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Application of Fixed Point in Algebra Fuzzy Normed Spaces

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Abstract. In the present paper first we recall the definition of algebra fuzzy metric space and some basic properties of algebra fuzzy metric space are introduced. Our goal is to prove the fixed point theorem in fuzzy complete algebra fuzzy metric space. Finally, the application to this theorem introduced.

1.Introduction

Kider in 2011 [1] introduced a fuzzy normed space. Also he proved this fuzzy normed space has completion in [2]. Again Kider introduced a new fuzzy normed space in 2012 [3]. Kider and Hussain in 2014 [4] introduced a new type of fuzzy metric space called standard fuzzy metric space and they study continuous, uniform continuous mappings on a standard fuzzy metric spaces. Again Kider in 2014 [5] study completeness of Hausdorff standard fuzzy metric spaces.

Kider and Kadhum in 2017 [6] introduced the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties were proved by Kadhum in 2017 [7]. Ali in 2018 [8] proved the basic properties of complete fuzzy normed algebra. Kider and Ali in 2018 [9] introduced the notion of fuzzy absolute value and study properties of finite dimensional fuzzy normed space. Again Kider and Ali in 2019 [10] introduced a new type of fuzzy normed algebra and they study properties of a complete fuzzy normed algebra.

The concept of general fuzzy normed space was presented by Kider and Gheeab in 2019 [11] [12] also they proved basic properties of this space and the general fuzzy normed space GFB (V, U). Kider and Kadhum in 2019 [13] introduce the notion of fuzzy compact linear operator and proved its basic properties a gain Kider and Kadhum in 2019 [14] study properties of fuzzy closed linear operator. For more information about fuzzy metric spaces also see [15, 16].

In 2020 Kider [17] introduced the notion of fuzzy soft metric space after that he investigated and proved some basic properties of this space again Kider in 2020 [18] introduced a new type of fuzzy metric space he called it algebra fuzzy metric space after that the basic properties of this space is proved.

In the present paper first we recall the notion of algebra fuzzy metric space which is a new type of fuzzy metric space then we recall basic properties of this space that will be used later in this paper. After that we introduce the main result in this paper it is the fixed point theorem with which prove to show that the important of this theorem and application to system of linear equation, solution of the differential equation, Fredholm integral equation and Volterra integral equation are introduced.

2. Basic Properties of Algebra fuzzy metric space

Definition 2.1: [18]

Let $\otimes: I \times I \to I$ be a binary operation function then $\otimes$ is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions $s, r, z, w \in I$ where $I = [0, 1]$
(i)$s \otimes r = r \otimes s$
(ii) $s \otimes [r \otimes z] = [s \otimes r] \otimes z$
(iii) $\otimes$ is continuous function
(iv)$s \otimes 0 = 0$
(v) $(s \otimes r) \leq (z \otimes w)$ whenever $s \leq z$ and $r \leq w$.

Lemma 2.2: [18]
If $\otimes$ is a continuous t-conorm on I then
(i)$1 \otimes 1 = 1$
(ii)$0 \otimes 1 = 1 \otimes 0 = 1$
(iii)$0 \otimes 0 = 0$
(iv)$a \otimes a \geq a$ for all $a \in [0, 1]$.

Example 2.3: [18]
The algebra product $a \otimes b = a + b - ab$ is a continuous t-conorm for all $a, b \in I$.

Definition 2.4: [18]
Assume that $S \neq \emptyset$, a fuzzy set $\widetilde{\mathrm{D}}$ in S is represented by $\widetilde{\mathrm{D}} = \{(s, \mu_{\widetilde{\mathrm{D}}}(s)): s \in S, 0 \leq \mu_{\widetilde{\mathrm{D}}}(s) \leq 1\}$ where $\mu_{\widetilde{\mathrm{D}}}(s): S \to [0,1]$ is a membership function.
The following definition the main definition

Definition 2.5: [18]
A triple $(S, m, \otimes)$ is said to be the algebra fuzzy metric space if $S \neq \emptyset$, $\otimes$ is a continuous t- conorm and $m: S \times S \to [0, 1]$ satisfying the following conditions:

(1)$0 \leq m(s, r) \leq 1$;

(2) $m(s, r) = 0$ if and only if $s = r$;

(3) $m(s, r) = m(r, s)$;

(4) $m(s, t) \leq m(s, r) \otimes m(r, t)$
For all s, r, t $\in$ S then the triple $(S, m, \otimes)$ is said to be the algebra fuzzy metric space
Example 2.6:[18]
If (S, d) is a metric space and $t \otimes r = t + r - tr$ for all t, r $\in [0, 1]$. Put $m_d(s, u) = \frac{d(s,u)}{1+d(s,u)}$ for all s, u $\in$ S. Then $(S, m_d, \otimes)$ is algebra fuzzy metric space. m_d is known as the algebra fuzzy metric comes from d.

Example 2.7: [18]
If $S \neq \emptyset$ put $m_D(s, u) = \begin{cases} 0 & if\ s = u \\ 1 & if\ s \neq u \end{cases}$
Then $(S, m_D, \otimes)$ is algebra fuzzy metric space known as the discrete algebra fuzzy metric space.

Definition 2.8: [18]
If $(S, m, \otimes)$ is algebra fuzzy metric space then $fb(s, j) = \{u \in S: m(s, u) < j\}$ is known as an open fuzzy ball with center s$\in$S and radius j $\in(0, 1)$. Similarly closed fuzzy ball is defined by
$fb[s, j] = \{u \in S: m(s, u) \leq j\}$.

Definition 2.9: [18]

If $(S, m, \otimes)$ is algebra fuzzy metric space and W⊆S is known as fuzzy open if $fb(w, j) \subseteq W$ for any arbitrary w∈W and for some j ∈(0, 1).
Also $D \subseteq S$ is known as fuzzy closed if D^C is fuzzy open then the fuzzy closure of D, $\overline{\mathrm{D}}$ is defined to be the smallest fuzzy closed set contains D.

Definition 2.10: [18]
If $(S, m, \otimes)$ is algebra fuzzy metric space then $D \subseteq S$ is known as fuzzy dense in S whenever $\overline{\mathrm{D}}$=S.

Theorem 2.11: [18]
If $fb(s, j)$ is open fuzzy ball in algebra fuzzy metric space $(S, m, \odot)$ then it is a fuzzy open set.

Proposition 2.12: [18]
In algebra fuzzy metric space $(S, m, \otimes)$ $s_n \to s$ if and only if $m(s_n, s) \to 0$.

Definition 2.13: [18]
In algebra fuzzy metric space $(S, m, \otimes)$ a sequence (s_n) is fuzzy Cauchy if for each r∈(0, 1) then we can find N such that $m(s_n, s_m) < r$, for each m, n ≤ N.
Definition 2.14: [18]
An algebra fuzzy metric space $(S, m, \otimes)$ is known as fuzzy complete if (s_n) is fuzzy Cauchy sequence then $s_n \to s \in$ S.

Theorem 2.15: [18]
In algebra fuzzy metric space $(S, m, \otimes)$ $if\ s_n \to s \in S$ then (s_n) is fuzzy Cauchy sequence.

Definition 2.16: [18]
Anon empty set D in algebra fuzzy metric space (S, m, ⊗) is known as fuzzy bounded whenever we can find s∈(0, 1) with $D \subset fb(d, s)$ for some d ∈S. Also a sequence (d_n) in algebra fuzzy metric space $(S, m, \otimes)$ is fuzzy bounded if we can find s∈(0, 1) with $\{d_1, d_2, \dots, d_n, \dots\} \subseteq fb(d, s)$ for some d ∈S.

Lemma 2.17:[18]
In algebra fuzzy metric space $(S, m, \otimes)$ If the sequence (s_n) ∈S with $s_n \to s$ ∈S. Then (s_n) is fuzzy bounded.

Lemma 2.18: [18]
In algebra fuzzy metric space $(S, m, \otimes)$ if (s_n) ∈ S with $s_n \to$s ∈S and $s_n \to$ d ∈S as n→ ∞. Then s=d.

Theorem 2.19: [18]
In algebra fuzzy metric space $(S, m, \otimes)$ when D⊂S then d$\in\overline{\mathrm{D}}$ if and only if there is $(\mathrm{d_n})$ ∈ D with $\mathrm{d_n} \to$ d.

Definition 2.20: [18]
If $(S, m_S, \otimes)$ and $(V, m_V, \otimes)$ are two algebra fuzzy metric spaces and W ⊆ S. Then a function $T: S \to V$ is called fuzzy continuous at w∈W. If for every $0 < r < 1$, we can find some $0 < t < 1$, with $\mathrm{m_V}[T(w), T(s)] < r$ as s∈W and $\mathrm{m_S}(w, s) < t$.
Also f is said to be fuzzy continuous on W if it is fuzzy continuous at every point of W.

Theorem 2.21: [18]
If $(S, m_S, \otimes)$ and $(V, m_V, \otimes)$ are two algebra fuzzy metric spaces and $W \subseteq S$. Then a function $T: S \to V$ is fuzzy continuous at w∈W if and only if whenever $\mathrm{w_n} \to w$ in W then $T(w_n) \to T(w)$ in V.

Theorem 2.22: [18]
The function $T: S \to V$ is fuzzy continuous on S if and only if $T^{-1}(D)$ is fuzzy open in S for all fuzzy open subset D of V where $(S, m_S, \otimes)$ and $(V, m_V, \otimes)$ are algebra fuzzy metric spaces.

3.Fixed Point Theorem

Definition 3.1:
If $(U, m, \otimes)$ is algebra fuzzy metric space then the function $S: U \to U$ is known as a fuzzy contraction on U if we can find r∈(0, 1) with $m[S(u), S(v)] \leq r\, m(u, v)$ for all u, v ∈U. The constant r is called the fuzzy contraction constant.
The proof of the next result is clear and hence is omitted.

Theorem 3.2:
Suppose that $(U, m, \otimes)$ is algebra fuzzy metric space. If the function $S: U \to U$ is a fuzzy contraction on U then S is fuzzy continuous.
The following theorem is the key of all results in this section
Theorem 3.3:
Suppose that $(U, m, \otimes)$ is fuzzy complete algebra fuzzy metric space where $U \neq \emptyset$, $a \otimes b = a + b - ab$ for all a, b∈I and assume that the function $S: U \to U$ is a fuzzy contraction on U that is we can find r∈(0, 1) with $m[S(u), S(v)] \leq r\, m(u, v)$ for all u, v ∈U. Then S has exactly one fixed point.

Proof: The construction of the iterative sequence (u_k) follows by Choosing u_0 ∈U and define
$$S(u_0) = u_1, S(u_1) = S^2(u_0) = u_2, \dots, S^k(u_0) = u_k, \dots \quad (1)$$
Now (u_k) is a fuzzy Cauchy sequence follows from the next steps
$$m(u_{n+1}, u_n) = m[S(u_n), S(u_{n-1})] \leq r\, m(u_n, u_{n-1}) = r\, m[S(u_{n-1}), S(u_{n-2})]$$
$$\leq r^2\, m(u_{n-1}, u_{n-2}) = r^2\, m[S(u_{n-2}), S(u_{n-3})]$$
And so on we have
$$m(u_{n+1}, u_n) \leq r^n\, m(u_1, u_0) \quad (2)$$
Therefore if we take n>j ≥N for some N ∈ ℕ we have
$$m(u_j, u_n) \leq m(u_j, u_{j+1}) \otimes m(u_{j+1}, u_{j+2}) \otimes, \dots, \otimes m(u_{n-1}, u_n)$$
$$\leq m(u_j, u_{j+1}) + m(u_{j+1}, u_{j+2}) +, \dots, m(u_{n-1}, u_n) - [m(u_j, u_{j+1}) m(u_{j+1}, u_{j+2}) \dots m(u_{n-1}, u_n)]$$
$$\leq [r^j + r^{j+1} + \dots + r^{n-j}]\, m(u_0, u_1) - [r^j\, r^{j+1} \dots r^{n-j}]\, m(u_0, u_1)$$
$$m(u_j, u_n) \leq [(r^j \tfrac{1-r^{n-j}}{1-r}) - (r^{[(n-j)j+(n-j)]})]\, m(u_0, u_1) \quad (3)$$
Now choose t ∈(0, 1) so that $[(r^j \tfrac{1-r^{n-j}}{1-r}) - (r^{[(n-j)j+(n-j)]})]\, m(u_0, u_1) < t$.
Thus $m(u_j, u_n) < t$ for all n>j ≥N that is (u_k) is a fuzzy Cauchy sequence but U is fuzzy complete so we can find u∈U such that (u_k) is fuzzy converges to u.
The proof of u is a fixed point of S follows from the following steps
$$m(u, S(u)) \leq m(u, u_n) \otimes m(u_n, S(u))$$
$$\leq m(u, u_n) + m(u_n, S(u)) - m(u, u_n)\, m(u_n, S(u))$$
$$\leq m(u, u_n) + r\, m(u_{n-1}, u) - m(u, u_n)\, r\, m(u_{n-1}, u)$$
By taking limit to both sides as n→ ∞ and using u_k →u getting $m(u, S(u)) = 0$ which implies that S(u)=u.
Finally, to show that u is the only fixed point of S. Assume that S(u)=u and S(y) = y that is S has two fixed points u and s. Now
$m(u, y) = m(S(u), S(y)) \leq r\, m(u, y)$. Thus $m(u, y) = 0$ since r ∈(0, 1). Hence u=y.

Corollary 3.4:

In Theorem 3.3 the iterative sequence (u_k) where u_0 is arbitrary fuzzy converges to the unique fixed point u of S. Error estimates are the prior

$$m(u_j, u) \leq [\tfrac{r^j}{1-r}]\, m(u_0, u_1) \tag{4}$$

and the posterior estimates

$$m(u_j, u) \leq [\tfrac{r}{1-r}]\, m(u_{j-1}, u_j) \tag{5}$$

Proof: The inequality (4) follows from inequality (3)
$m(u_j, u_n) \leq [(r^j \frac{1-r^{n-j}}{1-r}) - (r^{(n-j)j+(n-j)})]$ m(u_0, u_1). Since r ∈(0, 1) we have $(1-r^{n-j})$ ∈(0, 1) hence by letting n→ ∞ we have inequality (4)

$$m(u_j, u) \leq [\frac{r^j}{1-r}]\, m(u_0, u_1).$$

Now we derive (5) by taking j=1 and writing y_0 for u_0 and y_1 for u_1 we have from (4)
m(y_1, u) ≤ [$\frac{r}{1-r}$] m(y_0, y_1). Putting y_0=u_{j-1} we have $y_1 = S(y_0) = u_j$thus we obtain (5).

Theorem 3.5:
Suppose that $S: U \to U$ is a function where $(U, m, \otimes)$ is algebra fuzzy metric space and $a \otimes b = a + b - ab$ for all a, b∈I. Assume that $m(u_0, S(u_0)) < (1-r)t$ and S is a fuzzy contraction on the closed fuzzy ball $fb[u_0, t]$ that is we can find r∈(0, 1) with $m[S(u), S(v)] \leq r\, m(u, v)$ for all u, v ∈ $fb[u_0, t]$. Then the iterative sequence (u_k) fuzzy converges to $u \in fb[u_0, t]$, this u is fixed point of S and it is the only fixed point of S in $fb[u_0, t]$.

Proof:
$(u_k) \in fb[u_0, t]$ follows from the next step. In inequality (3) put j=0 and use
$m(u_0, S(u_0)) < (1-r)t$ we have

$$\begin{aligned} m(u_0, u_n) &\leq [\frac{1}{(1-r)} - r^n] m(u_0, u_1) \\ &= [\frac{1}{(1-r)}]\, m(u_0, u_1) - r^n m(u_0, u_1) \\ &< t - (1-r)\, r^n t < t. \end{aligned}$$

Hence $(u_k) \in fb[u_0, t]$ also $u \in fb[u_0, t]$ since (u_k) is fuzzy converges to u and $fb[u_0, t]$ is closed. Now from the proof of Theorem 3.3 the other assertion of the theorem follows.

Definition 3.6:
Let ℝ be the field of real numbers and ⊗ be a continuous t-norm. A fuzzy set $a_{\mathbb{R}}$ is a function from ℝ to I is called algebra fuzzy absolute value on ℝ if it satisfies the following conditions for all r, s ∈ ℝ**;**
(1) $a_{\mathbb{R}}(r) \in I$
(2) $a_{\mathbb{R}}(r) = 0$ if and only if r=0
(3) $a_{\mathbb{R}}(sr) \leq a_{\mathbb{R}}(s) \cdot a_{\mathbb{R}}(r)$
(4) $a_{\mathbb{R}}(s + r) \leq a_{\mathbb{R}}(s) \otimes a_{\mathbb{R}}(r)$
Then the triple $(\mathbb{R}, a_{\mathbb{R}}, \otimes)$ is called algebra fuzzy absolute value space.

Example 3.7:
Let *U*=*C[a, b]* and *t*⊗*s*=*t* + *s* −*ts* for all t, s ∈I. Define $m(f, g) = max_{t\in[a,b]}\, a_{\mathbb{R}}[f(t) - g(t)]$ then $(U, m, \otimes)$ is algebra fuzzy metric space.

Proof:
(1)Since $a_{\mathbb{R}}[f(t) - g(t)] \in (0, 1)$ for all t∈[a, b] it is clear that m(f, g) ∈(0, 1)

(2)m(f, g) = 0 if and only if $max_{t\in[a,b]}a_{\mathbb{R}}[f(t) - g(t)] = 0$ if and only if $a_{\mathbb{R}}[f(t) - g(t)] = 0$ for all t ∈ [a, b] if and only if f(t) –g(t) =0 for all t ∈ [a, b] if and only if f(t) =g(t) for all t ∈ [a, b] if and only if f=g.
(3)it is clear that $m(f, g) = m(g, f)$.

$$(4)m(f,g) = max_{t\in[a,b]}\, a_{\mathbb{R}}[f(t) - h(t) + h(t) - g(t)]$$
$$\leq max_{t\in[a,b]}\, a_{\mathbb{R}}[f(t) - h(t)] \otimes max_{t\in[a,b]}\, a_{\mathbb{R}}[h(t) - g(t)]$$
$$\leq m(f,h) \otimes m(h,g)$$

For all f, g, h ∈U. Hence (U, m, ⊗) is algebra fuzzy metric space.

Theorem 3.8:
If $a \otimes b = a + b - ab$ for all a, b ∈I define m: $\mathbb{R}^2$ →I by $m(u,v) = a_{\mathbb{R}}(u - v)$ for all u, v∈ ℝ. Then $(\mathbb{R}, m, \otimes)$ is algebra fuzzy metric space.

Proof:
(1)it is clear that m(u, v) ∈I for all u, v∈ ℝ.
(2) $m(u,v) = 0$ if and only if $a_{\mathbb{R}}(u-v)=0$ if and only if u−v=0 if and only if u=v.
(3)it is clear that $m(u,v) = m(v,u)$

$$(4)m(u,z) = a_{\mathbb{R}}(u - z) = a_{\mathbb{R}}(u - v + v - z)$$
$$\leq a_{\mathbb{R}}(u - v) \otimes a_{\mathbb{R}}(v - z)$$
$$\leq m(u,v) \otimes m(v,z)$$

Hence $(\mathbb{R}, m, \otimes)$ is algebra fuzzy metric space

Theorem 3.9:
If $s \otimes r = s + r - sr$ for all s, r ∈I then $(\mathbb{R}, m, \otimes)$ is a fuzzy complete where $m(u,v) = a_{\mathbb{R}}(u - v)$ for all u, v∈ ℝ.

Proof:
Let (r_n) be a fuzzy Cauchy sequence in ℝ then (r_n) has a monotonic subsequence (r_{n_j}) but (r_n) is a fuzzy bounded hence (r_{n_j}) is a fuzzy bounded thus (r_{n_j}) fuzzy approaches to r∈ ℝ that is every s∈(0, 1) we can find N ∈ ℕ with $a_{\mathbb{R}}(r_{n_k} - r_k) \leq s$. Since (r_n) is a fuzzy Cauchy sequence in ℝ so $a_{\mathbb{R}}(r_m - r_n) \leq s$ for all m, n ≥N. That is for k ≥N,
$a_{\mathbb{R}}(r_{n_k} - r_k) \leq s$. Now for all k≥N

$$a_{\mathbb{R}}(r_k - r) \leq a_{\mathbb{R}}\left(r_k - r_{n_k} + r_{n_k} - r\right)$$
$$\leq a_{\mathbb{R}}\left(r_k - r_{n_k}\right) \otimes a_{\mathbb{R}}(r_{n_k} - r)$$
$$\leq s \otimes s$$

Now we can find t∈(0, 1) with s ⊗ s <t that is $a_{\mathbb{R}}(r_k - r) \leq$ t. Thus (r_n) approaches to r∈ ℝ. Hence (ℝ, m, ⊗) is a fuzzy complete.

Example 3.10:
If $s \otimes t = s + t - st$ for all s, t ∈I define m: $\mathbb{R}^n \rightarrow I$ by $m(u,v) = max_j\, a_{\mathbb{R}}(\alpha_j - \beta_j)$ for all $u = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $v = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, m, \otimes)$ is algebra fuzzy metric space.

Proof:
(1)it is clear that m(u, v) ∈I for all u, v∈ $\mathbb{R}^n$.
(2) $m(u,v) = 0$ if and only if $max_j\, a_{\mathbb{R}}(\alpha_j - \beta_j)$ if and only if $a_{\mathbb{R}}(\alpha_j - \beta_j)$ for all j=1, 2, …,n if and only if $\alpha_j - \beta_j$=0 for all j=1, 2, …,n if and only if $\alpha_j = \beta_j$ for all j=1, 2, …,n if and only if u=v.
(3)it is clear that $m(u,v) = m(v,u)$

$$(4)\ m(u,z) = max_j\ a_{\mathbb{R}}(\alpha_j - \gamma_j)$$
$$= max_j\ a_{\mathbb{R}}(\alpha_j - \beta_j + \beta_j - \gamma_j)$$

$$\leq \; max_j \; a_{\mathbb{R}}(\alpha_j - \beta_j) \; \otimes \; max_j \; a_{\mathbb{R}}(\beta_j - \gamma_j)$$
$$\leq \; m(u, v) \; \otimes \; m(v, z)$$

Where $z = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^n$.
Hence $(\mathbb{R}^n, m, \otimes)$ is algebra fuzzy metric space

Example 3.11:
If $s \otimes t = s + t - st$ for all s, t ∈I then $(\mathbb{R}^n, m, \otimes)$ is fuzzy complete where m: $\mathbb{R}^n \to$I is defined by $m(u, v) = max_j \; a_{\mathbb{R}}(\alpha_j - \beta_j)$ for all $u = (\alpha_1, \alpha_2, \ldots, \alpha_n),\; v = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$.

Proof:
Consider a fuzzy Cauchy sequence (u_k) in $\mathbb{R}^n$ putting $u_k = (r_{1k}, r_{2k}, \ldots r_{nk})$. Since (u_k) is fuzzy Cauchy so for any s∈ (0, 1) we can find N∈ ℕ with

$$m(u_k, u_j) = max_i \; a_{\mathbb{R}}(r_{ik} - r_{ij}) < s \tag{6}$$

for all k, j ≥N and i=1, 2, …, n. This implies that $a_{\mathbb{R}}(r_{ik} - r_{ij}) < r$.
This shows that for any fixed i, 1≤ i ≤n the sequence (r_{i1}, r_{i2}, …)is a fuzzy Cauchy sequence in $(\mathbb{R}, m, \otimes)$. But $(\mathbb{R}, m, \otimes)$ is fuzzy complete by Theorem 3.10 so it fuzzy converges that is $r_{ik} \to r_i$ as k→ ∞. Now we use these n limits to define u=(r_1, r_2, …, r_n) it is clear that u∈ $\mathbb{R}^n$. From (6) by letting j→ ∞ we have

$$m(u_k, u) = m(u_k, \lim_{j\to\infty} u_j) = max_i \; a_{\mathbb{R}}(r_{ik} - r_i) < s$$

Thus (u_k) is fuzzy converges to u. Hence $(\mathbb{R}^n, m, \otimes)$ is fuzzy complete.

Example 3.12:
Let $U = C[a, b]$ and $t \otimes s = t + s - ts$ for all t, s ∈I. then $(U, m, \otimes)$ is fuzzy complete where

$$m(f, g) = max_{t\in[a,b]} \; a_{\mathbb{R}}[f(t) - g(t)]$$

Proof:
Let (f_k) be a fuzzy Cauchy sequence in *C[a, b]*. Then for any s ∈ (0, 1)we can find N∈ ℕ such that for m, n ≥N we have

$$m(f_k, f_n) = max_{t\in[a,b]} \; a_{\mathbb{R}}[f_k(t) - f_n(t)] < s \tag{7}$$

Hence for fixed t=t_0 ∈[a, b], $a_{\mathbb{R}}[f_k(t_0) - f_n(t_0)] < s$. This shows that ($f_1$ (t_0), f_2(t_0), …) is fuzzy Cauchy sequence in $(\mathbb{R}, m, \otimes)$. But $(\mathbb{R}, m, \otimes)$ is fuzzy complete by Example 3.9 so it is fuzzy converge to f(t_0). In this way we can associate with each t ∈[a, b] a unique f(t) ∈ ℝ. This defines a function f on [a, b] and we show that f ∈C[a, b] and (f_k) fuzzy converges to f. From (7) with n→ ∞ we have $max_{t\in[a,b]} a_{\mathbb{R}}[f_{k(t)} - f(t)] < s$ for all k≥N. Hence for every t∈[a, b],
$a_{\mathbb{R}}[f_k(t) - f(t)] < s$ for all k≥N.
This shows that (f_k(t)) fuzzy converge to f(t) uniformly on [a, b]. Since all members of (f_k) are fuzzy continuous on [a, b] and the fuzzy convergence is uniform the limit function f is fuzzy continuous on [a, b]. Hence $f \in C[a, b]$. Also (f_k) fuzzy converges to f. Hence $(C[a, b], m, \otimes)$ is fuzzy complete.

4.Application of Fixed Point Theorem

The following result is an application of Theorem 3.3 to system of linear equations

Theorem 4.1:
If the system of n linear equations

$$u = Cu + b \qquad (C = (c_{jk}), b \text{ given}) \tag{8}$$

in n unknowns α_1, α_2, …, α_n, (the component of u) satisfies

$$\sum_{k=1}^{n} a_{\mathbb{R}}(c_{jk}) < 1, (j = 1, 2, \ldots, n) \tag{9}$$

It has only one solution u. This solution can be obtained as the limit of the iterative sequence ($u^{(0)}$, $u^{(1)}$, $u^{(2)}$, …) where $u^{(j)}$ is arbitrary and

$u^{(j+1)} = Cu^{(j)} + b, \; (j = 0, 1, , , ,)$ (10)

Error bounds are

$$m(u^{(j)}, u) \;\le\; [(\frac{r}{1-r})] m(u^{(j-1)}, u^{(j)}) \;\le\; [(\frac{r^j}{1-r})] m(u^{(0)}, u^{(1)})$$

Proof:

Let $v = (\beta_1, \beta_2, \dots, \beta_n), w = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbb{R}^n$. From Example 3.10 and Example 3.11, $(\mathbb{R}^n, m, \otimes)$ is fuzzy complete algebra fuzzy metric space.

Now define S: $\mathbb{R}^n \to \mathbb{R}^n$by v= S(u)=Cu+b where C=$(c_{jk})$ is a fixed n×n real matrix and b$\in \mathbb{R}^n$. First we will show that S is a fuzzy contraction β_j=$\sum_{k=1}^{n} c_{jk}\, \alpha_j$+ σ_j where b=(σ_1, σ_2, …, σ_n). Putting S(w)=z where z=(δ_1, δ_2, …, δ_n). Now

$$m(v, z) = m(S(u), S(w)) = max_j a_{\mathbb{R}}(\beta_j \,-\, \delta_j) \;=\; max_j a_{\mathbb{R}}(\sum_{k=1}^{n} c_{jk}\,(\alpha_j - \delta_j))$$
$$\le\; max_j[a_{\mathbb{R}}(\alpha_j \,-\, \delta_j)\, a_{\mathbb{R}}(c_{jk})] \;\le\; max_j a_{\mathbb{R}}(\alpha_j \,-\, \delta_j)\, max_j a_{\mathbb{R}}(c_{jk})$$

Thus $m(v, z) \;\le\; r\, m(u, w)$ where $r = max_j \mathrm{a}_{\mathbb{R}}(c_{jk})$.

Now the assertion of the theorem follows from the proof of Theorem 3.3.

Definition 4.2:

Suppose that $(\mathbb{R}, a_{\mathbb{R}}, \otimes)$ is algebra fuzzy absolute value space. The function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be satisfying a fuzzy Lipschitz condition if $\mathrm{a}_{\mathbb{R}}$[f(t) – f(s)] ≤ k $\mathrm{a}_{\mathbb{R}}$(t – s) for all t, s ∈ $\mathbb{R}$ where the constant k is called fuzzy Lipschitz constant.

The next result is an application of Theorem 3.3 to differential equations.

Theorem 4.3:

Let f be fuzzy continuous function on the rectangle $D = \{(t, u)\colon |t - t_0| \le a, |u - u_0| \le b\}$and fuzzy bounded on D that is $a_{\mathbb{R}}(f(t, u)) \le c$ for all (t, u)∈D**.** Suppose that f satisfies a fuzzy Lipschitz condition on D with respect to its second argument, that is there is a constant k such that for (t, u), (t, v) ∈D, $\mathrm{a}_{\mathbb{R}}(f(t, u) - f(t, v)) \le k \mathrm{a}_{\mathbb{R}}(u - v)$. Then the initial value problem $\frac{du}{dt} = f(t, u),\ u(t_0) = u_0$ has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 \,+\, \beta]$ where β <min{a, $\frac{b}{c}$, $\frac{1}{k}$}

Proof:

Let $C(J)$ be the algebra fuzzy metric space of all real-valued continuous functions on the interval $J = [t_0 - \beta, t_0 \,+\, \beta]$ with fuzzy metric defined by $m(f, g) = \; max_{t\in J} a_{\mathbb{R}}[f(t) - g(t)]$. Then by Example 3.7 and Example 3.12 (C(J), m, ⊗) is fuzzy complete algebra fuzzy metric space. Let $\acute{C}$ be the subspace of C(J) consisting of all those functions u ∈C(J) that satisfy $\mathrm{a}_{\mathbb{R}}$(u(t)− u_0) ≤ cβ, It is clear that $\acute{C}$ is closed in C(J). $\acute{C}$ is fuzzy complete.

By integration we see that $\frac{du}{dt}$=f(t, u) can be written u=S(u) where S: $\acute{C} \to \acute{C}$ is defined by $S(u(t)) = u_0 \,+\, \int_{t_0}^{t} f(\tau, u(\tau)) d\tau$. S is defined for all u ∈ $\acute{C}$ since cβ <b so that if u ∈ $\acute{C}$ then τ ∈J and (τ, u(τ))∈ D and the integration exists since f is fuzzy continuous on D. To see S maps $\acute{C}$ into itself

$$a_{\mathbb{R}}[S(u(t)) - u_0] \le\; a_{\mathbb{R}}[\int_{t_0}^{t} f(\tau, u(\tau) d\tau] \le c a_{\mathbb{R}}[t - t_0] \le c\beta.$$

Now we will show that S is a fuzzy contraction on $\acute{C}$. By the fuzzy Lipschitz condition, we have

$$a_{\mathbb{R}}[S(u(t)) -\; S(v(t)] \;= a_{\mathbb{R}}[\int_{t_0}^{t} [f(\tau, u(\tau)) - f(\tau, v(\tau)] d\tau$$
$$\le a_{\mathbb{R}}[t - t_0]\; max_{\tau\in J} k\; a_{\mathbb{R}}[u(\tau) -\; v(\tau)]$$
$$\le k\beta m(u, v)$$

But the last expression does not depends on t. we can take the maximum on the left and have $m[S(u), S(v)] \leq r\, m(u, v)$ where r=kβ. Thus S is a fuzzy contraction on $\acute{C}$. Then Theorem 3.3 implies that S has a unique fixed point u ∈ $\acute{C}$ that is a fuzzy continuous function u on J satisfying $u = S(u)$. Or

$$u(t) = u_0 + \int_{t_0}^{t} f(\tau, u(\tau) d\tau \tag{11}$$

Since $(\tau, u(\tau))$ ∈D where f is fuzzy continuous (8) may be differentiated.
Hence u is even differentiable and satisfies $\frac{du}{dt}$=f(t, u), u(t_0)= u_0.
Conversely every solution of $\frac{du}{dt}$=f(t, u), u(t_0)= u_0 must satisfy (11).
The next result is an application of Theorem 3.3 to integral equations

Theorem 4.4:
Suppose k and v in the Fredholm integral equation

$$u(t) - \mu \int_a^b k(t,\tau)u(\tau)d\tau = v(t) \tag{12}$$

are fuzzy continuous on J×J where J=[a, b] respectively and assume that μ satisfies

$$a_{\mathbb{R}}(\mu) < \frac{1}{c(b-a)} \tag{13}$$

Where c is defined in

$$a_{\mathbb{R}}(k(t,\tau)) \leq c \ \text{ for all } (t,\tau) \in G \tag{14}$$

Then (12) has a unique solution u on J. This function u is the limit of the iterative sequence (u_0, u_1, …) where u_0 is any fuzzy continuous function on J for n=0, 1, …

$$u_{n+1}(t) = v(t) + \mu \int_a^b k(t,\tau)u_n(\tau)d\tau \tag{15}$$

Proof:
The kernel function k of (12) is a given function on the square G=J×J and v is a given function on J. We assume that v∈C[a, b] and k is fuzzy continuous on G. Then k is fuzzy bounded on G by (14).
Now (12) can be written S(u)=u where

$$S(u) = v(t) + \mu \int_a^b k(t,\tau)u(\tau)d\tau \tag{16}$$

Formula (16) defines an operator $S: C[a,b] \to C[a,b]$. Now

$$\begin{aligned} m(S(u), S(y)) &= max_{t\in J}\, a_{\mathbb{R}}[Su(t) - Sy(t)] \\ &= a_{\mathbb{R}}(\mu) \max_{t\in J} a_{\mathbb{R}}[\int_a^b k(t,\tau)[u(\tau) - y(\tau)]d\tau \\ &\leq a_{\mathbb{R}}(\mu) max_{t\in J} \int_a^b a_{\mathbb{R}}\{k(t,\tau)[u(\tau) - y(\tau)]\}d\tau \\ &\leq a_{\mathbb{R}}(\mu) max_{t\in J} \int_a^b a_{\mathbb{R}}\{k(t,\tau)\} a_{\mathbb{R}}\{[u(\tau) - y(\tau)]\}d\tau \\ &\leq a_{\mathbb{R}}(\mu) c\, max_{\sigma\in J} a_{\mathbb{R}}[u(\sigma) - y(\sigma)] \int_a^b d\tau \\ &\leq a_{\mathbb{R}}(\mu)\, c\, m(u,y)(b-a) \end{aligned}$$

This can be written $m(S(u), S(y)) \leq r\, m(u, y)$ where $r = a_{\mathbb{R}}(\mu)\, c(b-a)$
We see that S becomes a fuzzy contraction from (13). Then Theorem 3.3 implies that S has a unique fixed point u on J. This function u is the limit of the iterative sequence (u_0, u_1, …) where u_0 is any fuzzy continuous function on J for n=0, 1, … $u_{n+1}(t) = v(t) + \mu \int_a^b k(t,\tau)u_n(\tau)d\tau$.

Theorem 4.5:
Suppose that v in Volterra integral equation

$$u(t) - \mu \int_a^t k(t,\tau)u(\tau)d\tau = v(t) \tag{17}$$

is fuzzy continuous on J and the kernel k is fuzzy continuous on the triangular region D in the tτ-plane by a≤ τ ≤t, a≤t≤b. Then equation (17) has a unique solution u on J for every μ.

Proof:
We see that equation (16) can be written u=S(u) with $S: C(J) \to C(J)$defined by

$$S(u(t)) = v(t) + \mu \int_a^t k(t,\tau)u(\tau)d\tau \tag{18}$$

since k is fuzzy continuous on R and R is fuzzy closed also k is fuzzy bounded function on D so

$$a_{\mathbb{R}}(k(t,\tau)) \le c \text{ for all } (t,\tau) \in D \tag{19}$$

Now for all u, v ∈C(J) we have

$$a_{\mathbb{R}}[Su(t) - Sv(t)] = a_{\mathbb{R}}(\mu)a_{\mathbb{R}}[\int_a^t k(t,\tau)(u(\tau) - v(\tau))d\tau]$$

$$a_{\mathbb{R}}[Su(t) - Sv(t)] \le a_{\mathbb{R}}(\mu)c\, m(u,v) \int_a^t d\tau = a_{\mathbb{R}}(\mu)\, c\, m(u,v)\,(t-a) \tag{20}$$

We show by induction that

$$a_{\mathbb{R}}[S^m u(t), S^m v(t)] \le a_{\mathbb{R}}(\mu)^{\,m} c^m \frac{(t-a)^m}{m!} m(u,v) \tag{21}$$

For m=1 this is (20). Assume that (21) holds for any m we obtain

$$\begin{aligned} a_{\mathbb{R}}\,[S^{m+1}u(t), S^{m+1}v(t)] &= a_{\mathbb{R}}(\mu)a_{\mathbb{R}}[\int_a^t k(t,\tau)(S^m u(\tau) - S^m v(\tau))d\tau] \\ &\le a_{\mathbb{R}}(\mu)c \int_a^t a_{\mathbb{R}}(\mu)^{\,m} c^m \frac{(t-a)^m}{m!} d\tau\, m(u,v) \\ &\le a_{\mathbb{R}}(\mu)^{\,m+1} c^{m+1} \frac{(t-a)^{m+1}}{(m+1)!} m(u,v) \end{aligned}$$

Which complete the inductive proof of (21). Using (t−a) ≤(t−b) on the right hand of (21) then taking the maximum over t∈J on the left we obtain from (20)

m(S^mu, S^mv) ≤ r_mm(u, v) where $r_m = a_{\mathbb{R}}(\mu)^{\,m} c^m \frac{(t-a)^m}{m!}$

For any fixed μ and sufficiently large m we have r_m <1. Hence the corresponding S^m is a fuzzy contraction on *C(J).* The assertion of our theorem now follows from Theorem 3.3.

Lemma 4.6:
Let $S: U \to U$ be a function where $(U, m, \otimes)$ is a fuzzy complete algebra fuzzy metric spaces. Suppose that S^m is a fuzzy contraction on U for some m∈ ℕ. Then S has a unique fixed point.

Proof:
By assumption S^m=T is a fuzzy contraction on U by Theorem 3.3 this function T has a unique fixed point y that is T(y)=y. Hence T^n(y)=y. Now Theorem 3.3 also implies that for every u∈U T^n(u) →y as n→ ∞. For the particular u=S(y) since T^n= S^{nm} we obtain

$$y = \lim_{n\to\infty} T^n Sy = \lim_{n\to\infty} ST^n y = \lim_{n\to\infty} Sy = Sy.$$

This shows that y is a fixed point of S. Since every fixed point of S is also a fixed point of T we see that S cannot have more than one fixed point.

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