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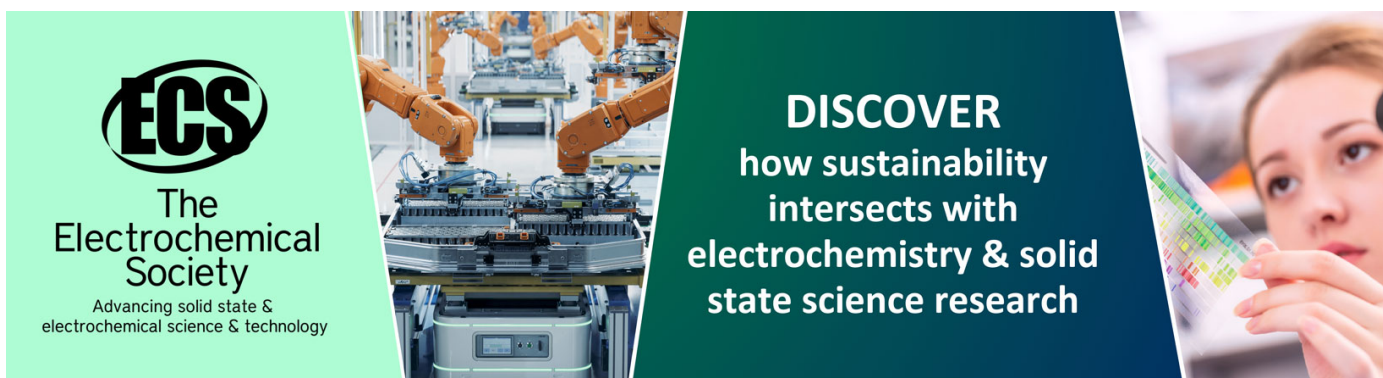
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Shapley value of n-person prisoner's dilemma

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Abstract. n-Person prisoner's dilemma is a Markovian sequence of static games that are realized randomly depending on players' previous choices. The dynamic Shapley value is constructed in this case. The work is a continuation of the paper published earlier by Grinikh A. L. [1]. In this paper, we consider the new approach of dynamics of game which depends on the strategies of all players.

1. Introduction

The numerical examples of n-person prisoner's dilemma was firstly introduced by H. Hamburger in the paper "N-person prisoner's dilemma" (1973), however, the player's payoff functions were set as a numerical example. Researches consider this game in the same setting. In this paper we consider the general case of payoff function introduced earlier by A. Grinikh in "Stochastic n-person Prisoner's Dilemma: the Time-Consistency of Core and Shapley Value" (2019), but in the mentioned paper the parameters of each step game do not depend on choices made by players at the previous step.

In this paper we consider a model of "n-person prisoner's dilemma" game. This game is a conflict of interests of n prisoners, each of them is jailed for complicity in the commission of a crime. Players have some information about the participation of each of the other members of the organized criminal group. The Judge is prepared to take into consideration confessions from each of the criminal group members, who agrees to cooperate with investigation, by further reducing his duration of detention in custody and detention. These testimonies will help to prove that the other members of the criminal group participated in the crime. Therefore, each "confessed" player provides the prolongation of remand in custody period for all members of the criminal group.

N is a set of players in n-person prisoner's dilemma game, $|N|=n$. A multi-step game consists of a number of static "n-person prisoner's dilemma" games played at each step, where the parameters of each static game depend on the player's strategies. In this statement, each static game meets the following conditions:

- each player has two possible pure strategies x_i : "to stay silent" (hereinafter " C_i ") and "to betray" (hereinafter " D_i "), $x_i \in \{C_i, D_i\}$;
- Nash equilibrium in pure strategies in a static game is the strategy profile that consists in choosing the strategy " D_i " by each player, which does not give a Pareto-optimal outcome;
- the payoffs of each player $h_i^{Y_j}(x_1, \dots, x_i, \dots, x_n)$ at each step of game are higher if all players choose the strategy " C_i " than if all of them choose the strategy " D_i ";
- the " D_i " strategy strictly dominates the " C_i " strategy for all $i \in N$ players.

2. n-Person Prisoner's dilemma

Let $\gamma_j = \{N, X_1 \dots X_n, h_1^{Y_j}(X_1 \dots X_n), \dots, h_n^{Y_j}(X_1 \dots X_n)\}$ be a static n-person prisoner's dilemma game. Here $N = \{1, \dots, n\}$ is a set of players. The set of pure strategies for the player i is denoted as $X_i =$



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$\{C, D\}$, where the strategy “C” means “to stay silent” and “D” corresponds to the strategy “to betray”. Let x be the number of players, which choose the strategy “C”. Then the payoff function of player i in static game $\gamma_j \in \{\gamma_1; \gamma_2\}$, $h_i^{\gamma_j}(x_1, \dots, x_n)$, can be represented as:

$$h_i^{\gamma_j}(x_1, \dots, x_n) = \begin{cases} C_i^{\gamma_j}(x) = a_1^{\gamma_j}x + b_1^{\gamma_j}, & \text{if } x_i = C \\ D_i^{\gamma_j}(x) = a_2^{\gamma_j}x + b_2^{\gamma_j}, & \text{if } x_i = D \end{cases} \quad (1)$$

Let Γ_K be a dynamic game that consists of K steps. The static game that is played at this step corresponds to the model of n -person prisoner's dilemma game and depends on the strategy profiles chosen at the previous step. At each step one of two static games γ_1 or γ_2 is played.

- γ_1 is always played at the first step and after if $(n - x) \geq x$ at the previous step, which means that the number of players that chose «to stay silent» is not less than the number of players that chose “to betray” at the previous step;
- γ_2 is played, if $(n - x) < x$ at the previous step. In this case the number of players that chose “to stay silent” is less than the number of players that chose “to betray”.

The player's payoff function $H_i^{\Gamma_K}(z_1, \dots, z_K)$ in the dynamic game Γ_K is the sum of player's payoffs at each step of dynamic game.

3. The Shapley Value

Let $\bar{X} = (\bar{X}_1, \dots, \bar{X}_n)$ be the strategy profile that maximize the sum of all players' payoffs in the game Γ_K . Then the path $\bar{z} = (\bar{z}_K, \dots, \bar{z}_1)$ is the cooperative trajectory of game Γ_K , if it is implemented by this strategy profile $(\bar{X}_1, \dots, \bar{X}_n)$:

$$\max_{z_K, \dots, z_1} \sum_{i=1}^n H_i^{\Gamma_K}(z_K, \dots, z_1) = \sum_{i=1}^n H_i^{\Gamma_K}(\bar{z}). \quad (2)$$

Definition 1:

The cooperative game $\Gamma_K(V)$ is the pair (N, V) , where N is the set of players and V is a characteristic function that is defined by formula (3).

$$V^{\Gamma_K}(S) = \max_{i \in S} \min_{j \in N \setminus S} \sum_{i \in S} H_i^{\Gamma_K}(X_1, \dots, X_i, \dots, X_n) \quad (3)$$

In particular,

$$V^{\Gamma_K}(N) = \max_{i \in N} \sum_{i \in N} H_i^{\Gamma_K}(X_1, \dots, X_i, \dots, X_n) \quad (4)$$

Since all the players in the considered game are symmetric, we will use the symmetry axiom of Shapley value. It follows:

$$Sh_i(V^{\Gamma_K}) = \frac{V^{\Gamma_K}(N)}{n} \quad (5)$$

There are three possible types of cooperative trajectory in the game Γ_K :

- γ_1 is played at each step of game Γ_K ;
- γ_1 is played at the first step of game Γ_K , γ_2 is played at the second and the following steps of game Γ_K ;
- γ_1 is played at each odd step of game Γ_K , γ_2 is played at each even step of game Γ_K .

3.1. The first case

Let $\bar{x}_{\gamma_1}$ and $\bar{x}_{\gamma_2}$ be the number of players that choose the strategy “to stay silent” to maximize the sum of players' payoffs in the static games γ_1 and γ_2 .

1. If $a_1^{\gamma_j} \geq a_2^{\gamma_j}$ or $\frac{a_2^{\gamma_j}n + b_1^{\gamma_j} - b_2^{\gamma_j}}{2a_2^{\gamma_j} - 2a_1^{\gamma_j}} \geq n$, then $\bar{x}_{\gamma_j} = \left\lfloor \frac{a_2^{\gamma_j}n + b_1^{\gamma_j} - b_2^{\gamma_j}}{2a_2^{\gamma_j} - 2a_1^{\gamma_j}} \right\rfloor$;
2. In other cases, $\bar{x}_{\gamma_j} = n$.

If $0 < \bar{x}_{\gamma_1} \leq \frac{n}{2}$ and $V^{\gamma_1} > V^{\gamma_2}$, then γ_1 will be played at each step of game Γ_K , $\left[\frac{a_2^{\gamma_1} n + b_1^{\gamma_1} - b_2^{\gamma_1}}{2a_2^{\gamma_1} - 2a_1^{\gamma_1}} \right]$ will be the number of players that choose the strategy “to stay silent” to maximize the sum of players’ payoffs at each step of game Γ_K . In this case, the Shapley value can be written as:

$$Sh_i(V^{\Gamma_K}) = K \left(a_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}^2}{n} + b_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}}{n} + a_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \bar{x}_{\gamma_1} + b_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \right). \quad (6)$$

Then $\bar{x}_{\gamma_1}$ players get:

$$C_i(\bar{x}_{\gamma_1}) = a_1^{\gamma_1} \bar{x}_{\gamma_1} + b_1^{\gamma_1}, \quad (7)$$

and other players get:

$$D_i(\bar{x}_{\gamma_1}) = a_2^{\gamma_1} \bar{x}_{\gamma_1} + b_2^{\gamma_1} \quad (8)$$

at the first step of cooperative trajectory of game Γ_K .

The components of Shapley value of remaining subgame Γ_{K-1} for each of players can be written as:

$$Sh_i(V^{\Gamma_{K-1}}) = (K-1) \left(a_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}^2}{n} + b_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}}{n} + a_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \bar{x}_{\gamma_1} + b_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \right). \quad (9)$$

Definition 2:

The finite sequence of vectors $\beta = (\beta^1, \dots, \beta^i, \dots, \beta^n)$ is called imputation distribution procedure (IDP) [4] in $\Gamma_K(V)$ for an imputation $d(\Gamma_K) = (d_1(\Gamma_K), \dots, d_n(\Gamma_K))$, if $d_i(\Gamma_K) = \sum_{z=0}^{K-1} \beta_{K-z}^i$, where $\beta^i = (\beta_1^i, \dots, \beta_z^i, \dots, \beta_K^i)$

Definition 3:

An optimality principle $D(\Gamma_K)$ is called time consistent [5], if for each given $d(\Gamma_K) \in D(\Gamma_K)$ there exists an IDP $\beta = (\beta_1, \dots, \beta_j, \dots, \beta_K)$ such that for any $0 \leq z \leq K-1$:

$$d_z(\Gamma_K) = d(\Gamma_K) - \sum_{j=1}^{z-1} \beta_j \in D(\Gamma_{K-z+1}). \quad (10)$$

Since:

$$Sh_i(V^{\Gamma_{K-1}}) + C_i(\bar{x}_{\gamma_1}) \neq Sh_i(V^{\Gamma_K}); \quad (11)$$

$$Sh_i(V^{\Gamma_{K-1}}) + D_i(\bar{x}_{\gamma_1}) \neq Sh_i(V^{\Gamma_K}). \quad (12)$$

C_i and D_i cannot be used as an IDP. Therefore, we introduce an IDP as follows:

$$\beta_i^z = Sh_i(V^{\gamma_1}), \quad (13)$$

Using this IDP, each player receives the Shapley value for a one-step game at each step of game.

$$z \cdot Sh_i(V^{\gamma_1}) + Sh_i(V^{\Gamma_{K-z}}) = z \cdot Sh_i(V^{\gamma_1}) + (K-z) \cdot Sh_i(V^{\gamma_1}) = K \cdot Sh_i(V^{\gamma_1}) = Sh_i(V^{\Gamma_K}) \quad (14)$$

This, shows the time-consistency of IDP and the Shapley value.

3.2. Example 1

Consider an example of game Γ_K , where $0 < \bar{x}_{\gamma_1} \leq \frac{n}{2}$ and $V^{\gamma_1} > V^{\gamma_2}$, which is played in $K = 5$ steps. Payoff functions for games γ_1 and γ_2 are defined as:

$$h_i^{\gamma_1}(x_1, \dots, x_4) = \begin{cases} C_i^{\gamma_1}(x) = 1\,500x, & \text{if } x_i = C \\ D_i^{\gamma_1}(x) = 3\,500x + 5\,000, & \text{if } x_i = D \end{cases}$$

$$h_i^{\gamma_2}(x_1, \dots, x_4) = \begin{cases} C_i^{\gamma_2}(x) = 1\,700x + 500, & \text{if } x_i = C \\ D_i^{\gamma_2}(x) = 1\,400x + 5\,000, & \text{if } x_i = D \end{cases}$$

Then, we can consider the sum of players' payoffs at each of possible step-game for different combinations of players who choose strategy "to stay silent", that means for different values of x (see table 1).

$$Sh_i(V^{\Gamma_K}) = \frac{30000K}{4} = 7\,500\,K = 37\,500.$$

$$Sh_i(V^{\Gamma_K}) = \frac{30000K}{4} = 7\,500\,K = 7\,500 \cdot 5 = 37\,500.$$

In this example:

$$Sh_i(V^{\gamma_1}) = 7\,500 = \beta_z^i \text{ is the imputation distribution procedure;}$$

$$C_i^{\gamma_1}(\bar{x}_{\gamma_1}) = 1\,500 \cdot 2 = 3\,000;$$

$$D_i^{\gamma_1}(\bar{x}_{\gamma_1}) = 3\,500 \cdot 2 + 5\,000 = 12\,000.$$

Table 1.

The sum of all players' payoffs for the games γ_1 and γ_2 in the example 1.

x	0	1	2	3	4
$\sum_{i \in N} h_i^{\gamma_1}$	20 000	27 000	30 000	29 000	24 000
$\sum_{i \in N} h_i^{\gamma_2}$	20 000	21 400	23 400	26 000	29 200

Since $Sh_i(V^{\gamma_1}) \neq C_i^{\gamma_1}(\bar{x}_{\gamma_1})$ and $Sh_i(V^{\gamma_1}) \neq D_i^{\gamma_1}(\bar{x}_{\gamma_1})$, then $Sh_i(V^{\gamma_1}) \neq h_i(\bar{x}_{\gamma_1})$. Therefore, the Shapley value for the whole dynamic game $\Gamma_K(V)$ is time-consistent, but the Shapley value of static game, which is played at various steps, is an imputation distribution procedure in accordance with this optimality principle.

3.3. The second case

If $\frac{n}{2} < \bar{x}_{\gamma_1}$, $\frac{n}{2} < \bar{x}_{\gamma_2}$ and $V^{\gamma_1} < V^{\gamma_2}$, then the game γ_1 is played at the first step of game Γ_K , γ_2 is played at the second and the following steps of cooperative trajectory of game Γ_K . In this case, the Shapley value of the whole game is written as:

$$Sh_i(V^{\Gamma_K}) = \left(a_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}^2}{n} + b_1^{\gamma_1} \frac{\bar{x}_{\gamma_1}}{n} + a_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \bar{x}_{\gamma_1} + b_2^{\gamma_1} \left(1 - \frac{\bar{x}_{\gamma_1}}{n} \right) \right) +$$

$$(K-1) \left(a_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}^2}{n} + b_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}}{n} + a_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n} \right) \bar{x}_{\gamma_2} + b_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n} \right) \right). \quad (15)$$

In this type of game Γ_K , an IDP is written as:

$$\beta_i^z = \begin{cases} Sh_i(V^{\gamma_1}), & \text{if } z = 1; \\ Sh_i(V^{\gamma_2}), & \text{if } z \neq 1. \end{cases}$$

If $\frac{n}{2} \geq \bar{x}_{\gamma_1}$, $\frac{n}{2} < \bar{x}_{\gamma_2}$ and $V^{\gamma_1} < V^{\gamma_2}$, then the game has also the second type of cooperative trajectory, but the Shapley value of the whole game in this case is given as:

$$Sh_i(V^{\Gamma_K}) = \left(a_1^{\gamma_1} \frac{\left(\left[\frac{n}{2} \right] + 1 \right)^2}{n} + b_1^{\gamma_1} \frac{\left[\frac{n}{2} \right] + 1}{n} + a_2^{\gamma_1} \frac{\left(\left[\frac{n}{2} \right] + 1 \right) (n - \left[\frac{n}{2} \right] - 1)}{n} + b_2^{\gamma_1} \frac{n - \left[\frac{n}{2} \right] - 1}{n} \right) +$$

$$(K-1) \left(a_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}^2}{n} + b_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}}{n} + a_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n}\right) \bar{x}_{\gamma_2} + b_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n}\right) \right). \quad (16)$$

However, in this case the Shapley value of step-game cannot be used as an IDP, since $Sh_i(V^{\Gamma_K}) < Sh_i(V^{\gamma_1}) + (K-1)Sh_i(V^{\gamma_2})$, because $V^{\gamma_1} > \left(a_1^{\gamma_1} \frac{(n+2)^2}{4} + b_1^{\gamma_1} \frac{n+2}{2} + a_2^{\gamma_1} \frac{n(n-2)}{4} + b_2^{\gamma_1} \frac{n-2}{2}\right)$. The time-consistent IDP will be as follows:

$$\beta_i^z = \begin{cases} a_1^{\gamma_1} \frac{(n+2)^2}{4n} + b_1^{\gamma_1} \frac{n+2}{2n} + a_2^{\gamma_1} \frac{n^2-4}{4n} + b_2^{\gamma_1} \frac{n-2}{2n}, & \text{if } z = 1 \text{ and } n \text{ is even;} \\ a_1^{\gamma_1} \frac{(n+1)^2}{4n} + b_1^{\gamma_1} \frac{n+1}{2n} + a_2^{\gamma_1} \frac{n^2-1}{4n} + b_2^{\gamma_1} \frac{n-1}{2n}, & \text{if } z = 1 \text{ and } n \text{ is odd;} \\ Sh_i(V^{\gamma_2}), & \text{if } z \neq 1 \end{cases} \quad (17)$$

3.4. Example 2

Consider an example of game Γ_K with the second type of cooperative trajectory, for which $\frac{n}{2} \geq \bar{x}_{\gamma_1}$, $\frac{n}{2} < \bar{x}_{\gamma_2}$ and $V^{\gamma_1} < V^{\gamma_2}$. Assume that this game has 5 steps ($K = 5$). The players' payoff functions are written as:

$$h_i^{\gamma_1}(x_1, \dots, x_4) = \begin{cases} C_i^{\gamma_1}(x) = 300x + 200, & \text{if } x_i = C \\ D_i^{\gamma_1}(x) = 800x + 1100, & \text{if } x_i = D \end{cases}$$

$$h_i^{\gamma_2}(x_1, \dots, x_4) = \begin{cases} C_i^{\gamma_2}(x) = 1500x + 100, & \text{if } x_i = C \\ D_i^{\gamma_2}(x) = 2200x + 4500, & \text{if } x_i = D \end{cases}$$

$$Sh_i(V^{\Gamma_K}) = \frac{6800 + 24900 + 24900 + 24900 + 24900}{4} = 26600$$

$\sum_{z=1}^5 Sh_i(V^{\gamma_z}) = 26650 \neq Sh_i(V^{\Gamma_K})$. Therefore, Shapley values for the static games cannot be an IDP in this case. Since the number of players for this game is even, we can identify IDP as:

$$\beta_i^z = \begin{cases} 1700, & \text{if } z = 1; \\ 6225, & \text{if } z \neq 1. \end{cases}$$

Table 2.

The sum of all players' payoffs for the games γ_1 and γ_2 in the example 2.

x	0	1	2	3	4
$\sum_{i \in N} h_i^{\gamma_1}$	4400	6200	7000	6800	5600
$\sum_{i \in N} h_i^{\gamma_2}$	18000	21700	24000	24900	24400

3.5. The third case

The last case of the cooperative trajectory can be achieved if the coefficients of the game meet the following conditions:

- $\frac{n}{2} < \bar{x}_{\gamma_1}$;
- $\frac{n}{2} \geq \bar{x}_{\gamma_2}$;
- $V^{\gamma_2} \geq a_1^{\gamma_1} \left[\frac{n}{2}\right]^2 + b_1^{\gamma_1} \left[\frac{n}{2}\right] + a_2^{\gamma_1} \left[\frac{n}{2}\right] \left(n - \left[\frac{n}{2}\right]\right) + b_2^{\gamma_1} \left(n - \left[\frac{n}{2}\right]\right)$;

- $V^{\gamma_1} \geq a_1^{\gamma_2} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)^2 + b_1^{\gamma_2} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + a_2^{\gamma_2} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(n - \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + b_2^{\gamma_2} \left(n - \left\lfloor \frac{n}{2} \right\rfloor - 1 \right).$

In this case, the components of the Shapley value will be given as:

$$Sh_i(V^{I_K}) = \left(K - \left\lfloor \frac{K}{2} \right\rfloor \right) \left(a_1^{\gamma_1} \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)^2}{n} + b_1^{\gamma_1} \frac{\left\lfloor \frac{n}{2} \right\rfloor + 1}{n} + a_2^{\gamma_1} \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(n - \left\lfloor \frac{n}{2} \right\rfloor - 1 \right)}{n} + b_2^{\gamma_1} \frac{n - \left\lfloor \frac{n}{2} \right\rfloor - 1}{n} \right) + \left\lfloor \frac{K}{2} \right\rfloor \left(a_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}^2}{n} + b_1^{\gamma_2} \frac{\bar{x}_{\gamma_2}}{n} + a_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n} \right) \bar{x}_{\gamma_2} + b_2^{\gamma_2} \left(1 - \frac{\bar{x}_{\gamma_2}}{n} \right) \right). \quad (18)$$

Since in this case along the cooperative trajectory players maximize the joint payoff in each step-game, the IDP for this case will be the following:

$$\beta_i^z = \begin{cases} Sh_i(V^{\gamma_1}), & \text{if } z \text{ is odd;} \\ Sh_i(V^{\gamma_2}), & \text{if } z \text{ is even.} \end{cases} \quad (19)$$

4. Conclusion

We have found Shapley values for the problem statement of “n-person prisoner’s dilemma” game for various combinations of model coefficients. A possible variant of imputation distribution procedure is introduced for one of model cases. An example of specification of values for the considering type of n-person prisoner’s dilemma with three players is shown.

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