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# Separation Axioms with Grill-Topological Open Set

# Separation Axioms with Grill-Topological Open Set

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**Abstract.** The concepts $\mathbb{G}$-g-closedness and $\mathbb{G}$-g-openness were used to popularize presented modern classes of separation axioms in grill topological spaces. Many relationships between multiple kinds of these classes are summarized, too.

## 1. Introduction

A nonempty collection $\mathbb{G}$ of nonempty subsets of a topological space $X$ is named a Grill if

**i.** $A \in \mathbb{G}$ and $A \subseteq \mathcal{B} \subseteq X$ then $\mathcal{B} \in \mathbb{G}$

**ii.** $A, \mathcal{B} \subseteq X$ and $A \cup \mathcal{B} \in \mathbb{G}$ then $A \in \mathbb{G}$ *or* $\mathcal{B} \in \mathbb{G}$ [1]. Let $X$ be a nonempty set. Then the following families are grills on $X$. [1-3]

- $\{\emptyset\}$ and $P(X)\backslash\{\emptyset\}$ are trivial examples of grills on $X$.
- $\mathbb{G}_{\infty}$ the grill of all infinite subsets of $X$.
- $\mathbb{G}_{co}$ the grill of all uncountable subsets of $X$.
- $\mathbb{G}_p = \{ \Lambda: \Lambda \in P(X), p \in \Lambda \}$ is a specific point grills on $X$.
- $\mathbb{G}_{A} = \{ \mathscr{B}: \mathscr{B} \in P(X), \mathscr{B} \cap A^c \neq \emptyset\}$, and

If $(X, t)$ is a topological space, then the family of all non-nowhere dense subsets called $\mathbb{G}_{\varsigma} = \{ A \subseteq X: int_t\, cl_t(A) \neq \emptyset\}$. Is the one of kinds of grill on $X$.

Let $\mathbb{G}$ be a grill on a topological space$(X, t)$. The operator $\varphi$: $P(X) \rightarrow P(X)$ was defined by $\varphi(A) = \{x \in X \backslash U \cap A \in \mathbb{G}$, for all $U \in t(x)\}$, $t(x)$ denotes the neighborhood of $x$. A mapping $\Psi$: $P(X) \rightarrow P(X)$ is defined as $\Psi(A) = A \cup \varphi(A)$ for all $A \in P(X)$ [4-6].

The map $\Psi$ satisfies Kuratowski closure axioms: [3,4]

(i) $\Psi(\emptyset) = \emptyset$,

(ii) If $A \subseteq \mathcal{B}$, then $\Psi(A) \subseteq \Psi(\mathcal{B})$,

(iii) If $A \subseteq X$, then $\Psi(\Psi(A)) = \Psi(A)$,

(iv) If $A, \mathscr{B} \subseteq X$, then $\Psi(A \cup \mathcal{B}) = \Psi(A) \cup \Psi(\mathcal{B})$.

In this research, we presented $\mathbb{G}$-g-closed set and it is complement $\mathbb{G}$-g-open set some notions have been presented by new kinds of separation axioms like: $\mathbb{G}$-g-$T_0$-space, $\mathbb{G}$-g-$T_1$-space, $\mathbb{G}$-g-$T_2$-space. ■

## 2. Separation Axioms with Grill-Topological Open Set

Definition 2. 1: In grill topological space (Ӿ, ŧ, 𝔾), let 𝔻 ⊆ Ӿ. 𝔻 is said to be grill-g-closed set denoted by "𝔾-g-closed", if (𝔻 - Ư) ∉ 𝔾 then, ($cl$(𝔻) - Ư) ∉ 𝔾 where every, Ư ⊆ Ӿ and Ư ∈ ŧ.
Now, $D^c$ is a grill-g-open set denoted by "𝔾-g-open" .The family of all "𝔾-g-closed" sets denoted by 𝔾**g**C(Ӿ). The family of all "𝔾-g-open" sets denoted by 𝔾**g**O(Ӿ).

Example 2. 2: Consider the space (Ӿ, ŧ, 𝔾), where Ӿ = { $ƒ_1$, $ƒ_2$, $ƒ_3$}, ŧ = {Ӿ, ∅, { $ƒ_1$}, { $ƒ_2$}, { $ƒ_1$, $ƒ_2$}}, and 𝔾 = {Ӿ, { $ƒ_1$}, { $ƒ_1$, $ƒ_2$}, { $ƒ_1$, $ƒ_3$}}. So,
P(Ӿ) = { Ӿ, ∅, { $ƒ_1$}, { $ƒ_2$}, { $ƒ_3$}, { $ƒ_1$, $ƒ_2$}, { $ƒ_1$, $ƒ_3$}, { $ƒ_2$, $ƒ_3$}}
𝔾gC(Ӿ) = { Ӿ, ∅, { $ƒ_1$}, { $ƒ_2$}, { $ƒ_3$}, { $ƒ_1$, $ƒ_2$}, { $ƒ_1$, $ƒ_3$}, { $ƒ_2$, $ƒ_3$}}, 𝔾gO(Ӿ) = { Ӿ, ∅, { $ƒ_1$}, { $ƒ_2$}, { $ƒ_3$}, { $ƒ_1$, $ƒ_2$}, { $ƒ_1$, $ƒ_3$}, { $ƒ_2$, $ƒ_3$}}.

Remark 2. 3: For any (Ӿ, ŧ, 𝔾) then
**i.** Every closed set is a 𝔾-g-closed set.
**ii.** Every open set is a 𝔾-g-open set.
The converse of Remark 2. 3 is not true. See Example 2. 2.
{$ƒ_1$} is a 𝔾-g-closed set, but {$ƒ_1$} is not closed set.
{$ƒ_1$, $ƒ_3$} is a 𝔾-g-open set, but {$ƒ_1$, $ƒ_3$} is not open set.

Definition 2. 4: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-$Ŧ_0$-space shortly "𝔾g-$Ŧ_0$-space" if for each ɱ ≠ ɳ and ɱ, ɳ ∈ Ӿ, there exist Ư ∈ 𝔾g-o(Ӿ) whenever, ɱ ∈ Ư and ɳ ∉ Ư or ɱ ∉ Ư and ɳ ∈ Ư.

Example 2. 5: Consider the space (Ӿ, ŧ, 𝔾), where Ӿ = { $ƒ_1$, $ƒ_2$ }, ŧ = {Ӿ, ∅, { $ƒ_1$}} and 𝔾 = {Ӿ, { $ƒ_2$}}. 𝔾gC(Ӿ) = { Ӿ, ∅, { $ƒ_2$}}, 𝔾gO(Ӿ) = { Ӿ, ∅, { $ƒ_1$}}. Hence (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_0$-space.

Proposition 2. 6: If (Ӿ, ŧ ) is a $Ŧ_0$-space then (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_0$-space.
Proof: By Remark 2. 3. (ii).For each Ư ∈ ŧ, then Ư is a 𝔾g-open set ,so the proof is over.

Definition 2. 7: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-$Ŧ_1$-space shortly "𝔾g-$Ŧ_1$-space" if for each ɱ, ɳ ∈ Ӿ and ɱ ≠ ɳ. Then there are 𝔾g-open sets $Ư_1$, $Ư_2$ whenever ɱ ∈ $Ư_1$ , ɳ ∉ $Ư_1$ and ɳ ∈ $Ư_2$ , ɱ ∉ $Ư_2$.

Example 2. 8: A space (Ӿ, ŧ, 𝔾) when Ӿ = ₦, where ₦ is the set of all natural numbers, ŧ= $ŧ_{cof}$ = { Ư ⊆ ₦, $Ư^c$ is a finite set} ∪ ∅ *and* 𝔾 = P(Ӿ)\{∅}. So (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space. Since for each ɱ, ɳ ∈ Ӿ and ɱ ≠ ɳ. Then there are 𝔾g-open sets (₦ - {ɳ}), (₦ - {ɱ}) whenever {ɳ} and {ɱ} are two finite sets such that (ɱ ∈ (₦ - {ɳ})) , (ɳ ∉ (₦ - {ɳ})) and (ɱ ∉ (₦ - {ɱ})) , (ɳ ∈ (₦ - {ɱ})).

Proposition 2. 9**:** If (Ӿ, ŧ ) is a $Ŧ_1$-space then (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space.
Proof: By Remark 2. 3.(ii).For each Ư ∈ ŧ, then Ư is a 𝔾g-open set ,so the proof is over.

Proposition 2. 10: If (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space then it is a 𝔾g-$Ŧ_0$-space.

Proof: Let ɱ, ɳ ∈ Ӿ such that ɱ ≠ ɳ since (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space, then there exists $Ư_1$, $Ư_2$∈ 𝔾g-o(Ӿ) such that, ɱ ∈ $Ư_1$ , ɳ ∉ $Ư_1$ and ɳ ∈ $Ư_2$ , ɱ ∉ $Ư_2$. Then there exist Ư ∈ 𝔾g-o(Ӿ) whenever, ɱ ∈ Ư , ɳ ∉ Ư and ɱ ∉ Ư , ɳ ∈ Ư .
The converse of proposition 2. 10 is not true for example 2. 11.

Example 2. 11: The grill topological space (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_0$-space, where Ӿ = {$ƒ_1$, $ƒ_2$}, ŧ = {Ӿ, ∅, { $ƒ_2$}} and 𝔾 = {Ӿ, { $ƒ_1$}}. 𝔾gC(Ӿ) = { Ӿ, ∅, { $ƒ_1$}}, 𝔾gO(Ӿ) = { Ӿ, ∅, {$ƒ_2$}}. The grill topological space (Ӿ, ŧ, 𝔾) is not 𝔾g-$Ŧ_1$-space, since for any elements $ƒ_1$ ≠ $ƒ_2$, there is no 𝔾g-open set Ư containing $ƒ_1$ which does not contain $ƒ_2$.

Definition 2. 12: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-Ŧ$_2$-space shortly "𝔾g-Ŧ$_2$-space" if for each ɱ ≠ ɲ. There are 𝔾g-open sets Ư$_1$ , Ư$_2$ whenever ɱ ∈ Ư$_1$ , ɲ ∈ Ư$_2$ and Ư$_1$ ∩ Ư$_2$ = ∅.

Example 2. 13: Consider the space (Ӿ, ŧ, 𝔾), where Ӿ = { ʄ$_1$, ʄ$_2$, ʄ$_3$}, ŧ = {Ӿ, ∅, { ʄ$_1$}, {ʄ$_2$}, { ʄ$_1$, ʄ$_2$}}, and 𝔾 = {Ӿ, { ʄ$_1$}, { ʄ$_1$, ʄ$_2$ }, { ʄ$_1$ , ʄ$_3$}}. So, 𝔾gC(Ӿ) = P(Ӿ) = 𝔾gO(Ӿ). Hence (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space.

Proposition 2. 14: If (Ӿ, ŧ ) is a Ŧ$_2$-space then (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space.

Proof: By Remark 2. 3. (ii).For each Ư ∈ ŧ, then Ư is a 𝔾g-open set ,so the proof is over.

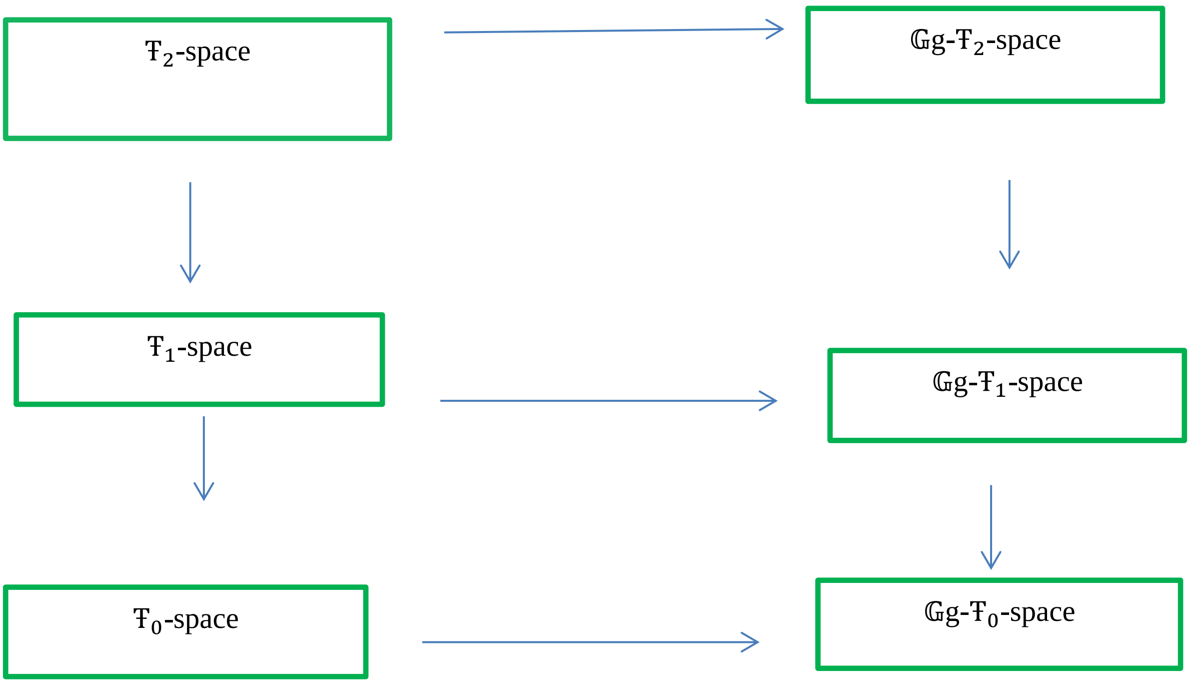
Proposition 2. 15: If (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space then it is a 𝔾g-Ŧ$_1$-space.

Proof: Let ɱ, ɲ ∈ Ӿ whenever, ɱ ≠ ɲ since (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space, then there are 𝔾g-open sets Ư$_1$, Ư$_2$ such that ɱ ∈ Ư$_1$, ɲ ∈ Ư$_2$ and Ư$_1$ ∩ Ư$_2$ = ∅. Implies ɱ ∈ Ư$_1$ and ɲ ∉ Ư$_1$ or ɲ ∈ Ư$_2$ and ɱ ∉ Ư$_2$ .
The converse of proposition 2. 15 is not true see example 2. 8.
A space (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_1$-space. Which is not 𝔾g-Ŧ$_2$-space, since there is no two 𝔾g-open sets, Ư$_1$, Ư$_2$ such that Ư$_1$ ∩ Ư$_2$ = ∅.

Remark 2. 16: We have formerly noted that (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_i$-space whenever, it is a Ŧ$_i$-space (for every $i$ ∈ { 0, 1, 2 }).
The converse of Remark 2. 16 is not true in general for example 2. 17.

Example 2. 17: Consider the space (Ӿ, ŧ, 𝔾), where Ӿ = { ʄ$_1$, ʄ$_2$, ʄ$_3$}, ŧ = {Ӿ, ∅}, and 𝔾 = P(Ӿ)\{∅}. So, 𝔾gC(Ӿ) = P(Ӿ) = 𝔾gO(Ӿ). Then the space (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_i$-space (for every $i$ ∈ { 0, 1, 2 }). But the space (Ӿ, ŧ) is not Ŧ$_i$-space (for every $i$ ∈ { 0, 1, 2 }).
The following diagram shows the relations between the various kinds of concepts of our formerly mentioned.

**Diagram (2. 1)**



Relationships among $Ŧ_i$-space and 𝔾g-$Ŧ_i$-space

## 3. Separation Axioms with Some Types of Open Functions

Definition 3. 1: The function Ꜧ: ( Ӿ, ŧ, 𝔾) →( Ұ, ţ, Ǥ ) is called
**i.** 𝔾-g-open function, shortly "𝔾go-function" if Ꜧ (Ư) ∈ Ǥg-o(Ұ) whenever, Ư ∈ 𝔾g-o(Ӿ).
**ii.** $𝔾^*$-g-open function, shortly "$𝔾^*$go-function" if Ꜧ (Ư) ∈ Ǥg-o(Ұ) whenever, Ư ∈ ŧ.
**iii.** $𝔾^{**}$-g-open function, shortly "$𝔾^{**}$go-function" if Ꜧ (Ư) ∈ ţ whenever, Ư ∈ 𝔾g-o(Ӿ).

Proposition 3. 2: If (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_i$-space and Ꜧ is an onto, 𝔾go-function from ( Ӿ, ŧ, 𝔾) to ( Ұ, ţ, Ǥ ) then Ұ is a Ǥg-$Ŧ_i$-space (for every $i$ ∈ { 0, 1, 2 }).

Proof: If $i$ = 0: Let ɱ, ƞ ∈ Ұ such that ɱ ≠ ƞ. Since Ꜧ is an onto function, then $Ꜧ^{-1}$(ɱ) ≠ ∅ ≠ $Ꜧ^{-1}$(ƞ) and $Ꜧ^{-1}$(ɱ), $Ꜧ^{-1}$(ƞ) ∈ Ӿ and $Ꜧ^{-1}$(ɱ) ≠ $Ꜧ^{-1}$(ƞ). Since (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_0$-space, then there exist Ư ∈ 𝔾g-o(Ӿ) whenever $Ꜧ^{-1}$(ɱ) ∈ Ư , $Ꜧ^{-1}$(ƞ) ∉ Ư or $Ꜧ^{-1}$(ɱ) ∉ Ư, $Ꜧ^{-1}$(ƞ) ∈ Ư. Since Ꜧ is a 𝔾go-function, then Ꜧ(Ư) is a Ǥg-open set such that ɱ ∈ Ꜧ(Ư) and ƞ ∉ Ꜧ(Ư) or ɱ ∉ Ꜧ(Ư) and ƞ ∈ Ꜧ(Ư). Hence Ұ is a Ǥg-$Ŧ_0$-space. If $i$ = 1: Let ɱ, ƞ ∈ Ұ such that ɱ ≠ ƞ. Since Ꜧ is an onto function, then $Ꜧ^{-1}$(ɱ) ≠ ∅ ≠ $Ꜧ^{-1}$(ƞ) and $Ꜧ^{-1}$(ɱ), $Ꜧ^{-1}$(ƞ) ∈ Ӿ and $Ꜧ^{-1}$(ɱ) ≠ $Ꜧ^{-1}$(ƞ). So (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space, then there exists $Ư_1$, $Ư_2$ ∈ 𝔾g-o(Ӿ) such that $Ꜧ^{-1}$(ɱ) ∈ $Ư_1$ , $Ꜧ^{-1}$(ƞ) ∉ $Ư_1$ and $Ꜧ^{-1}$(ƞ) ∈ $Ư_2$ , $Ꜧ^{-1}$(ɱ) ∉ $Ư_2$. By the condition Ꜧ is a 𝔾go-function, Ꜧ($Ư_1$), Ꜧ($Ư_2$) are Ǥg-open sets in Ұ such that ɱ ∈ Ꜧ($Ư_1$) , ƞ ∉ Ꜧ($Ư_1$) and ƞ ∈ Ꜧ($Ư_2$) , ɱ ∉ Ꜧ($Ư_2$). Hence ( Ұ, ţ, Ǥ ) is a Ǥg-$Ŧ_1$-space.
If $i$ = 2: The same proof above , but Ꜧ($Ư_1$) ∩ Ꜧ($Ư_2$) = ∅. Hence ( Ұ, ţ, Ǥ ) is a Ǥg-$Ŧ_2$-space.

Corollary 3. 3: If (Ӿ, ŧ ) is a $Ŧ_i$-space and Ꜧ is an onto, $𝔾^*$go-function from ( Ӿ, ŧ, 𝔾) to ( Ұ, ţ, Ǥ ) then ( Ұ, ţ, Ǥ ) is a Ǥg-$Ŧ_i$-space, where $i$ ∈ { 0, 1, 2 }.

Proof: Follows from Ꜧ(Ư) is a Ǥg-open in ( Ұ, ţ, Ǥ ) for all open set Ư in Ӿ .

Corollary 3. 4: If (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_i$-space and Ꜧ is an onto, $𝔾^{**}$go-function from ( Ӿ, ŧ, 𝔾) to ( Ұ, ţ, Ǥ ) then Ұ is a $Ŧ_i$-space, where $i$ ∈ { 0, 1, 2 }.

Proof: Follows from Ꜧ(Ư) is an open set in Ұ for all 𝔾g-open set Ư in ( Ӿ, ŧ, 𝔾).

Corollary 3. 5: If Ꜧ is an onto and open function from ( Ӿ, ŧ ) to ( Ұ, ţ ) and (Ӿ, ŧ) is a $Ŧ_i$-space, then ( Ұ, ţ, Ǥ ) is a Ǥg-$Ŧ_i$-space, where $i$ ∈ { 0, 1, 2 }, for any grill Ǥ on ( Ұ, ţ, Ǥ ).

Definition 3. 6: The function Ꜧ: ( Ӿ, ŧ, 𝔾) →( Ұ, ţ, Ǥ ) is called
**i.** 𝔾-g-continuous function, shortly "𝔾g-continuous function" if $Ꜧ^{-1}$(Ư) ∈ 𝔾g-o(Ӿ) for all Ư ∈ ţ.
**ii.** Strongly-𝔾-g-continuous function, shortly "Strongly-𝔾g-continuous function" if $Ꜧ^{-1}$(Ư) ∈ ŧ for every, Ư ∈ Ǥg-o(Ұ).
**iii.** 𝔾-g-irresolute function, shortly "𝔾g-irresolute function" if $Ꜧ^{-1}$(Ư) ∈ 𝔾g-o(Ӿ) for every, Ư ∈ Ǥg-o(Ұ).

Proposition 3. 7: If ( Ұ, ţ ) is a $Ŧ_i$-space and Ꜧ: ( Ӿ, ŧ, 𝔾) → ( Ұ, ţ, Ǥ ) is an injective, 𝔾g-continuous function then ( Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_i$-space, where $i$ ∈ { 0, 1, 2 }.

Proof: If $i$ = 0: Let ɱ, ƞ ∈ Ӿ such that ɱ ≠ ƞ. Since Ꜧ is an injective function, then Ꜧ(ɱ) ≠ Ꜧ(ƞ), where, Ꜧ(ɱ), Ꜧ(ƞ) ∈ Ұ. So, ( Ұ, ţ ) is a $Ŧ_0$-space, then there exist Ư ∈ ţ whenever, Ꜧ(ɱ) ∈ Ư, Ꜧ(ƞ) ∉ Ư or Ꜧ(ɱ)

$\notin$ Ư, Ђ(ɳ) $\in$ Ư. By Ђ is a 𝔾g-continuous function, then Ђ$^{-1}$(Ư) $\in$ 𝔾g-o(Ӿ) whenever, ɱ $\in$ Ђ$^{-1}$(Ư), ɳ $\notin$ Ђ$^{-1}$(Ư) or ɱ $\notin$ Ђ$^{-1}$(Ư), ɳ $\in$ Ђ$^{-1}$(Ư). Hence ( Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_0$-space.
If $i$ = 1: Let ɱ, ɳ $\in$ Ӿ such that ɱ $\neq$ ɳ. Since Ђ is an injective function, then Ђ(ɱ) $\neq$ Ђ(ɳ), where, Ђ(ɱ), Ђ(ɳ) $\in$ Ұ. So, ( Ұ, ţ ) is a Ŧ$_1$-space, then there exists Ư$_1$, Ư$_2$ $\in$ ţ whenever, Ђ(ɱ) $\in$ Ư$_1$ , Ђ(ɳ) $\notin$ Ư$_1$ and Ђ(ɳ) $\in$ Ư$_2$ , Ђ(ɱ) $\notin$ Ư$_2$ . Since Ђ is a 𝔾g-continuous function, then Ђ$^{-1}$(Ư$_1$) and Ђ$^{-1}$(Ư$_2$) are 𝔾g-open sets whenever, ɱ $\in$ Ђ$^{-1}$(Ư$_1$) , ɳ $\notin$ Ђ$^{-1}$(Ư$_1$) and ɳ $\in$ Ђ$^{-1}$(Ư$_2$) , ɱ $\notin$ Ђ$^{-1}$(Ư$_2$). Hence (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_1$-space.
If $i$ = 2: The same proof above but Ђ(Ư$_1$) $\cap$ Ђ(Ư$_2$) = $\emptyset$. Hence ( Ұ, ţ, G ) is a Gg-Ŧ$_2$-space.

Remark 3. 8: Let Ђ: ( Ӿ, ŧ, 𝔾) $\rightarrow$( Ұ, ţ, G ) is a function
If Ђ is a continuous function, then Ђ is a 𝔾g- continuous function

Corollary 3. 9: If ( Ұ, ţ ) is a Ŧ$_i$-space and Ђ: ( Ӿ, ŧ, 𝔾) $\rightarrow$( Ұ, ţ, G ) is an injective continuous function, then ( Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_i$-space, where $i \in$ { 0, 1, 2 }.

Proof: Since every continuous function is a 𝔾g-continuous function by Proposition 3. 7, then Corollary 3. 5, is applicable

Proposition 3. 10: If ( Ұ, ţ, G ) is a Gg-Ŧ$_i$-space and Ђ: ( Ӿ, ŧ, 𝔾) $\rightarrow$ ( Ұ, ţ, G ) is an injective strongly-𝔾g-continuous function then ( Ӿ, ŧ ) is a Ŧ$_i$-space, where $i \in$ { 0, 1, 2 }.

Proof: Follows from, Ђ$^{-1}$(Ư) $\in$ ŧ for each Ư $\in$ Gg-o(Ұ).

Proposition 3. 11: If ( Ұ, ţ, G ) is a Gg-Ŧ$_i$-space and Ђ: ( Ӿ, ŧ, 𝔾) $\rightarrow$ ( Ұ, ţ, G ) is an injective 𝔾g-irresolute function then ( Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_i$-space, where $i \in$ { 0, 1, 2 }.

Proof: Since Ђ$^{-1}$(Ư) $\in$ 𝔾g-o(Ӿ) for each Ư $\in$ Gg-o(Ұ).

**4. Gg-Convergence Sequence**
Definition 4. 1: Let ( Ӿ, ŧ, 𝔾) be a grill topological space, where ӿ$_0$ $\in$ Ӿ and (Ş$_ɳ$)$_{ɳ \in ₦}$ be a sequence in Ӿ. Then (Ş$_ɳ$)$_{ɳ \in ₦}$ is called Gg-Convergence to ӿ$_0$ shortly Ş$_ɳ$ $\rightsquigarrow$ ӿ$_0$ if for every Gg-open set Ư where, ӿ$_0$ $\in$ Ư there exist Ꝁ $\in$ ₦ where, Ş$_ɳ$ $\in$ Ư for every ɳ $\geq$ Ꝁ. A sequence (Ş$_ɳ$)$_{ɳ \in ₦}$ is called Gg-divergence, if it is not Gg-Convergence.

Theorem 4. 2: If (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space then every Gg-Convergence sequence in Ӿ has a unique limit point.
Proof: Let (Ş$_ɳ$)$_{ɳ \in ₦}$ be a sequence in Ӿ where, Ş$_ɳ$ $\rightsquigarrow$ ӿ$_1$ and Ş$_ɳ$ $\rightsquigarrow$ ӿ$_2$; ӿ$_1$ $\neq$ӿ$_2$ where, ӿ$_1$, ӿ$_2$ $\in$ Ӿ. Since (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space then there exists Ư$_1$,Ư$_2$ $\in$ 𝔾go(Ӿ) such that ӿ$_1$ $\in$ Ư$_1$ and ӿ$_2$ $\in$ Ư$_2$ where Ư$_1$ $\cap$ Ư$_2$ = $\emptyset$. Since Ş$_ɳ$ $\rightsquigarrow$ ӿ$_1$ and ӿ$_1$ $\in$ Ư$_1$ $\in$ 𝔾go(Ӿ) implies there exist Ꝁ$_1$ $\in$ ₦; Ş$_ɳ$$\in$ Ư$_1$ for all ɳ $\geq$ Ꝁ$_1$. So, Ş$_ɳ$ $\rightsquigarrow$ ӿ$_2$ and ӿ$_2$ $\in$ Ư$_2$ $\in$ 𝔾go(Ӿ) implies there exist Ꝁ$_2$ $\in$ ₦; Ş$_ɳ$$\in$ Ư$_2$ for all ɳ $\geq$ Ꝁ$_2$. Hence, Ư$_1$ $\cap$ Ư$_2$ $\neq$ $\emptyset$, that is contradiction.
The prerequisite that a space Ӿ is a 𝔾g-Ŧ$_2$-space is very necessary to make Theorem 4. 2 is proper.

Example 4. 3**:** Let (Ӿ, ŧ, 𝔾) be a grill topological space, where, Ӿ = { ƒ$_1$, ƒ$_2$, ƒ$_3$}, ŧ = {Ӿ, $\emptyset$}, and 𝔾 = {Ӿ, { ƒ$_3$}, { ƒ$_1$, ƒ$_3$ }, { ƒ$_2$ , ƒ$_3$}}. Then 𝔾gC(Ӿ) = { Ӿ, $\emptyset$,{ ƒ$_3$}, { ƒ$_1$, ƒ$_3$}, { ƒ$_2$, ƒ$_3$}}, 𝔾gO(Ӿ) = { Ӿ, $\emptyset$, { ƒ$_1$}, { ƒ$_2$}, { ƒ$_1$, ƒ$_2$}}.The sequence (Ş$_ɳ$)$_{ɳ \in ₦}$ in Ӿ, where Ş$_ɳ$ = ƒ$_3$ for all ɳ $\in$ ₦. The sequence (Ş$_ɳ$)$_{ɳ \in ₦}$ has one limit point such that Ş$_ɳ$ $\rightsquigarrow$ ƒ$_3$, but (Ӿ, ŧ, 𝔾) is not 𝔾g-Ŧ$_2$-space.

Proposition 4. 4: If a sequence (Ş$_ɳ$)$_{ɳ \in ₦}$ is a Gg-convergence to ӿ$_0$ in (Ӿ, ŧ, 𝔾), then it is a convergence to ӿ$_0$.

Proof: Let Ư be an open set in Ӿ where, $x_0 \in$ Ư. By Remark 2. 3. (ii). Ư is a Gg-open set in Ӿ where, $x_0 \in$ Ư. Since $(Ş_n)_{n \in ₦}$ is a Gg-convergent to $x_0$, then there exist Ƙ ∈ ₦ where, $Ş_n \in$ Ư for every $n \geq$ Ƙ. Hence $(Ş_n)_{n \in ₦}$ is a convergent to $x_0$.
The converse of Proposition 4. 4 is not true for example 4. 5.

Example 4. 5: Let (Ӿ, ŧ, 𝔾) be a grill topological space, where, Ӿ = ₦ set of all natural numbers ŧ = {Ӿ, ∅}, and 𝔾 = P(Ӿ)\{∅}, 𝔾gC(Ӿ) = P(Ӿ) = 𝔾gO(Ӿ). The sequence $(Ş_n)_{n \in ₦}$ where $Ş_n = n$ for all $n \in$ ₦ is a convergent to $n$ for all $n \in$ ₦, which is not Gg-convergence for any element in ₦.

Proposition 4. 6: Let Ђ: ( Ӿ, ŧ, 𝔾) → ( Ұ, ţ, Ǥ ) be an injective and 𝔾g-irresolute function and $(Ş_n)_{n \in ₦}$ be a sequence in Ӿ. Then Ђ($Ş_n$) ⇝ Ђ( $x_0$) in ( Ұ, ţ, Ǥ ) whenever, $Ş_n$ ⇝ $x_0$ in ( Ӿ, ŧ, 𝔾).

Proof: Let Ư is a Gg-open set in Ұ where Ђ( $x_0$) ∈ Ư. Since Ђ is a 𝔾g-irresolute function, then $Ђ^{-1}$(Ư) is a Gg-open set where, $x_0 \in Ђ^{-1}$(Ư). Since $(Ş_n)_{n \in ₦}$ is a Gg-convergent to $x_0$, then there exist Ƙ ∈ ₦ where, $Ş_n \in Ђ^{-1}$(Ư) for all $n \geq$ Ƙ. Since Ђ is an injective function, then there exist Ƙ ∈ ₦ where, Ђ($Ş_n$) ∈ Ư for all $n \geq$ Ƙ. Hence Ђ($Ş_n$) is a Gg-convergent to Ђ( $x_0$).

Theorem 4. 7: Let Ђ: ( Ӿ, ŧ, 𝔾) → ( Ұ, ţ, Ǥ ) be an injective and 𝔾g-continuous function and $(Ş_n)_{n \in ₦}$ be a sequence in Ӿ. Then Ђ($Ş_n$) → Ђ( $x_0$) in ( Ұ, ţ, Ǥ ) whenever, $Ş_n$ ⇝ $x_0$ in ( Ӿ, ŧ, 𝔾).

Proof: Let Ư is an open set in ( Ұ, ţ, Ǥ ) where, Ђ( $x_0$) ∈ Ư. Since Ђ is a 𝔾g- continuous function, then $Ђ^{-1}$(Ư) is a Gg-open set in ( Ӿ, ŧ, 𝔾) where, $x_0 \in Ђ^{-1}$(Ư). Since $(Ş_n)_{n \in ₦}$ is a Gg-convergent to $x_0$, then there exist Ƙ ∈ ₦ where, $Ş_n \in Ђ^{-1}$(Ư) for all $n \geq$ Ƙ. Since Ђ is an injective function, then there exist Ƙ ∈ ₦ where, Ђ($Ş_n$) ∈ Ư for all $n \geq$ Ƙ. Hence Ђ($Ş_n$) is a convergent to Ђ( $x_0$).

Proposition 4. 8: Let Ђ: ( Ӿ, ŧ, 𝔾) → ( Ұ, ţ, Ǥ ) be an injective and strongly-𝔾g-continuous function and $(Ş_n)_{n \in ₦}$ be a sequence in Ӿ. Then Ђ($Ş_n$) ⇝ Ђ( $x_0$) in ( Ұ, ţ, Ǥ ) whenever, $Ş_n$ → $x_0$ in ( Ӿ, ŧ, 𝔾).

Proof: Let Ư is a Gg-open set in ( Ұ, ţ, Ǥ ) where Ђ( $x_0$) ∈ Ư. Since Ђ is a strongly-𝔾g-continuous function, then $Ђ^{-1}$(Ư) is an open set in Ӿ where, $x_0 \in Ђ^{-1}$(Ư). Since $(Ş_n)_{n \in ₦}$ is a convergent to $x_0$, then there exist Ƙ ∈ ₦ where, $Ş_n \in Ђ^{-1}$(Ư) for all $n \geq$ Ƙ. Since Ђ is an injective function, then there exist Ƙ ∈ ₦ where, Ђ($Ş_n$) ∈ Ư for all $n \geq$ Ƙ. Hence Ђ($Ş_n$) is a Gg-convergent to Ђ( $x_0$).

## References

[1] Choquet G 1947 Sur les notions de filter et grille *Comptes Rendus Acad. Sci. Paris* **224** 171-173
[2] Roy B and Mukherjee M N 2007 On a type of compactness via grills *Matematicki Vesnik*. **59** 113-120
[3] Roy B and Mukherjee M N 2007 on a typical topology induced by a grill *Soochow J. Math* **33** (4) 771-786
[4] Shawqi A Hazza, Sobhy A EL-Sheikh, Ali Kandil and Mohamed Ahmed Abdelhakem 2015 on ideals and grills in topological spaces *South Asian Journal of Mathematics* **5** (6) 233-238
[5] Thenmozhi P, Kaleeswari M and Maheswari N 2015 Regular Generalized Closed Sets in Grill Topological Spaces *International Journal of Science Research* 2319-7064
[6] Mustafa M O and Esmaeel R B 2020 Some Properties in Grill–Topological Open and Closed Sets Ready to be Published in *Journal of Physics: Conference Series (*$6^{th}$ *International Conferen of Iraqi AL-Khwarizmi Association)*