \gamma_1 E$

By similar way as above we get:

$$E(t) \leq \frac{\theta}{\gamma_1} \cdot \frac{H}{q}, \text{ as } t \to \infty \tag{6}$$

Hence all the solution of system (1) that initiate in $\mathfrak{R}_+^4$ are confined in the region

$$\Omega = \left\{(X, Y_1, Y_2, E) \in \mathfrak{R}_+^4 : N \leq \frac{H}{q}, 0 \leq E \leq \frac{\theta}{\gamma_1} \cdot \frac{H}{q}\right\} \tag{7}$$

Thus, these solutions are uniformly bounded and the proof is complete.   ∎

3.3. Existence of equilibrium points.

It is easy to verify that the system (1) has at most six biologically feasible equilibrium points. The existence conditions of each of them along with their local stability analyses are discussed as follows

1. The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exists.

2. The first axial equilibrium point $E_1 = (\breve{X}, 0,0,0)$, where $\breve{X} = \frac{k_1}{r}(r - m_1)$ exists provided

$$m_1 < r. \tag{8}$$

3. second axial equilibrium point with no infection occurred $E_2 = (0, \ddot{Y}_1, 0,0)$,

where $\ddot{Y}_1 = \frac{c}{s}(s - m_2 - d_1)$ exists, provided

$$m_2 + d_1 < s. \tag{9}$$

4. The first planar equilibrium point $E_3 = (\hat{X}, \hat{Y}_1, 0,0)$, where $\hat{Y}_1 = \frac{n\hat{X}+c}{s}(s - m_2 - d_1)$ exists

provided as the same case above by condition (9). while $\hat{X}$, represents a positive root of the following

quadratic equation

$$A_1 X^2 + A_2 X + A_3 = 0, \tag{10}$$

here

$$A_1 = r$$
$$A_2 = rk_2 + \alpha_1 k_1 n \left(\frac{s-m_2-d_1}{s}\right) - k_1(r - m_1)$$
$$A_3 = \alpha_1 k_1 \left(\frac{s-m_2-d_1}{s}\right) c - k_1 k_2(r - m_1)$$

Clearly, E_3 exists uniquely in interior of XY −plane when $A_3 < 0$.

5. Prey-free equilibrium point with Migration permitted and infection occurred

$E_4 = (0, \bar{Y}_1, \bar{Y}_2, \bar{E})$, where $\bar{Y}_2 = \frac{\gamma_1 \bar{E}}{\theta(1-\bar{E})}$ and $\bar{Y}_1 = \frac{\gamma_1(\alpha_2+d_2+\mu)}{\beta\gamma_1+\gamma\theta(1-\bar{E})}$ exist, provided $\bar{E} < 1$,

While $\bar{E}$, represents a positive root of the following quadratic equation

$$B_1 E^3 + B_2 E^2 + B_3 E + B_4 = 0 \tag{11}$$

Here

$$B_1 = \gamma^2\theta^2[c\alpha_2\gamma_1 - cD]$$
$$B_2 = \gamma\theta[(2\gamma + s - m_2 - d_1)cD\theta + sD\gamma_1 + 2\beta\gamma_1 cD - 2(\beta\gamma_1 + \gamma\theta)\alpha_2\gamma_1 c]$$
$$B_3 = \begin{cases} sD^2\theta + (\beta\gamma_1 + \gamma\theta)^2\alpha_2\gamma_1 c - (s - m_2 - d_1)(\beta\gamma_1 + 2\gamma\theta)cD\theta \\ -sD\gamma(\gamma\beta + \gamma_1\theta) - (2\beta\gamma_1 + \gamma\theta)\gamma\theta cD - \beta^2\gamma_1^2 cD \end{cases}$$
$$B_4 = (s - m_2 - d_1)cD\theta[\beta\gamma_1 + \gamma\theta] - sD^2\theta$$

with $D = \gamma_1(\alpha_2 + d_2 + \mu)$. Clearly, E_4 exists uniquely in interior of Positive octant of $Y_1 Y_2 E$ − space

when $B_4 > 0$ with one of the following conditions:

$$B_3 > 0 \text{ Or } B_2 < 0$$

6. The positive equilibrium point $E_5 = (\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}})$ of system (1) can be determined by equating the

right hand side of system (1) to the zero and solve the resulting algebraic system. Straightforward

computation gives that:

$$\bar{\bar{Y}}_2 = \frac{\gamma_1 \bar{\bar{E}}}{\theta(1-\bar{\bar{E}})}$$

$$\bar{\bar{Y}}_1 = \frac{\gamma_1(\alpha_{2+}d_2+\mu)}{\beta\gamma_1+\gamma\theta(1-\bar{\bar{E}})}$$

$$\bar{\bar{X}} = \frac{sD[D\theta(1-\bar{\bar{E}})+\gamma_1\bar{\bar{E}}G]}{nG[D\theta(1-\bar{\bar{E}})(s-\gamma\bar{\bar{E}}-m_2-d_1)-D\beta\gamma_1\bar{\bar{E}}+\alpha_2\gamma_1\bar{\bar{E}}G]} - \frac{c}{n}$$

$$\text{(12)}$$

while $\bar{\bar{E}}$ is a unique positive root of the following sixth order polynomial equation

$$C_1E^6 + C_2E^5 + C_3E^4 + C_4E^3 + C_5E^2 + C_6E + C_7 = 0 \tag{13a}$$

Here

$$C_1 = w_1[D\theta(s-m_2-d_1)]^2\gamma^2\theta^2$$

$$C_2 = \begin{Bmatrix} w_2\gamma\gamma_1\theta(D\theta\gamma - \alpha_2\gamma\gamma_1\theta) + \alpha_1Dn^2k_1(D\theta\gamma - \alpha_2\gamma\gamma_1\theta)^2\gamma\theta \\ -w_2[(\beta\gamma_1+\gamma\theta)(D\theta\gamma - \alpha_2\gamma\gamma_1\theta) + \gamma\theta w_3][\gamma\theta(D\theta\gamma - \alpha_2\gamma\gamma_1\theta) + 1] \end{Bmatrix}$$

$$C_3 = \begin{Bmatrix} 2w_1[(\beta\gamma_1+\gamma\theta)w_3 + D\gamma\theta^2(s-m_2-d_1)] + w_1[(\beta\gamma_1+\gamma\theta)w_4 + w_3\gamma\theta]^2 - rs^2D^2\gamma\gamma_1\theta \\ -w_2[w_4w_5\gamma\theta + [(\beta\gamma_1+\gamma\theta)w_4 + w_3\gamma\theta]\gamma\gamma_1\theta] - \alpha_1Dn^2k_1w_4[2\gamma\theta w_3 + w_4(\beta\gamma_1+\gamma\theta)] \end{Bmatrix}$$

$$C_4 = \begin{Bmatrix} -2w_1w_4D\gamma\theta^2(\beta\gamma_1+\gamma\theta)(s-m_2-d_1) - w_2\left[w_4\gamma D\theta^2 - \begin{bmatrix} (\beta\gamma_1+\gamma\theta)w_3 \\ -D\gamma\theta^2(s-m_2-d_1) \end{bmatrix}\gamma\gamma_1\theta\right] \\ +\alpha_1Dn^2k_1[2w_4D\gamma\theta^2(s-m_2-d_1) + w_3[w_4(\beta\gamma_1+\gamma\theta) + w_3\gamma\theta] + (\beta\gamma_1+\gamma\theta)w_3w_4] \\ +2w_5rs^2D^2\gamma\gamma_1\theta \end{Bmatrix}$$

$$C_5 = \begin{Bmatrix} [(\beta\gamma_1+\gamma\theta)w_4 + w_3\gamma\theta][2w_1D\theta(\beta\gamma_1+\gamma\theta)(s-m_2-d_1) + w_2(D\theta+1)] \\ +[w_3(\beta\gamma_1+\gamma\theta) + D\gamma\theta^2(s-m_2-d_1)]\begin{bmatrix} [w_3(\beta\gamma_1+\gamma\theta) + D\gamma\theta^2(s-m_2-d_1)]w_1 \\ -w_2w_5 \end{bmatrix} \\ -rs^2D^2(w_5^2 - 2\gamma\gamma_1D\theta^2) - w_2D\gamma\gamma_1\theta^2(\beta\gamma_1+\gamma\theta)(s-m_2-d_1) \\ -2\alpha_1Dn^2k_1D\theta(s-m_2-d_1)[(\beta\gamma_1+\gamma\theta)w_4 + w_3\gamma\theta] \end{Bmatrix}$$

$$C_6 = \begin{Bmatrix} 2w_1D\theta(\beta\gamma_1+\gamma\theta)(s-m_2-d_1)[w_3(\beta\gamma_1+\gamma\theta) + D\gamma\theta^2(s-m_2-d_1)] \\ +w_2D\theta[[(\beta\gamma_1+\gamma\theta)[w_3 - (s-m_2-d_1)w_5] + D\gamma\theta^2(s-m_2-d_1)]] \\ -\alpha_1Dn^2k_1D\theta(s-m_2-d_1)[D\theta(s-m_2-d_1)\gamma\theta - 2(\beta\gamma_1+\gamma\theta)w_3] + 2w_5r\theta s^2D^3 \end{Bmatrix}$$

$$C_7 = D\theta(\beta\gamma_1+\gamma\theta)(s-m_2-d_1)\begin{bmatrix} w_1D\theta(\beta\gamma_1+\gamma\theta)(s-m_2-d_1) \\ +w_2D\theta - \alpha_1Dn^2k_1D\theta(s-m_2-d_1) \end{bmatrix}$$

with

$$w_1 = (nk_2 - c)[(r-m_1)nk_1 - rc]$$

$$w_2 = sD[(r-m_1)nk_1 - rc - (nk_2 - c)]$$

$$w_3 = D\theta\gamma E + D\theta(s-m_2-d_1) + D\beta\gamma_1 - \alpha_2\gamma_1(\beta\gamma_1+\gamma\theta)$$

$$w_4 = D\gamma\theta - \alpha_2\gamma\gamma_1\theta$$

$$w_5 = \gamma\theta\gamma_1 + \gamma_1^2\beta - D\theta$$

So by Descartes rule of sign [17], equation (13a) has a unique positive root provided that one set of the following sets of conditions hold:

$$C_7 < 0, C_i > 0; \quad i = 1, \dots, 5 \tag{13b}$$

$$C_j < 0, C_i > 0; \quad i = 1,2,3, j = 5,6,7 \tag{13c}$$

$$C_i < 0, C_1 > 0; \quad i = 3, \dots, 7 \tag{13d}$$

$$C_i < 0, C_7 > 0; \quad i = 1, \dots, 5 \tag{13e}$$

$$C_i < 0\,, C_j > 0\,; \quad i = 1,2,3\,, j = 5,6,7 \tag{13f}$$

$$C_1 < 0\,, C_i > 0\,; \quad i = 3, \dots, 7 \tag{13g}$$

Consequently, the positive point $E_5 = \left(\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}}\right)$ exist uniquely in the $\mathfrak{R}_+^4$ provided that in addition

to satisfy one of conditions (13b)-(13g), the following conditions should be hold.

$$\left.\begin{array}{l} cG < sD\left[(D + \gamma\gamma_1\bar{\bar{E}})\theta(1 - \bar{\bar{E}}) + \beta\gamma_1^2\bar{\bar{E}}\right] \\ D\theta(1 - \bar{\bar{E}})(\gamma\bar{\bar{E}} + m_2 + d_1) + D\beta\gamma_1\bar{\bar{E}} < D\theta s(1 - \bar{\bar{E}}) + \alpha_2\gamma_1\bar{\bar{E}}[\beta\gamma_1 + \gamma\theta(1 - \bar{\bar{E}})] \end{array}\right\} \tag{14}$$

where

$$G = \left[\beta\gamma_1 + \gamma\theta(1 - \bar{\bar{E}})\right]\left[D\theta(1 - \bar{\bar{E}})(s - \gamma\bar{\bar{E}} - m_2 - d_1) - D\beta\gamma_1\bar{\bar{E}} + \alpha_2\gamma_1\bar{\bar{E}}[\beta\gamma_1 + \gamma\theta(1 - \bar{\bar{E}})]\right]$$

4. Local stability analysis:

In this section, the local stability of the equilibrium points of system (1) are established using the linearization method. It is easy to verify that the variational matrix of system (1), at the general point (X, Y_1, Y_2, E), can be written as $J = \left(a_{ij}\right)_{4\times4}$; $i, j = 1,2,3,4$, where

$$J = \begin{bmatrix} r - m_1 - \frac{2rX}{k_1} - \frac{k_2\alpha_1 Y_1}{(k_2+X)^2} & -\frac{\alpha_1 X}{k_2+X} & 0 & 0 \\ \frac{nsY_1(Y_1+Y_2)}{(nX+c)^2} & s - \frac{s(2Y_1+Y_2)}{nX+c} - \gamma E - \beta Y_2 - m_2 - d_1 & -\frac{sY_1}{nX+c} - \beta Y_1 + \alpha_2 & -\gamma Y_1 \\ 0 & \gamma E + \beta Y_2 & \beta Y_1 - \alpha_2 - d_2 - \mu & \gamma Y_1 \\ 0 & 0 & \theta(1-E) & -\theta Y_2 - \gamma_1 \end{bmatrix} \tag{15}$$

Therefore, the variational matrix of the system (1) at $E_0 = (0,0,0,0)$ is given by;

$$J(E_0) = \begin{bmatrix} r - m_1 & 0 & 0 & 0 \\ 0 & s - m_2 - d_1 & \alpha_2 & 0 \\ 0 & 0 & -\alpha_2 - d_2 - \mu & 0 \\ 0 & 0 & \theta & -\gamma_1 \end{bmatrix} \tag{16}$$

Then the eigenvalues of $J(E_0)$ are given by;

$$\left.\begin{array}{l} \lambda_1 = -\gamma_1 \\ \lambda_2 = r - m_1 \\ \lambda_3 = s - m_2 - d_1 \\ \lambda_4 = -\alpha_2 - d_2 - \mu \end{array}\right\} \tag{17}$$

So, $E_0 = (0,0,0,0)$ is an asymptotically stable equilibrium if

$$r < m_1 \text{ and } s < m_2 + d_1 \tag{18}$$

The variational matrix of the system (1) at $E_1 = \left(\check{X}, 0,0,0\right)$ is given by;

$$J(E_1) = \begin{bmatrix} r - m_1 - \frac{2r\check{X}}{k_1} & -\frac{\alpha_1\check{X}}{k_2+\check{X}} & 0 & 0 \\ 0 & s - m_2 - d_1 & \alpha_2 & 0 \\ 0 & 0 & -\alpha_2 - d_2 - \mu & 0 \\ 0 & 0 & \theta & -\gamma_1 \end{bmatrix} \tag{19}$$

Then

eigenvalues are given by;

$$\left.\begin{array}{l} \check{\lambda}_1 = -\gamma_1 \\ \check{\lambda}_2 = -\alpha_2 - d_2 - \mu \\ \check{\lambda}_3 = s - m_2 - d_1 \\ \check{\lambda}_4 = r - m_1 - \frac{2r\check{X}}{k_1} \end{array}\right\} \tag{20}$$

So, $E_1 = (\check{X}, 0, 0, 0)$ is a locally asymptotically stable equilibrium if

$$s < m_2 + d_1 \text{ and } r < m_1 + \frac{2r\check{X}}{k_1}. \tag{21}$$

The variational matrix of the system (1) at $E_2 = (0, \ddot{Y}_1, 0, 0)$ is given by;

$$J(E_2) = \begin{bmatrix} r - m_1 - \frac{\alpha_1 \ddot{Y}_1}{k_2} & 0 & 0 & 0 \\ \frac{ns\ddot{Y}_1^2}{c^2} & s - \frac{2s\ddot{Y}_1}{c} - m_2 - d_1 & -\frac{s\ddot{Y}_1}{c} - \beta\tilde{Y}_1 + \alpha_2 & -\gamma\ddot{Y}_1 \\ 0 & 0 & \beta\ddot{Y}_1 - \alpha_2 - d_2 - \mu & \gamma\ddot{Y}_1 \\ 0 & 0 & \theta & -\gamma \end{bmatrix} \tag{22}$$

The characteristic equation is given by;

$$\left[r - m_1 - \frac{\alpha_1 \ddot{Y}_1}{k_2} - \lambda\right]\left[s - \frac{2s\ddot{Y}_1}{c} - m_2 - d_1 - \lambda\right]\left[\lambda^2 + \ddot{A}_1 \lambda + \ddot{A}_2\right] = 0 \tag{23}$$

Here

$$\ddot{A}_1 = -(\ddot{a}_{33} + \ddot{a}_{44})$$
$$\ddot{A}_2 = \ddot{a}_{33}\ddot{a}_{44} - \ddot{a}_{34}\ddot{a}_{43}$$

Based on the above characteristic equation (23), the eigenvalues are given by

$$\left.\begin{array}{l} \ddot{\lambda}_1 = r - m_1 - \frac{\alpha_1 \ddot{Y}_1}{k_2} \\ \ddot{\lambda}_2 = s - \frac{2s\ddot{Y}_1}{c} - m_2 - d_1 < 0 \\ \ddot{\lambda}_{3,4} = -\frac{\ddot{A}_1}{2} \mp \frac{1}{2}\sqrt{\ddot{A}_1^2 - 4\ddot{A}_2} \end{array}\right\} \tag{24}$$

So, $E_2 = (0, \ddot{Y}_1, 0, 0)$ is a locally asymptotically stable equilibrium if

$$r < m_1 \text{ and } c(\gamma_1\beta + \gamma\theta)(s - m_2 - d_1) < \gamma_1 s(\alpha_2 + d_2 + \mu) \tag{25}$$

The variational matrix of the system (1) at $E_3 = (\hat{X}, \hat{Y}_1, 0, 0)$ is given by;

$$J(E_3) = \begin{bmatrix} r - m_1 - \frac{2r\hat{X}}{k_1} - \frac{k_2\alpha_1\hat{Y}_1}{(k_2+\hat{X})^2} & -\frac{\alpha_1\hat{X}}{k_2+\hat{X}} & 0 & 0 \\ \frac{ns\hat{Y}_1}{(n\hat{X}+c)^2} & s - \frac{2s\hat{Y}_1}{n\hat{X}+c} - m_2 - d_1 & -\frac{s\hat{Y}_1}{n\hat{X}+c} - \beta\hat{Y}_1 + \alpha_2 & -\gamma\hat{Y}_1 \\ 0 & 0 & \beta\hat{Y}_1 - \alpha_2 - d_2 - \mu & \gamma\hat{Y}_1 \\ 0 & 0 & \theta & -\gamma_1 \end{bmatrix}. \tag{26}$$

The characteristic equation is given by;

$$\left[\lambda^2 + \hat{A}_1 \lambda + \hat{A}_2\right]\left[\lambda^2 + \hat{B}_1 \lambda + \hat{B}_2\right] = 0, \tag{27}$$

here

$$\hat{A}_1 = -(\hat{a}_{11} + \hat{a}_{22})$$
$$\hat{A}_2 = \hat{a}_{11}\hat{a}_{22} - \hat{a}_{12}\hat{a}_{21}$$
$$\hat{B}_1 = -(\hat{a}_{33} + \hat{a}_{44})$$
$$\hat{B}_2 = \hat{a}_{33}\hat{a}_{44} - \hat{a}_{34}\hat{a}_{43}$$

Based on the above characteristic equation (27), the eigenvalues are given by;

$$\left.\begin{array}{l} \hat{\lambda}_{1,2} = -\dfrac{\hat{A}_1}{2} \mp \dfrac{1}{2}\sqrt{\hat{A}_1^2 - 4\hat{A}_2} \\[3mm] \hat{\lambda}_{3,4} = -\dfrac{\hat{B}_1}{2} \mp \dfrac{1}{2}\sqrt{\hat{B}_1^2 - 4\hat{B}_2} \end{array}\right\} \tag{28}$$

So, $E_3 = (\hat{X}, \hat{Y}_1, 0, 0)$ is a locally asymptotically stable equilibrium if

$$r < m_1 \text{ and } (n\hat{X} + c)(\gamma_1\beta + \gamma\theta)(s - m_2 - d_1) < \gamma_1 s(\alpha_2 + d_2 + \mu) \tag{29}$$

The variational matrix of the system (1) at $E_4 = (0, \bar{Y}_1, \bar{Y}_2, \bar{E})$ is given by;

$$J(E_4) =$$

$$\begin{bmatrix} r - m_1 - \dfrac{\alpha_1 \bar{Y}_1}{(k_2+\hat{X})^2} & 0 & 0 & 0 \\[3mm] \dfrac{ns\bar{Y}_1(\bar{Y}_1+\bar{Y}_2)}{c^2} & s\left(1 - \dfrac{2\bar{Y}_1+\bar{Y}_2}{c}\right) - \gamma\bar{E} - \beta\bar{Y}_2 - m_2 - d_1 & -\dfrac{s\bar{Y}_1}{c} - \beta\bar{Y}_1 + \alpha_2 & -\gamma\bar{Y}_1 \\[3mm] 0 & \gamma\bar{E} + \beta\bar{Y}_2 & \beta\bar{Y}_1 - \alpha_2 - d_2 - \mu & \gamma\bar{Y}_1 \\[3mm] 0 & 0 & \theta(1 - \bar{E}) & -\theta\bar{Y}_2 - \gamma_1 \end{bmatrix} \tag{30}$$

The characteristic equation is given by;

$$(\bar{a}_{11} - \lambda)[\lambda^3 + \bar{A}_1\lambda^2 + \bar{A}_2\lambda + \bar{A}_3] = 0 \tag{31}$$

Here

$$\bar{A}_1 = -(\bar{a}_{22} + \bar{a}_{33} + \bar{a}_{44})$$
$$\bar{A}_2 = \bar{a}_{22}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{32} + \bar{a}_{22}\bar{a}_{44} + \bar{a}_{33}\bar{a}_{44} - \bar{a}_{34}\bar{a}_{43}$$
$$\bar{A}_3 = -\bar{a}_{22}\bar{a}_{33}\bar{a}_{44} - \bar{a}_{24}\bar{a}_{32}\bar{a}_{43} + \bar{a}_{22}\bar{a}_{34}\bar{a}_{43} + \bar{a}_{23}\bar{a}_{32}\bar{a}_{44}$$

So either $(\bar{a}_{11} - \lambda) = 0$, which gives the eigenvalue in the $X-$direction by $\bar{\lambda}_X = \bar{a}_{11}$ or

$$\lambda^3 + \bar{A}_1\lambda^2 + \bar{A}_2\lambda + \bar{A}_3 = 0.$$

Now, according to the Routh-Hawirtiz Criterion all the eigenvalues of $J(E_4)$ have roots with negative real parts if and only if $\bar{A}_i(i = 1,3) > 0$ and $\Delta = \bar{A}_1\bar{A}_2 - \bar{A}_3 > 0$. So, $E_4 = (0, \bar{Y}_1, \bar{Y}_2, \bar{E})$ is a locally asymptotically stable equilibrium if

$$\left.\begin{array}{l} r < m_1 + \dfrac{\alpha_1 \bar{Y}_1}{(k_2+\hat{X})^2} \\[3mm] s < s\left(\dfrac{2\bar{Y}_1+\bar{Y}_2}{c}\right) + \gamma\bar{E} + \beta\bar{Y}_2 + m_2 + d_1 \\[3mm] \alpha_2 < \dfrac{s\bar{Y}_1}{c} + \beta\bar{Y}_1 \\[3mm] \gamma\bar{E} + \beta\bar{Y}_2 < \theta\bar{Y}_2 + \gamma_1 \end{array}\right\} \tag{32}$$

From ecological point of view this equilibrium is very important. The reason is quite obvious that in this case all four populations will exist simultaneously. This provides actual interaction and competition among all different populations. The variational matrix of the system (1) at $E_5 = \left(\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}}\right)$ can be written as;

$$J_5 = \left(b_{ij}\right)_{4\times4} \tag{33}$$

where

$$b_{11} = r - m_1 - \frac{2r\bar{\bar{X}}}{k_1} - \frac{k_2\alpha_1\bar{\bar{Y}}_1}{\left(k_2+\bar{\bar{X}}\right)^2} \ , \ b_{12} = -\frac{\alpha_1\bar{\bar{X}}}{k_2+\bar{\bar{X}}} \ , \ b_{21} = \frac{ns\bar{\bar{Y}}_1\left(\bar{\bar{Y}}_1+\bar{\bar{Y}}_2\right)}{\left(n\bar{\bar{X}}+c\right)^2}$$

$$b_{22} = s - \frac{s\left(2\bar{\bar{Y}}_1+\bar{\bar{Y}}_2\right)}{nX+c} - \gamma\bar{\bar{E}} - \beta\bar{\bar{Y}}_2 - m_2 - d_1, \ b_{23} = -\frac{s\bar{\bar{Y}}_1}{n\bar{\bar{X}}+c} - \beta\bar{\bar{Y}}_1 + \alpha_2$$

$$b_{32} = \gamma\bar{\bar{E}} + \beta\bar{\bar{Y}}_2 \ , \ b_{33} = \beta\bar{\bar{Y}}_1 - \alpha_2 - d_2 - \mu \ , \ b_{34} = \gamma\bar{\bar{Y}}_1 \ , \ b_{43} = \theta\left(1 - \bar{\bar{E}}\right)$$

$$b_{24} = -\gamma\bar{\bar{Y}}_1 \ , \ b_{44} = -\theta\bar{\bar{Y}}_2 - \gamma_1, \ b_{13} = b_{14} = b_{31} = b_{41} = b_{42} = 0$$

The characteristic equation is given by;

$$\left[\lambda^4 + \bar{\bar{A}}_1\lambda^3 + \bar{\bar{A}}_2\lambda^2 + \bar{\bar{A}}_3\lambda + \bar{\bar{A}}_4\right] = 0 \tag{34}$$

here

$$\bar{\bar{A}}_1 = -(b_{11} + b_{22} + b_{33} + b_{44})$$

$$\bar{\bar{A}}_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} + b_{11}b_{44} + b_{22}b_{33} - b_{23}b_{32} + b_{33}b_{44} - b_{34}b_{43}$$

$$\bar{\bar{A}}_3 = \left\{\begin{matrix} -b_{11}(b_{22}b_{33} - b_{23}b_{32}) + b_{21}b_{12}b_{33} - b_{11}(b_{33}b_{44} - b_{34}b_{43}) - b_{11}b_{22}b_{44} \\ +b_{21}b_{12}b_{44} - b_{22}(b_{33}b_{44} - b_{34}b_{43}) + b_{32}(b_{23}b_{44} - b_{24}b_{43}) \end{matrix}\right\}$$

$$\bar{\bar{A}}_4 = (b_{11}b_{22} - b_{21}b_{12})(b_{33}b_{44} - b_{34}b_{43}) - b_{11}b_{32}(b_{23}b_{44} - b_{24}b_{43})$$

Now, according to the Routh-Hawirtiz Criterion all the eigenvalues of $J(E_5)$ have roots with negative real parts if and only if $\bar{\bar{A}}_i(i = 1,3,4) > 0$ and $\Delta = \left(\bar{\bar{A}}_1\bar{\bar{A}}_2 - \bar{\bar{A}}_3\right)\bar{\bar{A}}_3 - \bar{\bar{A}}_1^2\bar{\bar{A}}_4 = D_1 - D_2 > 0$.

So, $E_5 = \left(\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}}\right)$ is a locally asymptotically stable equilibrium if the following conditions hold:

$$\left.\begin{matrix} r < m_1 \\ s < m_2 + d_1 \\ \gamma\bar{\bar{Y}}_1\theta\left(1 - \bar{\bar{E}}\right) < \left(\theta\bar{\bar{Y}}_2 + \gamma_1\right)\left(\alpha_2 + d_2 + \mu - \beta\bar{\bar{Y}}_1\right) \\ \beta\left(\bar{\bar{Y}}_1 + \bar{\bar{Y}}_2\right) + \gamma\bar{\bar{E}} < \alpha_2 + d_2 + \mu + \theta\bar{\bar{Y}}_2 + \gamma_1 \\ D_2 < D_1 \end{matrix}\right\} \tag{35}$$

5. Global stability analysis

In this section, the region of global stability (basin of attraction) of each equilibrium points of system (1) is presented as shown in the following theorems.

Theorem (5.1): Assume that, the vanishing equilibrium point E_0 is locally asymptotically stable in $\Re_+^4$. Then it is a globally asymptotically stable provided that the following conditions hold

$$\theta < d_2 + \mu \tag{36}$$

Proof: Consider the following positive definite function

$$V_0(X, Y_1, Y_2, E) = X + Y_1 + Y_2 + E$$

Clearly, $V_0 : \mathfrak{R}_+^4 \to \mathfrak{R}$ is a continuously differentiable function such that $V_0(0,0,0,0) = 0$ and $V_0(X, Y_1, Y_2, E) > 0, \forall (X, Y_1, Y_2, E) \neq (0,0,0,0)$. Further,

$$
\begin{aligned}
\frac{dV_0}{dt} = & \left[rX\left(1 - \frac{X}{k_1}\right) - \frac{\alpha_1 X Y_1}{k_2 + X} - m_1 X \right] \\
& + \left[sY_1\left(1 - \frac{Y_1 + Y_2}{nX + c}\right) - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - m_2 Y_1 - d_1 Y_1 \right] \\
& + \left[\gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu) Y_2 \right] + \left[\theta Y_2 (1 - E) - \gamma_1 E \right]
\end{aligned}
$$

Now, by doing some algebraic manipulation and using the condition (36), we get

$$\frac{dV_0}{dt} \leq -\frac{rX^2}{k_1} - \frac{sY_1^2}{nX + c} - \theta Y_2 E - \gamma_1 E \tag{37}$$

Consequently, due to condition above $\frac{dV_0}{dt} < 0$ is negative definite and hence V_0 is Lyapunov function with respect to E_0. Thus E_0 is a globally asymptotically stable and the proof is complete. ∎

Theorem (5.2): Assume that the equilibrium point E_1 is locally asymptotically. Then it is a globally asymptotically stable in the subregion of $\mathfrak{R}_+^4$ provided that

$$
\begin{aligned}
& \theta < d_2 + \mu \\
& (r - m_1)\left(X - \check{X}\right)^2 + sY_1 + \theta Y_2 < (m_2 + d_1)Y_1 + (d_2 + \mu)Y_2 + \gamma_1 E
\end{aligned}
\tag{38}
$$

Proof: Consider the following positive definite function

$$V_1(X, Y_1, Y_2, E) = \frac{(X - \check{X})^2}{2} + Y_1 + Y_2 + E$$

Clearly, $V_1 : \mathfrak{R}_+^4 \to \mathfrak{R}$ is a continuously differentiable function such that $V_1(\check{X}, 0,0,0) = 0$ and $V_1(X, Y_1, Y_2, E) > 0, \forall (X, Y_1, Y_2, E) \neq (\check{X}, 0,0,0)$. Further,

$$
\begin{aligned}
\frac{dV_1}{dt} = & \left(X - \check{X}\right)\left[rX - \frac{r}{k_1}X^2 - \frac{\alpha_1 X Y_1}{k_2 + X} - m_1 X \right] \\
& + \left[sY_1 - \frac{sY_1(Y_1 + Y_2)}{nX + c} - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - m_2 Y_1 - d_1 Y_1 \right] \\
& + \left[\gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu) Y_2 \right] + \left[\theta Y_2 - \theta Y_2 E - \gamma_1 E \right]
\end{aligned}
$$

Now, by doing some algebraic manipulations and using the condition (38), we get

$$\frac{dV_1}{dt} \leq (r - m_1)\left(X - \check{X}\right)^2 - (m_2 + d_1 - s)Y_1 - (d_2 + \mu - \theta)Y_2 - \gamma_1 E \tag{39}$$

Consequently, due to the condition above $\frac{dV_1}{dt} < 0$ is negative definite and hence V_1 is Lyapunov function with respect to E_1 in the region that satisfies the given condition. Thus E_1 is a globally asymptotically stable and the proof is complete. ∎

Theorem (5.3): Assume that the equilibrium point E_2 is locally asymptotically. Then it is a globally asymptotically stable in the sub region of $\Re_+^4$ that satisfied the following conditions

$$\frac{s\ddot{Y}_1}{nX+c} + \beta\ddot{Y}_1 + \theta < d_2 + \mu$$

$$\frac{cY_1^2}{\ddot{Y}_1^2} < nX + c$$

$$\gamma\ddot{Y}_1 < \gamma_1$$

$$\frac{s}{c}\ddot{Y}_1^2 + \left(\frac{s-(m_2+d_1)}{Y_1}\right)\left(Y_1 - \ddot{Y}_1\right)^2 < L \tag{40}$$

Where the symbol L is given in the proof.

Proof: Consider the following positive definite function

$$V_2(X, Y_1, Y_2, E) = X + \left(Y_1 - \ddot{Y}_1 - \ddot{Y}_1 \ln\frac{Y_1}{\ddot{Y}_1}\right) + Y_2 + E$$

Clearly, $V_2: \Re_+^4 \to \Re$ is a continuously differentiable function such that $V_2\left(0, \ddot{Y}_1, 0, 0\right) = 0$ and $V_2(X, Y_1, Y_2, E) > 0, \forall\left(0, \ddot{Y}_1, 0, 0\right) \neq (0,0,0,0)$. Further,

$$\frac{dV_2}{dt} = \left[rx\left(1 - \frac{X}{k_1}\right) - \frac{\alpha_1 XY_1}{k_2+X} - m_1 X\right]$$

$$+ \left(\frac{Y_1 - \ddot{Y}_1}{Y_1}\right)\left[sY_1\left(1 - \frac{Y_1+Y_2}{nX+c}\right) - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - m_2 Y_1 - d_1 Y_1\right]$$

$$+ [\gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu)Y_2] + [\theta Y_2 - \theta Y_2 E - \gamma_1 E]$$

Now, by doing some algebraic manipulation and using the condition (40), we get

$$\frac{dV_2}{dt} \leq (r - m_1)X + \left(\frac{s-(m_2+d_1)}{Y_1}\right)\left(Y_1 - \ddot{Y}_1\right)^2 - \left[(d_2 + \mu) - \left(\frac{s\ddot{Y}_1}{nX+c} + \beta\ddot{Y}_1 + \theta\right)\right]Y_2$$

$$+ \frac{s}{c}\ddot{Y}_1^2 - \left(\frac{\ddot{Y}_1^2}{cY_1^2} - \frac{1}{nX+c}\right)s\ddot{Y}_1 Y_1 - (\gamma_1 - \gamma\ddot{Y}_1)E \tag{41}$$

Where

$$L = (r - m_1)X + \left[(d_2 + \mu) - \left(\frac{s\ddot{Y}_1}{nX+c} + \beta\ddot{Y}_1 + \theta\right)\right]Y_2 + \left(\frac{\ddot{Y}_1^2}{cY_1^2} - \frac{1}{nX+c}\right)s\ddot{Y}_1 Y_1 + (\gamma_1 - \gamma\ddot{Y}_1)E$$

Consequently, due to condition above $\frac{dV_2}{dt} < 0$ is negative definite and hence V_2 is Lyapunov function with respect to E_2 in the region that satisfies the given condition. Thus E_2 is a globally asymptotically stable and the proof is complete.   ∎

Theorem (5.4): Assume that the equilibrium point E_3 is locally asymptotically stable. Then it is a globally asymptotically stable in the sub region of $\Re_+^4$ that satisfied the following conditions

$$\frac{s\hat{Y}_1}{nX+c} + \beta\hat{Y}_1 + \theta < d_2 + \mu$$
$$\gamma\hat{Y}_1 < \gamma_1$$
$$\hat{X} < X \tag{42}$$
$$\frac{\alpha_1\hat{Y}_1}{(k_2+X)(k_2+\hat{X})}X^2 < (\gamma_1 - \gamma\hat{Y}_1)E + \left[(d_2+\mu) - \left(\frac{s\hat{Y}_1}{nX+c} + \beta\hat{Y}_1 + \theta\right)\right]Y_2$$
$$q_{12}^2 < 4q_{11}q_{22}$$

Proof: Consider the following positive definite function

$$V_3(X, Y_1, Y_2, E) = \left(X - \hat{X} - \hat{X}\ln\frac{X}{\hat{X}}\right) + \left(Y_1 - \hat{Y}_1 - \tilde{Y}_1\ln\frac{Y_1}{\hat{Y}_1}\right) + Y_2 + E$$

Clearly, $V_3: \mathfrak{R}_+^4 \to \mathfrak{R}$ is a continuously differentiable function such that $V_3(\hat{X}, \hat{Y}_1, Y_2, E) = 0$ and $V_3(X, Y_1, Y_2, E) > 0, \forall (X, Y_1, Y_2, E) \in \mathfrak{R}_+^4$ and $(X, Y_1, Y_2, E) \neq (\hat{X}, \hat{Y}_1, Y_2, E)$.

Taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dV_3}{dt} = \left(\frac{X-\hat{X}}{X}\right)\left[rX - \frac{r}{k_1}X^2 - \frac{\alpha_1 XY_1}{k_2+X} - m_1X\right]$$
$$+ \left(\frac{Y_1-\hat{Y}_1}{Y_1}\right)\left[sY_1\left(1 - \frac{Y_1}{nX+c}\right) - \frac{sY_1Y_2}{nX+c} - \gamma Y_1E - \beta Y_1Y_2 + \alpha_2 Y_2 - (m_2+d_1)Y_1\right]$$
$$+ [\gamma Y_1E + \beta Y_1Y_2 - \alpha_2 Y_2 - (d_2+\mu)Y_2] + [\theta Y_2(1-E) - \gamma_1 E]$$

$$\frac{dV_3}{dt} = -\left[q_{11}(X-\hat{X})^2 + q_{12}(X-\hat{X})(Y_1-\hat{Y}_1) + q_{22}(Y_1-\hat{Y}_1)^2\right]$$
$$- \frac{\alpha_1\hat{X}Y_1}{(k_2+X)(k_2+\hat{X})}(X-\hat{X}) + \frac{\alpha_1\hat{Y}_1}{(k_2+X)(k_2+\hat{X})}X^2 - \frac{\alpha_1\hat{X}\hat{Y}_1}{(k_2+X)(k_2+\hat{X})}$$
$$- \frac{s}{nX+c}Y_1Y_2 - \frac{\alpha_2\hat{Y}_1}{Y_1}Y_2 - (\gamma_1 - \gamma\hat{Y}_1)E - \left[(d_2+\mu) - \left(\frac{s\hat{Y}_1}{nX+c} + \beta\hat{Y}_1 + \theta\right)\right]Y_2$$

Consequently by using (42) conditions we get that

$$\frac{dV_3}{dt} \leq -\left[\sqrt{q_{11}}(X-\hat{X}) + \sqrt{q_{22}}(Y_1-\hat{Y}_1)\right]^2 + \frac{\alpha_1\hat{Y}_1}{(k_2+X)(k_2+\hat{X})}X^2$$
$$-(\gamma_1 - \gamma\hat{Y}_1)E - \left[(d_2+\mu) - \left(\frac{s\hat{Y}_1}{nX+c} + \beta\hat{Y}_1 + \theta\right)\right]Y_2 \tag{43}$$

Where

$$q_{11} = \frac{r}{k_1}, q_{12} = \frac{\alpha_1 k_2}{(k_2+X)(k_2+\hat{X})} - \frac{nsY_1}{(nX+c)(n\hat{X}+c)},$$
$$q_{22} = \frac{sc}{(nX+c)(n\hat{X}+c)} + \frac{sc\hat{Y}_1}{(nX+c)(n\hat{X}+c)} + \frac{m_2+d_1}{Y_1} - \frac{s}{Y_1}$$

Obviously, $\frac{dV_3}{dt}$ is negative definite and hence V_3 is Layapunov function with respect to E_3. So E_3 is globally asymptotically stable in the sub region that satisfies the given condition. ∎

Theorem (5.5): Assume that the equilibrium point E_4 is locally asymptotically stable. Then it is a globally asymptotically stable in the sub region of $\Re_+^4$ that satisfied the following conditions

$$m_1 + \frac{ns\bar{Y}_1^3 + ns\bar{Y}_1^2\bar{Y}_2}{(nX+c)c} < r$$

$$\frac{ns\bar{Y}_1^2 + ns\bar{Y}_1\bar{Y}_2}{(nX+c)c} < \frac{\alpha}{k_2+X}$$

$$q_{12}^2 < q_{11}q_{22}$$ (44)

$$q_{13}^2 < q_{11}q_{33}$$

$$q_{23}^2 < q_{22}q_{33}$$

Proof: Consider the following positive definite function

$$V_4(X, Y_1, Y_2, E) = X + \frac{(Y_1 - \bar{Y}_1)^2}{2} + \frac{(Y_2 - \bar{Y}_2)^2}{2} + \frac{(E - \bar{E})^2}{2}$$

Clearly, $V_4 : \Re_+^4 \to \Re$ is a continuously differentiable function such that $V_4(0, \bar{Y}_1, \bar{Y}_2, \bar{E}) = 0$ and $V_4(X, Y_1, Y_2, E) > 0, \forall (X, Y_1, Y_2, E) \in \Re_+^4$ and $(X, Y_1, Y_2, E) \neq (0, \bar{Y}_1, \bar{Y}_2, \bar{E})$.

Taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dV_4}{dt} = \left[rX - \frac{r}{k_1}X^2 - \frac{\alpha_1 XY_1}{k_2+X} - m_1 X\right]$$
$$+ (Y_1 - \bar{Y}_1)\left[sY_1\left(1 - \frac{Y_1}{nX+c}\right) - \frac{sY_1Y_2}{nX+c} - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - (m_2 + d_1)Y_1\right]$$
$$+ (Y_2 - \bar{Y}_2)[\gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu)Y_2] + (E - \bar{E})[\theta Y_2 - \theta Y_2 E - \gamma_1 E]$$

$$\frac{dV_4}{dt} = -\left[\frac{q_{11}}{2}(Y_1 - \bar{Y}_1)^2 + q_{12}(Y_1 - \bar{Y}_1)(Y_2 - \bar{Y}_2) + \frac{q_{22}}{2}(Y_2 - \bar{Y}_2)^2\right]$$
$$- \left[\frac{q_{11}}{2}(Y_1 - \bar{Y}_1)^2 + q_{13}(Y_1 - \bar{Y}_1)(E - \bar{E}) + \frac{q_{33}}{2}(E - \bar{E})^2\right]$$
$$- \left[\frac{q_{22}}{2}(Y_2 - \bar{Y}_2)^2 + q_{23}(Y_2 - \bar{Y}_2)(E - \bar{E}) + \frac{q_{33}}{2}(E - \bar{E})^2\right]$$
$$- \left[r - \left(m_1 + \frac{ns\bar{Y}_1^3 + ns\bar{Y}_1^2\bar{Y}_2}{(nX+c)c}\right)\right]X - \frac{r}{k_1}X^2 - \left[\frac{\alpha}{k_2+X} - \frac{ns\bar{Y}_1^2 + ns\bar{Y}_1\bar{Y}_2}{(nX+c)c}\right]XY_1$$

Consequently by using (44) conditions we get that

$$\frac{dV_4}{dt} \leq -\left[\sqrt{\frac{q_{11}}{2}}(Y_1 - \bar{Y}_1) + \sqrt{\frac{q_{22}}{2}}(Y_2 - \bar{Y}_2)\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(Y_1 - \bar{Y}_1) + \sqrt{\frac{q_{33}}{2}}(E - \bar{E})\right]^2$$

$$- \left[\sqrt{\frac{q_{22}}{2}}(Y_2 - \bar{Y}_2) + \sqrt{\frac{q_{33}}{2}}(E - \bar{E})\right]^2 - \left[r - \left(m_1 + \frac{ns\bar{Y}_1^3 + ns\bar{Y}_1^2\bar{Y}_2}{(nX+c)c}\right)\right]X$$ (45)

Where

$$q_{11} = \frac{s(Y_1 + \bar{Y}_1 + \bar{Y}_2)}{(nX+c)} + \gamma\bar{E} + \beta\bar{Y}_2 + (m_2 + d_1) - s, q_{12} = \frac{csY_1}{(nX+c)} + \beta Y_1 - (\alpha_2 + \gamma\bar{E} + \beta\bar{Y}_2),$$

$$q_{22} = \alpha_2 + d_2 + \mu - \beta Y_1, q_{13} = \gamma Y_1, q_{23} = \theta\bar{E} - (\gamma Y_1 + \theta), q_{33} = \theta Y_2 + \gamma_1$$

Obviously, $\frac{dV_4}{dt}$ is negative definite and hence V_4 is Layapunov function with respect to E_4. So E_4 is globally asymptotically stable in the subregion that satisfies the given condition. ∎

Theorem (5.6): Assume that the equilibrium point E_5 is locally asymptotically stable. Then it is a globally asymptotically stable in the subregion of $\mathfrak{R}_+^4$ that satisfied the following conditions

$$
\begin{aligned}
&\frac{\alpha_1 \bar{\bar{Y}}_1}{(k_2+X)(k_2+\bar{\bar{X}})} < \frac{r}{k_1} \\
&\left(s + \beta \bar{\bar{Y}}_2\right) < \frac{s\left(Y_1 + \bar{\bar{Y}}_1 + Y_2\right)}{nX+c} + m_2 + d_1 + \gamma E \\
&q_{12}^2 < \frac{4}{3} q_{11} q_{22} \\
&q_{23}^2 < \frac{2}{3} q_{22} q_{33} \\
&q_{24}^2 < \frac{2}{3} q_{22} q_{44} \\
&q_{34}^2 < q_{33} q_{44}
\end{aligned}
\tag{46}
$$

Proof: Consider the following positive definite function

$$
V_5(X, Y_1, Y_2, E) = \left(X - \bar{\bar{X}} - \bar{\bar{X}} \ln \frac{X}{\bar{\bar{X}}}\right) + \frac{\left(Y_1 - \bar{\bar{Y}}_1\right)^2}{2} + \frac{\left(Y_2 - \bar{\bar{Y}}_2\right)^2}{2} + \frac{\left(E - \bar{\bar{E}}\right)^2}{2}
$$

Clearly, $V_5 : \mathfrak{R}_+^4 \to \mathfrak{R}$ is a continuously differentiable function such that $V_5\left(\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}}\right) = 0$ and $V_5(X, Y_1, Y_2, E) > 0, \forall (X, Y_1, Y_2, E) \in \mathfrak{R}_+^4$ and $(X, Y_1, Y_2, E) \neq \left(\bar{\bar{X}}, \bar{\bar{Y}}_1, \bar{\bar{Y}}_2, \bar{\bar{E}}\right)$.

$$
\begin{aligned}
\frac{dV_5}{dt} =\ & \left(X - \bar{\bar{X}}\right)\left[r - \frac{r}{k_1}X - \frac{\alpha_1 Y_1}{k_2+X} - m_1\right] \\
&+ \left(Y_1 - \bar{\bar{Y}}_1\right)\left[s Y_1\left(1 - \frac{Y_1}{nX+c}\right) - \frac{s Y_1 Y_2}{nX+c} - \gamma Y_1 E - \beta Y_1 Y_2 + \alpha_2 Y_2 - (m_2 + d_1)Y_1\right] \\
&+ \left(Y_2 - \bar{\bar{Y}}_2\right)\left[\gamma Y_1 E + \beta Y_1 Y_2 - \alpha_2 Y_2 - (d_2 + \mu)Y_2\right] + \left(E - \bar{\bar{E}}\right)\left[\theta Y_2 - \theta Y_2 E - \gamma_1 E\right]
\end{aligned}
$$

Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$
\begin{aligned}
\frac{dV_5}{dt} =\ & -\left[q_{11}\left(X - \bar{\bar{X}}\right)^2 + q_{12}\left(X - \bar{\bar{X}}\right)\left(Y_1 - \bar{\bar{Y}}_1\right) + \frac{q_{22}}{3}\left(Y_1 - \bar{\bar{Y}}_1\right)^2\right] \\
&- \left[\frac{q_{22}}{3}\left(Y_1 - \bar{\bar{Y}}_1\right)^2 + q_{23}\left(Y_1 - \bar{\bar{Y}}_1\right)\left(Y_2 - \bar{\bar{Y}}_2\right) + \frac{q_{33}}{2}\left(Y_2 - \bar{\bar{Y}}_2\right)^2\right] \\
&- \left[\frac{q_{22}}{3}\left(Y_1 - \bar{\bar{Y}}_1\right)^2 + q_{24}\left(Y_1 - \bar{\bar{Y}}_1\right)\left(E - \bar{\bar{E}}\right) + \frac{q_{44}}{2}\left(E - \bar{\bar{E}}\right)^2\right] \\
&- \left[\frac{q_{33}}{2}\left(Y_2 - \bar{\bar{Y}}_2\right)^2 + q_{34}\left(Y_2 - \bar{\bar{Y}}_2\right)\left(E - \bar{\bar{E}}\right) + \frac{q_{44}}{2}\left(E - \bar{\bar{E}}\right)^2\right]
\end{aligned}
$$

Consequently by using (46) conditions we get that

$$\frac{dV_5}{dt} \leq -\left[\sqrt{q_{11}}(X - \bar{\bar{X}}) + \sqrt{\frac{q_{22}}{3}}(Y_1 - \bar{\bar{Y}}_1)\right]^2 - \left[\sqrt{\frac{q_{22}}{3}}(Y_1 - \bar{\bar{Y}}_1) + \sqrt{\frac{q_{33}}{2}}(Y_2 - \bar{\bar{Y}}_2)\right]^2$$

$$- \left[\sqrt{\frac{q_{22}}{3}}(Y_1 - \bar{\bar{Y}}_1) + \sqrt{\frac{q_{44}}{2}}(E - \bar{\bar{E}})\right]^2 - \left[\sqrt{\frac{q_{33}}{3}}(Y_2 - \bar{\bar{Y}}_2) + \sqrt{\frac{q_{44}}{2}}(E - \bar{\bar{E}})\right]^2 \tag{47}$$

Where

$$q_{11} = \frac{r}{k_1} - \frac{\alpha_1 \bar{Y}_1}{(k_2+X)(k_2+\bar{\bar{X}})}, q_{12} = \frac{\alpha_1}{(k_2+X)} - \frac{sn\bar{\bar{Y}}_1(\bar{\bar{Y}}_1+\bar{\bar{Y}}_2)}{(nX+c)(n\bar{X}+c)},$$

$$q_{22} = \frac{s(Y_1+\bar{Y}_1+Y_2)}{nX+c} + m_2 + d_1 + \gamma E - (s + \beta\bar{\bar{Y}}_2), q_{23} = \frac{s\bar{\bar{Y}}_1}{nX+c} - (\beta Y_1 + \alpha_2 + \gamma E + \beta Y_2),$$

$$q_{33} = \alpha_2 + d_2 + \mu - \beta\bar{\bar{Y}}_1, q_{24} = \gamma\bar{\bar{Y}}_1, q_{34} = -\theta(1 - E) - \gamma\bar{\bar{Y}}_1, q_{44} = \theta\bar{\bar{Y}}_2 + \gamma_1$$

Obviously, $\frac{dV_5}{dt}$ is negative definite and hence V_5 is Layapunov function with respect to E_5. So E_5 is globally asymptotically stable in the sub region that satisfies the given condition.   ∎

6. Numerical Simulation

To visualize the above analytical findings and understand the effect of varying the parameters on the global dynamics of the system (1), numerical simulation is done in this section. The objectives of this study are confirming our obtained analytical results and detecting the set of control parameters that affect the dynamics of the system. Consequently, system (1) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that for the following set of hypothetical parameters the system (1) has a globally asymptotically stable positive equilibrium point as shown in the below figures:

$$r = 1.1, \ c = 0.5, \ \alpha_1 = 0.6, \ m_1 = 0.3, \ s = 1.2, \ k_2 = 0.6$$
$$m_2 = 0.3, \ d_1 = 0.1, \gamma = 1.2, \alpha_2 = 0.5, \ d_2 = 0.1, \ \mu = 0.2 \tag{48}$$
$$n = 0.7, \ k_1 = 0.7, \ \beta = 0.9, \theta = 0.6, \ \gamma_1 = 0.2$$

We obtained that the trajectories of system (1) with three different sets of positive initial conditions approach asymptotically to the positive equilibrium point

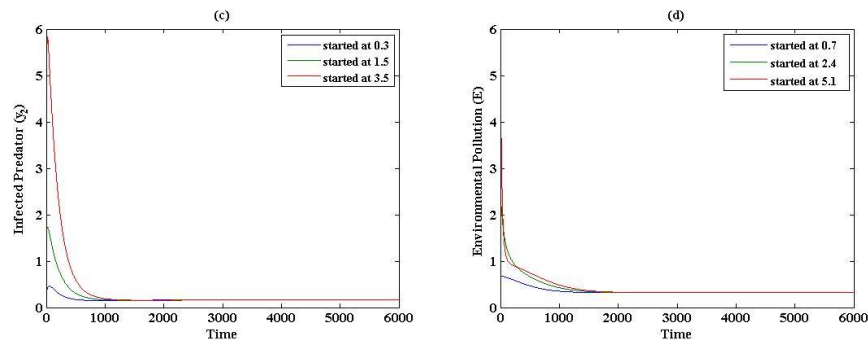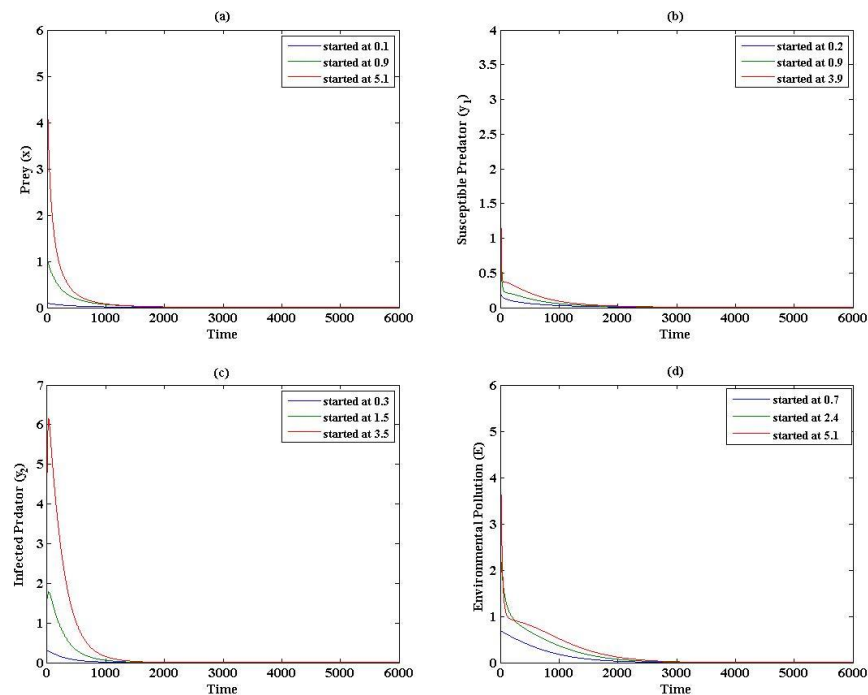$E_5 = (0.419, 0.239, 0.158, 0.322)$ as shown in figure.1
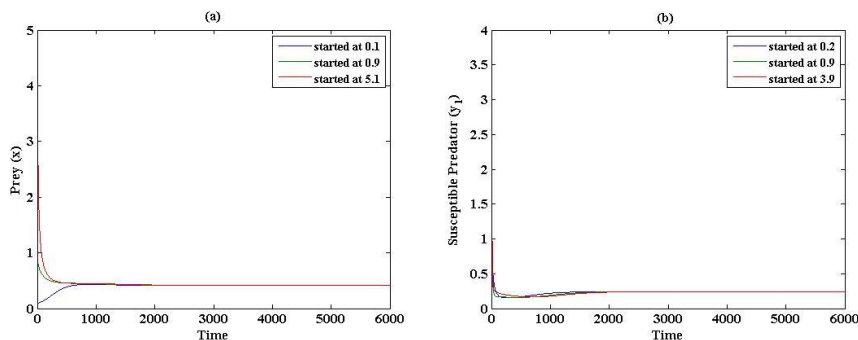
Figure 1. Globally asymptotically stable positive equilibrium point E_5 of system (1) for:

(a) Trajectories of X(t)   (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d) Trajectories of E(t)

Clearly, figure. 1 confirms our obtained analytical results regarding to existence that positive equilibrium point is a globally asymptotically stable. However, for the data by equation (48) with $r = 0.2$ and $s = 0.3$, the solution of system (1) approaches asymptotically to the vanishing equilibrium point with no migration as shown in the following typical, figure 2
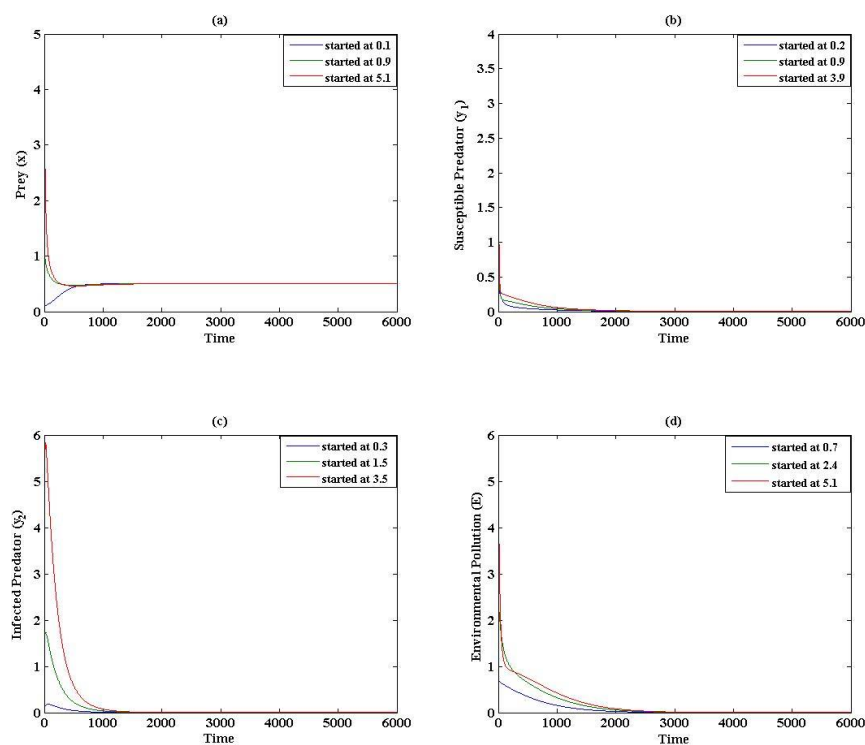

Figure 2- Globally asymptotically stable of vanishing equilibrium point E_0 of system (1) for:

(a) Trajectories of X(t)   (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d)

Trajectories of E(t)

Now in order to investigate the effect of varying parameters value at a time on the dynamical behavior of system (1), the following results are observed. According to the figure 3, it is clear that the solution of system (1) approaches asymptotically to the disease-free wherein migration of prey equilibrium point for the parameters values given in Eq. (48) with varying $s = 0.8, m_2 = 0.5$ and $d_1 = 0.4$, to obtain the trajectories of system (1) approach asymptotically to the $E_1 = (0.509,0,0,0)$ as shown in Figure. 3

Figure 3- Globally asymptotically stable of axial equilibrium point E_1 of system (1) for:

(a) Trajectories of X(t)   (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d) Trajectories of E(t)

Again, we choose the intrinsic growth rate and the environment coefficient values $r = 0.5, \gamma_1 = 0.8$ respectively, keeping other parameters fixed as given in equation (48), we get the trajectories of system (1) still approaches to the second axial equilibrium point while the healthy predator increases. Furthermore, the effect of environment is not most different, as shown in figure 4.
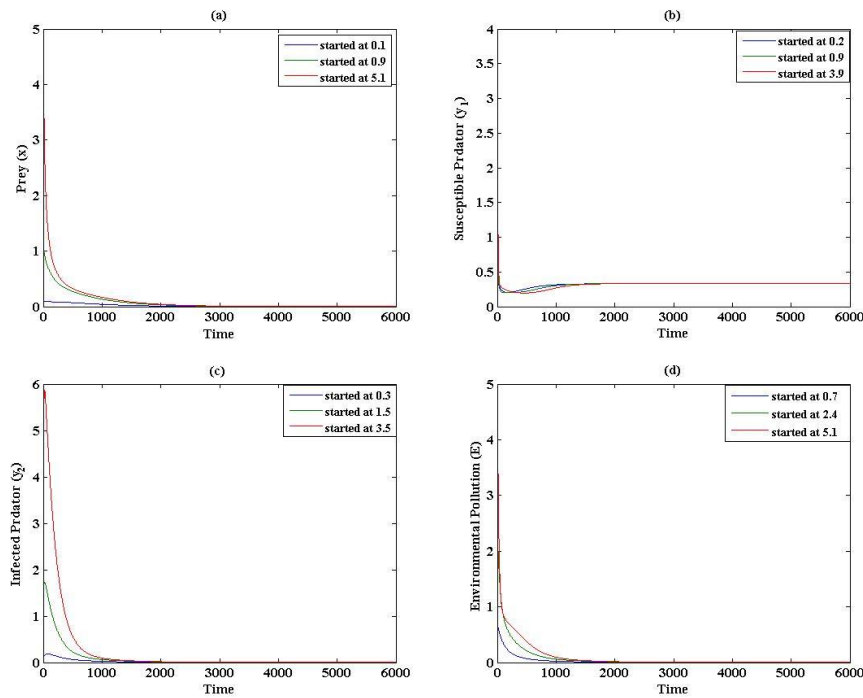
Figure 4- Globally asymptotically stable of equilibrium point E_2 of system (1)

(a) Trajectories of $X(t)$ (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d)

Trajectories of $E(t)$

On the other hand, system (1) for the following set of hypothetical data approaches asymptotically to the first planar equilibrium point as shown in figure 5,

$$
\begin{aligned}
&r = 1.1,\; m_1 = 0.3,\; s = 0.5,\; n = 0.5,\; c = 0.5,\; \gamma_1 = 0.2 \\
&m_2 = 0.3,\; d_1 = 0.01, \gamma = 0.02, \mu = 0.2,\; \theta = 0.6,\; \beta = 0.09 \\
&k_1 = 0.7,\; \alpha_1 = 0.06,\; \alpha_2 = 0.1,\; k_2 = 0.6,\; d_2 = 0.1
\end{aligned}
\tag{49}
$$

We obtained that the trajectories of system (1) with different sets of initial conditions approach asymptotically to the positive equilibrium point $E_3 = (0.497, 0.322, 0, 0)$ as shown in figure 5.
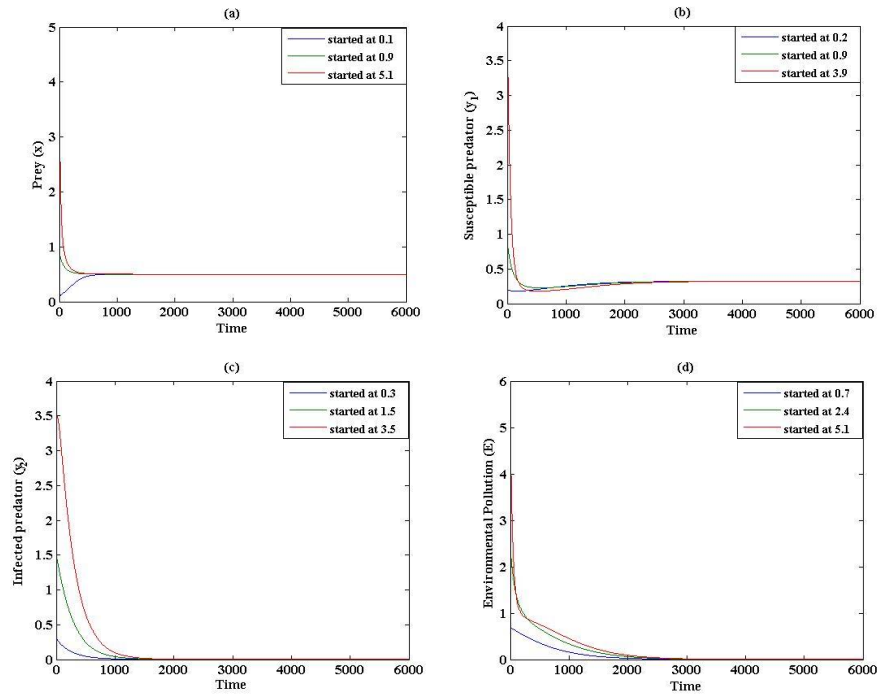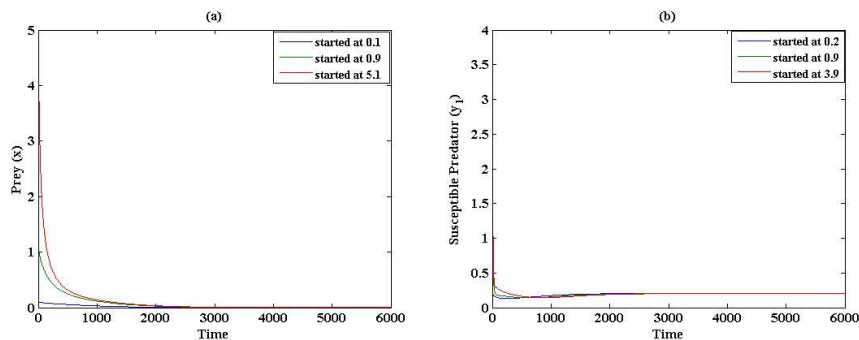
Figure 5- Globally asymptotically stable of interior equilibrium point E_3 of system (1)

(a) Trajectories of X(t)   (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d)

Trajectories of E(t)

Again, we choose the intrinsic growth rate coefficient values $r = 0.35$, keeping other parameter fixed as given in equation (48), we get the trajectories of system (1) still approaches to Prey-free equilibrium point with migration permitted and infection occurred. Furthermore, the effect of environment is not most different, as shown in figure 6.
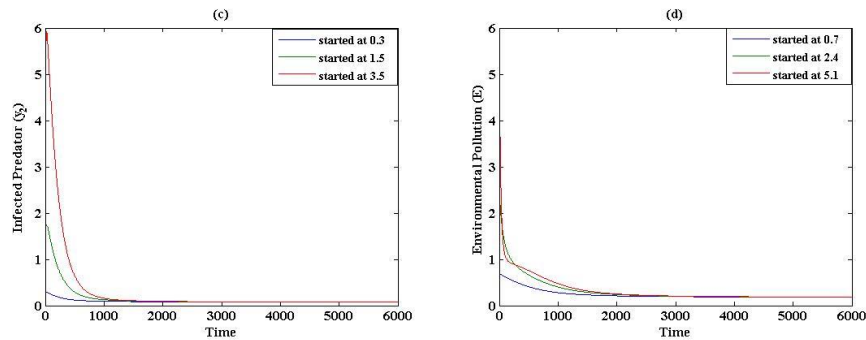
Figure 6- Globally asymptotically stable of equilibrium point E_4 of system (1) for:

(a) Trajectories of X(t)   (b) Trajectories of $Y_1(t)$ (c) Trajectories of $Y_2(t)$ (d)

Trajectories of E(t)

7 CONCLUSIONS AND DISCUSSION

In this paper, we consider a predator-prey model with modified Leslie-Gower and Holling type-II functional response. We discuss the structure of nonnegative equilibria and their local stability. Migration has been allowed among prey and healthy predator population. It is also remarkable that Holling type-II functional responses are more frequently used as compare to other functional responses. By the above discussion, we can note that each of the functional responses are useful and have their specific importance in ecology. However, in the present study we have considered Holling type-II functional response.

Finally, to complete our understanding to the global dynamical behavior of system (1), numerical simulation is used using hypothetical set of parameters values given by Eq. (48) and (49). In the following, the obtained numerical simulation results are summarized.

1. The trajectory of system (1) approaches asymptotically to positive equilibrium point starting from different initial points using the data Eq. (48), which indicates to existence of globally asymptotically stable positive equilibrium point.

2. Increasing the inhibition rate of disease or disease death rate above a specific value leads to extinction in predator species due to the lack in their food. Further increasing at least one of these parameters causes extinction in the infected prey specie and the trajectory of system (1) approaches asymptotically to free equilibrium point. Otherwise, the system still persists at a positive equilibrium point.

3. We observed that migration of prey (m_1) plays a leading role in the existence and stability of equilibria of systems (1).

4. (E_5) is most important equilibrium point since it provides the coexistence of all the four species simultaneously. For ecological balance of an eco-system coexistence of all the species in respective proportions is very important. The stability of (E_5) indicates the existence of all the species for a long time.

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