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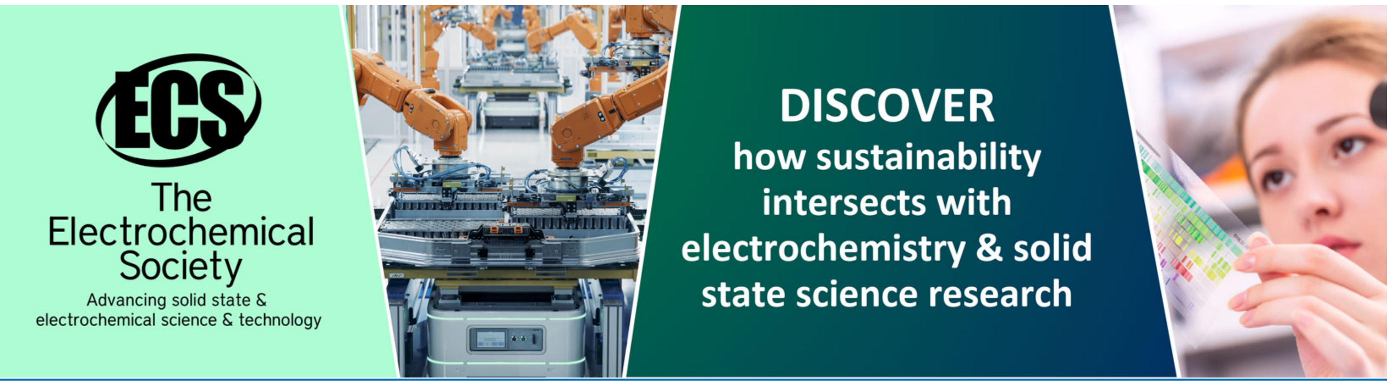

Some convergence results by using $K-$ iteration process in $CAT(H)$spaces

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Abstract: In this paper, we study the convergence and$\Delta -$convergence results of $K -$iteration process for Lipschizian self-mapping with $L \geq 1$in $CAT(H)$ spaces , $H > 0$.

1 −Introduction

Let D a positive number . A metric space (E, d) is said to be a$D -$ geodesic space if any two points of E with the distance less than D are joined by a geodesic .If $H > 0$,then E is said to be a $CAT(H)$ space if and only if it is $D_H -$ geodesic and any geodesic triangle$\Delta(x , y , w)$ in $E$with$d(x, y) + d(y, w) + d(w, x) < 2 D_H$ satisfies the $CAT(H)$ inequality .If $H < 0$,then E is said to be aCAT (H) space if and only if it is a geodesic space such that all of its geodesic triangle satisfy the $CAT(H$ inequalitysee ([1]). He et al .[2]definedMann iteration in$CAT(H)$spaces , $H > 0$ for self mapping V as follows , for any $x_1 \in E$

$$x_{n+1} = (1 - \gamma_n)x_n \oplus \gamma_n V x_n \ ; n \geq 0 \quad \dots (i)$$

Where $\{\gamma_n\}$is sequence in $(0,1)$ and proved the sequencedefined by (i) converges in $CAT(H)$ spaces.

Kifayat U., Kashif I. and Muhammed A. [3] introdused$K -$ iteration in $CAT(0)$, space for Suzuki generalized nonexpansive self mapping V as follows , for any $x_1 \in E$

$$x_{n+1} = V y_n$$
$$y_n = V(1 - \sigma_n)V x_n \oplus \sigma_n V z_n$$
$$z_n = (1 - \gamma_n)x_n \oplus \gamma_n V x_n \ ; n \geq 0 \quad \dots (ii)$$

Where $\{\sigma_n\}$and$\{\gamma_n\}$ are sequences in $(0,1)$ and proved the sequence defined by (ii) converges in $CAT(0$ space .

Raweerots S . [4] denote the set of fixed points of the mapping V by $F(V) = \{x \in E; Vx = x\}$.

On the other hand A sequence $\{y_n\}$ in the space A is said to be $\Delta -$ converges to y if y is the unique asymptotic center of $\{v_n\}$for everysubsequence $\{v_n\}$of $\{y_n\}$. we write $\Delta - \lim_{n\to\infty} y_n = y$. where the asymptotic center $C(\{y_n\})$of $\{y_n\}$is the set

$C(\{y_n\}) = \{x \in A: r(\{y_n\}) = r(y, \{y_n\})\}$ .see ([5])

Das and Debata [6]introdused Ishikawa iteration in $CAT(0)$ space for two nonexpansiveself mapping$s$ V and Was follows, forany$x_1 \in E$

$$x_{n+1} = (1 - \sigma_n)x_n \oplus \sigma_n W y_n$$
$$y_n = (1 - \gamma_n)x_n \oplus \gamma_n V x_n \ ; n \geq 0 \quad \dots (iii)$$

Where $\{\sigma_n\}$and $\{\gamma_n\}$ are sequences in (0,1) and proved the sequence defined by $(iii)\Delta-$ converges in $CAT(0)$ space . Jun[7]defined Ishikawa iteration in$CAT(H), H>0$ space for self mapping Vas follows , for any $x_1 \in E$

$$x_{n+1}=(1-\sigma_n)x_n \oplus \sigma_n V y_n$$
$$y_n=(1-\gamma_n)x_n \oplus \gamma_n V x_n \ ; n \geq 0 \quad \dots (v)$$

Where $\{\sigma_n\}$and$\{\gamma_n\}$ are sequences in (0,1) and proved the sequencedefined by $(v)\Delta-$ converges to a fixed point of$V$ in $CAT(H)$ space .

2- Preliminaries

In this section ,we provide some definitions and lemmas which will be used

Lemma (2.1)[8]: Let $H>0$ and(E,d) is a complete $CAT(H)$ spacewith diam (E) $=\frac{\pi/2-\mu}{\sqrt{H}}$for some $\mu \in (0,\pi/2\ )$. Then

$$d\big((1-\sigma)x \oplus \sigma y\,, w\big) \leq (1-\sigma)d(x,w)+\sigma d(y,w)$$
for all $x,y,w \in E$ and $\sigma \in (0,1)$.

Definition (2.2)[3]: Let G be a non$-$empty and convex subset of a CAT(H)spaceand $V: G \rightarrow G$ is a mapping for any $x_1 \in E$, the sequence $\{x_n\}$ define by

$$x_{n+1}=Vy_n$$

$$y_n = V((1-\sigma_n)Vx_n \oplus \sigma_n V z_n)$$

$$z_n=(1-\gamma_n)x_n \oplus \gamma_n V x_n \ ; n \geq 0 \qquad \dots (1)$$

is said to be $K-$ iteration sequence$CAT(H)$space, $\{\sigma_n\}$and $\{\gamma_n\}$ are sequences in (0,1).

Definition (2.3)[9]: Let G be a non$-$empty and convex subset of $CAT(H)$. A mapping $V: G \rightarrow G$ is said to be

(i) Lipschizian if $d(Vx,Vy) \leq L\, d(x,y)$ ... (2)
for all $x,y \in G$ and $L \geq 1$

Definition (2.4)[4]: A point$y \in E$ is a $\Delta-$ cluster point of $\{y_n\}$ if there exist asubsequence of $\{y_n\}$that $\Delta-$ converges to$y$,for a sequence$\{y_n\}$in A .

Lemma(2.5), [2] : Let(E,d) is a complete $CAT(H)$space . $q \in E$
Suppose that a sequence$\{y_n\}$ in A $\Delta-$ converges to$y$such that $r(q,y_n) < \frac{D_H}{2}$. Then $d(y,q) \leq \lim_{n\to\infty} \inf d(y_n\,,q)$

Definition (2.6)[4] : Let (E,d) be a complete metric space and Gbe a non$-$empty subset of E . Then a sequence $\{y_n\}$ in E isFej é r monotone with respect to G . If $d(y_{n+1}\,,q) \leq d(y_n\,,q)$, $n \geq 0$ andfor all $q \in G$.

Lemma(2.7), [2] :Let $H>0$ and(E,d) is a complete $CAT(H)$ space , G is a nonempty subset of E . Suppose that the sequence $\{y_n\}$ in EisFejér monotone with respect to G and the asymptotic radius $r(\{y_n\})$ of$\{y_n\}$ is less than $\frac{\pi}{2}$.If anyΔ $-$cluster point of$\{x_n\}$ in G. Then$\{y_n\}$ $\Delta-$ converges to a point in G .

Definition (2.8)[1]: Let G be a non$-$empty and convex subset of $CAT(H)$space. A sequence$\{y_n\}$inG issaid to be an approximate fixed point sequence for G if $\lim_{n\to\infty} d(\,y_n\,,Vy_n) = 0$.

3 $-$Main Theorem

In this section , The convergence and $\Delta-$convergenceresults of $K-$ iteration process has been proved .

Theorem(3.1): Let $H>0$ and(E,d) is acomplete $CAT(H)$ space with diam (E) $=\frac{\pi/2-\mu}{\sqrt{H}}$for some$\mu\in(0,\pi/2)$, G is a nonempty and convex subset of E and $V:G\to G$is Lipschizian mapping with$L\geq 1$, $F\neq\emptyset$.Let$\{x_n\}$define by condition (1) withγ_n and$\sigma_n\in(0,1)$. If G is complete and $\lim_{n\to\infty}\inf d(x_n,F)=0$ or $\lim_{n\to\infty}\sup d(x_n,F)=0$, then $\{x_n\}$converges to a unique point in F .

Proof: Let $q\in F$, from lemma (2.1), condition(1)and(2),

$$\begin{aligned} d(z_n,q) &= d((1-\gamma_n)x_n \oplus \gamma_n V x_n,q)\\ &\leq (1-\gamma_n)d(x_n,q)+\gamma_n d(Vx_n,q)\\ &\leq (1-\gamma_n)d(x_n,q)+\gamma_n L\,d(x_n,q)\\ &= (1-\gamma_n+\gamma_n L)d(x_n,q)\\ &\leq (1+\gamma_n L)d(x_n,q)\ \ldots(3)\end{aligned}$$

From lemma (2.1), condition(1), (2)and(3), we get

$$d(y_n,q)= d(V(1-\sigma_n)Vx_n \oplus \sigma_n V z_n,q)$$
$$\leq L\,d((1-\sigma_n)Vx_n \oplus \sigma_n V z_n,q)$$
$$\leq L[(1-\gamma_n)d(Vx_n,q)+\sigma_n d(Vz_n,q)]$$
$$\leq L[(1-\sigma_n)\,L\,d(x_n,q)+\sigma_n\,L\,d(z_n,q)]$$
$$=[L^2+\sigma_n\gamma_n L^3]d(x_n,q)$$
$$=L^2[1+\sigma_n\gamma_n L]d(x_n,q)\ \ldots(4)$$

From lemma (2.1), condition(1), (2), (3)and(4), we get

$$d(x_{n+1},q)=d(Vy_n,q)$$
$$\leq L\,d(y_n,q)$$
$$= L^3[1+\sigma_n\gamma_n L]d(x_n,q)$$

Hence , for all$n,m\,\epsilon\,N$ and every $q\in F$, there exist$s$ $W>0$ such that,

$$d(x_{n+m},q)\leq W\,d(x_n,q)\ .$$

Now we show that$\{x_n\}$ is acaushysequence in G,since $\lim_{n\to\infty} d(x_n,F)=0$, so for each $\delta>0$, $there\ e$ $n_1\in N$ such that

$$d(x_n,F)<\frac{\delta}{W+1}\text{for all } n>n_1$$

Thus , there exist$s$ $\rho\in F$ such that

$d(x_n,\rho)<\frac{\delta}{W+1}$for all $n>n_1$, we get

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},\rho)+d(x_n,\rho)\\ &\leq Wd(x_n,\rho)+d(x_n,\rho)\\ &\leq (W+1)d(x_n,\rho)\\ &\leq (W+1)\frac{\delta}{W+1}=\delta\ ,\end{aligned}$$

therefore$\{x_n\}$ is a caushysequence in G . From the completeness of G , we get $\lim_{n\to\infty}\{x_n\}$existsand equals$\rho\in G$, thereforefor all$\delta_1>0$

$$\text{there exist}s\ n_1\in N\text{such that } d(x_n,\rho)<\frac{\delta_1}{3(2+3\vartheta_1)}$$

Now , $\lim_{n\to\infty}\inf d(x_n,F)=0$ or $\lim_{n\to\infty}\sup d(x_n,F)=0$ gives that $\lim_{n\to\infty} d(x_n,F)=0$.So there exist$s$ $n_2\in N$with$n_2>n_1$

$d(x_n,\rho)<\frac{\delta_1}{3(2+3\vartheta_1)}$.

Thus there exist$s$ $\theta\in F$ such that

$d(x_{n_2},\theta)<\frac{\delta_1}{3(4+3\vartheta_1)}$, we obtain

$$\begin{aligned} d(V\rho,\rho) &\le d(V\rho,\theta)+d(\theta,x_{n_2})+d(x_{n_2},\rho)\\ &\le L\,d(\rho,\theta)+d(\theta,x_{n_2})+d(x_{n_2},\rho)\\ &\le \frac{\delta_1}{3(4+3\vartheta_1)\mathrm{L}}+\frac{\delta_1}{3(4+3\vartheta_1)}+\frac{\delta_1}{3(2+3\vartheta_1)}\\ &\le L\,(4+3\vartheta_1)\frac{\delta_1}{3(4+3\vartheta_1)\mathrm{L}}+(4+3\vartheta_1)\frac{\delta_1}{3(4+3\vartheta_1)}\\ &\quad +(2+\vartheta_1)\delta_1/3(2+\vartheta_1)\\ &= \frac{\delta_1}{3}+\frac{\delta_1}{3}+\frac{\delta_1}{3}=\delta_1 \end{aligned}$$

Since δ_1 is arbitrary so $d(V\rho,\rho)$, so thus$V\rho=\rho$, there fore$\rho\in F$.

Corollary (3.2) Let (E,d) is a complete $CAT(0)$ space , G is a nonempty and convex subset of E and $V\colon G\to G$ is Lipschizian mapping with $L\ge 1$, $F\neq\emptyset$. Let $\{x_n\}$define by condition(1) with $\gamma_n$and$\sigma_n\in(0,1)$. If G is complete and $\lim_{n\to\infty}\inf d(x_n,F)=0$ or $\lim_{n\to\infty}\sup d(x_n,F)=0$, the$\{x_n\}$converges to a unique point in F.

Corollary(3.3) Let E,G,V and$\{x_n\}$ be as in theorem(3.1) with $F(V)\neq\emptyset$ if

$(i)\{x_n\}$ is an approximate fixed point sequence for V
(ii)there exists a function $\tau\colon[0,\infty[\to[0,\infty[$ which is right continuous at 0, $\tau(0)=0$ and$\tau(d(x_n,Vx_n)\ge d(x_n,F)$, for all $n\in N$
then $\{x_n\}$converges to a unique point in F .

proof ;From (i) and (ii) , we get

$$\begin{aligned}\lim_{n\to\infty} d(x_n,F) &\le \lim_{n\to\infty}\tau(d(x_n,Vx_n)\\ &= \tau\lim_{n\to\infty}(d(x_n,Vx_n)\\ &=\tau(0)=0\end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n,F)=0$.

Thus $\lim_{n\to\infty}\inf d(x_n,F)=0$ and $\lim_{n\to\infty}\sup d(x_n,F)=0$.

By theorem(3.1) , $\{x_n\}$converges to a unique point in$F$.

Corollary (3.4) Let E,G,V and $\{x_n\}$ be as inCorollary(3.2) with $F(V)\neq\emptyset$ if
$(i)\{x_n\}$ is approximate fixed point sequence for V
(ii)there exists a function $\tau\colon[0,\infty[\to[0,\infty[$ which is right continuous at 0, $\tau(0)=0$ and$\tau(d(x_n,Vx_n)\ge d(x_n,F)$, for all $n\in N$
then $\{x_n\}$ converges to a unique point in F .

Theorem(3.5)Let E,G,V and$\{x_n\}$ be as in theorem(3.1) with $F(V)\neq\emptyset$if
(i) $d(x_0,F)<\frac{T_H}{4}$ for $x_0\in G$. $(ii)\{x_n\}$ is Fej é r monotone with respect to G. $(iii)\{x_n\}$ is an approximate fixed point sequence for V. then$\{x_n\}$ $\Delta-$converges to a point in F .

proof :Set $F_0=F\cap B_{\frac{\pi}{2}}(x_0)$.

Let $\{x_n\}$ is Fejér monotonewith respect to $F_0$and$q\in F$ such that $d(x_0,q)<\frac{\pi}{4}$. Then $q\in F_0$we get

$$d(x_{n+1}, q) \leq d(x_n, q) \leq d(x_0, q) < \frac{\pi}{4} \quad \text{, for all } n \geq 0. \qquad \ldots (5)$$

Hence $r(\{x_n\}) < \frac{\pi}{4}$

from lemma (2.7) ,let $\overline{q} \in G$ be a $\Delta-$ cluster point of $\{x_n\}$, then there exists asubsequence $\{x_{n_l}\}$ of $\{x_n\}$ which $\Delta-$ converges to $\overline{q}$. By (5) , we obtain $r(q, \{x_{n_l}\}) \leq d(x_0, q) < \frac{\pi}{4}$.

Using lemma (2.5) , we get

$$d(\overline{q}, x_0) \leq d(\overline{q}, q) + d(x_0, q) \leq \lim_{l\to\infty} \inf d(x_{n_l}, q) + d(x_0, q) < \frac{\pi}{4}.$$

That is $\overline{q} \in B_{\frac{\pi}{2}}(x_0)$. From condition (iii), we obtain

$$\lim_{l\to\infty} \sup d(V\overline{q}, x_{n_l}) \leq \lim_{l\to\infty} \sup d(V\overline{q}, Vx_{n_l}) + \lim_{l\to\infty} \sup d(Vx_{n_l}, x_{n_l})$$
$$\leq \lim_{l\to\infty} \sup d(\overline{q}, x_{n_l})$$

Hence $V\overline{q} \in C(\{x_{n_l}\})$ and $V\overline{q} = \overline{q}$. Then $\overline{q} \in F_0$ and using lemma (2.7) , we get $\{x_n\}$ $\Delta-$ converges to a point in F .

Corollary (3.6) Let E , G, V and $\{x_n\}$ be as in corollary(3.2) with $F(V) \neq \emptyset$ if (i) $d(x_0, F) < \frac{T_H}{4}$ for $x_0 \in G$. $(ii)\{x_n\}$ is Fejér monotone with respect to H. $(iii)\{x_n\}$ is an approximate fixed point sequence for V. then $\{x_n\}$ $\Delta-$ converges to a point in F .

4- Conclusions

The convergence and $\Delta-$convergenceresults of $K-$ iteration process has been proved when used Lipschizian self mapping with $L \geq 1$ in $CAT(H)$, $H > 0$ spaces . alsoe stablished the convergence and $\Delta-$convergence results has been in $CAT(0)$spaces .

5- Suggestion

$1-$ we can useuniformly Lipschizian mapping to established the convergence and $\Delta-$ convergesresults ofK $-$ Iteration .

$2-$ we can use another iteration to established the convergence and $\Delta-$ convergesresults .

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