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To cite this article: Petras Rupšys 2021 J. Phys.: Conf. Ser. 1854 012002

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# Modeling of stem taper evolution using stochastic differential equations

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Abstract. Stochastic differential equations (SDEs) were developed at the beginning of the twentieth century to quantify all aspects of stochastic processes. This study focusses to evaluate the applicability and efficiency of the SDEs for modeling tree diameter over bark at any particular height and total stem volume for birch tree species in the boreal forests of Lithuania. Newly developed models of the stem taper development are based on well-defined diffusion processes, such as the symmetric Vasicek type diffusion process, and asymmetric geometric type diffusion process. The stem taper models with the fixed- and mixed-effect parameters are examined. The fixed- and mixed-effect parameters of the SDEs stem are evaluated using maximum likelihood procedure. Results are illustrated using birch trees longitudinal measurements. These models are compared with traditionally used regression type stem taper models using statistical measures and residual analysis. Overall, the best goodness-of-fit statistics for tree diameter and volume predictions produced the SDEs stem taper models. All results are implemented in the Maple software.

Keywords: stochastic differential equation, probability density, approximated maximum likelihood procedure, stem taper.

# **1. Introduction**

Deterministic population dynamics defined by the ordinary differential equation growth models are usually unrealistic since in nature individuals coexist with others and are affected by different types of interactions. The phenomenon of the environment stochasticity can be modelled by allowing that some of parameters vary randomly in the shape of a mean zero process. Such growth process is described by a stochastic differential equation (SDE) resulting in a solution which is a diffusion process. SDEs now find applications in many disciplines including engineering, finance, population dynamics, biology and medicine [1,2,3,4,5].

A good amount of work has in the last five decades been done for stem taper construction from both linear and nonlinear regression modeling perspectives. Hence, numerous studies have been performed on evaluating stem diameter at any particular height. These dynamics are basically the classical deterministic regression models in diverse configurations, generalized by including additional stand level or tree variables and random effects. For many years, the segmented, variable-exponential and qexponential taper regression curves have been used in forestry for modeling the tree bole development [6,7,8]. The most of taper curves are species-specific, and, according to their results, the superiority of the q-exponential stem taper curves in improving the diameter at any particular height was concluded. For a more accurate predictions of the stem diameter outside the bark at any particular height with a



minimum variance led to the development of more advanced taper models phenomenally implemented by using the SDE.

For describing growth and yield processes in a forest stand, there is a purely stochastic analogy to the ordinary differential equation method. The stochastic calculus of the Itô type with certain mathematical restrictions concerning drift and diffusion functions capacitates for an exact form solution. In the modeling of tree development, the univariate [9,10] and multivariate [11,12] SDE, symmetric type (Vasicek) [13], and non-symmetric type (Gompertz, Bertalanffy and gamma) [14,15] have been adapted for the prediction of the tree height, diameter, crown width and other variables. The multivariate SDE extends univariate SDE to additionally incorporate more realistic distributions with the underlying covariance structure formalizing the changes in the state variables [16].

Taper equations are widely used in forestry to estimate the diameter at any given height along a tree bole and, therefore, to calculate the total or merchantable stem volume. This study focusses on a segmented model of a tree taper, which uses two different SDEs for different parts of the stem to overcome local bias. For greater understanding of the physical meaning of a stochastic tree stem taper equation, it is a benefit to consider a problem for which the underlying stem taper dynamic is known deterministic but this mechanism cannot be fully observed [17,18]. One of SDE's unquestionable advantage is the consistent description of the stem taper development process by means of the probability density function.

In this research, we suppose that in the lower section (less than 1.3 m) of a tree trunk can be modeled by the geometric Brownian motion SDE, and in the upper section (more than 1.3 m) of the stem can be modeled by the Vasicek type SDE. In particular, for butt swell at the height 1.3 m, the segmented SDE models were greatly superior.

The objectives of this research are to develop equations describing relative diameter and relative height stem taper relationships by applying segmented mixed-effect parameters SDE, describe the maximum likelihood procedure for parameter estimators, and compare the SDE stem taper models with traditionally used regression models for prediction of diameter at any specified height and stem volume. The results are illustrated using longitudinal measurements of birch trees in Lithuania.

# 2. Materials and methods

This research focusses on univariate continuous diffusion processes  $Y^i(x)$ , i = 1, ..., M, evolving in M different individuals (stems). It is supposed that the development of the tree relative diameter  $Y^i = \frac{D}{d^i}$  against the tree relative height  $x^i = \frac{h}{h^i}$  ( $x^i \in [0; 1]$ , in the sequel x) is expressed by the Itô-type [19] SDE, where D is the diameter at any particular height h,  $d^i$  is the diameter at breast height of the i-th tree, and  $h^i$  is the stem height of the i-th tree. In what follows, we will outline three different segmented stem taper models given by the SDE which solution possessed the exact transition probability density function.

# 2.1. SDE stem taper models

Two different SDEs are used for quantifying the stem development. The first SDE model of the relative diameter via the relative height is described using the geometric Brownian motion model as:

 $d Y^{i}(x) = \alpha_{B}Y^{i}(x)dx + \sigma_{B}Y^{i}(x)dW_{B}^{i}(x), \qquad (1)$ where i = 1, ..., M is the number of stems,  $Y^{i}(x^{i})$  is the value of the relative diameter at a particular relative height  $x^{i}$ ;  $\alpha_{B}$ , and  $\sigma_{B}$  are fixed effect parameters (for all M stems);  $y_{0}^{i}$  is the butt relative diameter; index B indicates a butt part of a stem; and  $W_{B}^{i}(x^{i}), i = 1, ..., M$ , are mutually independent standard Brownian motions. Stochastic process,  $Y^{i}(x)$ , is conditioned on two different initial values  $y_{x_{0}}^{i} \in \{\delta, 1\}$  at  $x_{0}^{i} \in \{0, \frac{1.3}{h^{i}}\}$ , thus is: if  $x_{0}=0$ , then  $y_{x_{0}}^{i} = \delta$  and  $\delta$  has a log-normal distribution where  $LN_{1}(\mu_{0}; \sigma_{0})$ , and if  $x_{0}^{i} = \frac{1.3}{h^{i}}$ , then  $y_{x_{0}}^{i} = 1$ ,  $P\left(Y^{i}\left(\frac{1.3}{h^{i}}\right) = 1\right) = 1$ . If the initial condition is fixed,  $P\left(Y^{i}(x_{0}^{i}) = y_{0}^{i}\right) = 1$ , then the diffusion process defined by Eq. (1) has the lognormal distribution

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 $LN_1(\mu_g(x|\alpha_B,\sigma_B,y_{x_0}^i,x_0^i);v_g(x|\sigma_B,x_0^i))$  with conditional mean, variance, and density, respectively, defined as:

$$\mu_{g}(x|\alpha_{B},\sigma_{B},y_{x_{0}}^{i},x_{0}^{i}) = ln(y_{x_{0}}^{i}) + \left(\alpha_{B} - \frac{\sigma_{B}^{2}}{2}\right)|x - x_{0}^{i}|$$

$$\nu_{\sigma}(x|\sigma_{B},x_{0}^{i}) = \sigma_{B}^{2}|x - x_{0}^{i}|$$
(2)
(3)

$$p_g(y, x | \alpha_B, \sigma_B, y_{x_0}^i, x_0^i) = \frac{1}{y_{\sqrt{2\pi\nu_g(x|\sigma_B, x_0^i)}}} exp\left(-\frac{(\ln y - \mu_g(x|\alpha_B, \sigma_B, y_{x_0}^i, x_0^i))^2}{2\nu_g(x|\sigma_B, x_0^i)}\right)$$
(4)

If  $y_{x_0}^i = \delta$ ,  $\delta \sim LN_1(\mu_0; \sigma_0)$ , then the diffusion process defined by Eq. (1) has the lognormal distribution  $LN_1(\mu'_g(x|\alpha_B, \sigma_B, y_{x_0}^i, x_0^i); v'_g(x|\sigma_B, x_0^i))$  with conditional mean, variance, and density, respectively, defined as:

$$\mu'_{g}(x|\alpha_{B},\sigma_{B},\mu_{0},x_{0}^{i}) = \mu_{0} + \left(\alpha_{B} - \frac{\sigma_{B}^{2}}{2}\right)\left|x - x_{0}^{i}\right|$$

$$(5)$$

$$\mu'_{g}(x|\sigma_{0},x_{0}^{i}) = \sigma_{0}^{2} + \sigma_{0}^{2}\left|x - x_{0}^{i}\right|$$

$$(6)$$

$$v'_{g}(x|\sigma_{B}, x_{0}^{t}) = \sigma_{0}^{2} + \sigma_{B}^{2}|x - x_{0}^{t}|$$
(6)

$$p_g(y, x | \alpha_B, \sigma_B, \mu_0, \sigma_0, x_0^i) = \frac{1}{y_{\sqrt{2\pi v_g'(x | \sigma_B, x_0^i)}}} exp\left(-\frac{(\ln y - \mu_g'(x | \alpha_B, \sigma_B, \mu_0, x_0^i))}{2v_g'(x | \sigma_B, x_0^i)}\right)$$
(7)

The Vasicek type SDE of the relative diameter against the relative height is given as:

 $d Y^{i}(x) = \beta_{T} (\alpha_{T} - Y^{i}(x)) dx + \sigma_{T} dW_{T}^{i}(x), i = 1, ..., M$ (8) where  $\alpha_{T}$ ,  $\beta_{T}$ , and  $\sigma_{T}$  are fixed effect parameters (identical for all M stems) and  $W_{T}^{i}(x)$  denotes independent standard Brownian motions. Diffusion process,  $Y^{i}(x)$ , conditioned on the initial value  $y_{x_{0}}^{i} \in$ {1.0; 0.0} at  $x_{0}^{i} \in \{\frac{1.3}{h^{i}}; 0.0\}$ , thus is  $P\left(Y^{i}\left(\frac{1.3}{h^{i}}\right) = 1.0\right) = 1$ ,  $P(Y^{i}(1) = 0.0) = 1$ , and has a normal distribution  $N_{1}(\mu_{V}(x|\alpha_{T},\beta_{T},y_{x_{0}}^{i},x_{0}^{i}); v_{V}(x|\beta_{T},\sigma_{T},x_{0}^{i}))$  with the conditional mean, variance, and

density, respectively  

$$\mu_V(x|\alpha_T,\beta_T,y_{x_0}^i,x_0^i) = \alpha_T - (y_{x_0}^i - \alpha_T) \cdot e^{-\beta_T|x-x_0^i|}$$
(9)

$$v_V(x|\beta_T, \sigma_T, x_0^i) = \frac{\sigma_T^2}{2\beta_T} \left( 1 - e^{-2\beta_T |x - x_0^i|} \right)$$
(10)

$$p_{V}(y, x | \alpha_{T}, \beta_{T}, \sigma_{T}, y_{x_{0}}^{i}, x_{0}^{i}) = \frac{1}{\sqrt{2\pi\nu_{V}(x | \beta_{T}, \sigma_{T}, x_{0}^{i})}} exp\left(-\frac{(y - \mu_{V}(x | \alpha_{T}, \beta_{T}, y_{x_{0}}^{i}, x_{0}^{i}))^{2}}{2\nu_{V}(x | \beta_{T}, \sigma_{T}, x_{0}^{i})}\right)$$
(11)

Using Eqs. 1–11 and fixing the initial conditions, we will define three different stem taper models. The proposed SDE stem taper models take the following forms:

*Model 1*: Eqs. (1) and (8), i = 1, ..., M, with starting-points at the stem but  $y_{x_0}^i = \delta$ ,  $\delta \sim LN_1(\mu_0; \sigma_0)$  and at the stem top  $P(Y^i(1.0) = 0.0) = 1$ :

$$dY^{i}(x) = \begin{cases} \alpha_{B}Y^{i}(x)dx + \sigma_{B}Y^{i}(x)dW^{i}_{B}(x), y^{i}_{x_{0}} = \delta, \delta \sim LN_{1}(\mu_{0};\sigma_{0}), \ 0 \leq \frac{h}{h^{i}} \leq \frac{1.3}{h^{i}} \\ \beta_{T}\left(\alpha_{T} - Y^{i}(x)\right)dx + \sigma_{T}dW^{i}_{T}(x), P\left(Y^{i}(1.0) = 0.0\right) = 1, \ \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(12)

*Model 2*: Eqs. (1) and (8), i = 1, ..., M, with starting-points at the stem but  $P\left(Y^i\left(\frac{1.3}{h^i}\right) = 1.0\right) = 1$  and at the stem top  $P(Y^i(1.0) = 0.0) = 1$ :

$$Y^{i}(x) = \begin{cases} \alpha_{B}Y^{i}(x)dx + \sigma_{B}Y^{i}(x)dW^{i}_{B}(x), P\left(Y^{i}\left(\frac{1.3}{h^{i}}\right) = 1.0\right) = 1, 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ \beta_{T}\left(\alpha_{T} - Y^{i}(x)\right)dx + \sigma_{T}dW^{i}_{T}(x), P\left(Y^{i}(1.0) = 0.0\right) = 1, \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(13)

*Model 3*: Eqs. (1) and (8), i = 1, ..., M, at the stem but  $P\left(Y^i\left(\frac{1.3}{h^i}\right) = 1.0\right) = 1$  and at the stem top  $P\left(Y^i\left(\frac{1.3}{h^i}\right) = 1.0\right) = 1$ :

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$$dY^{i}(x) = \begin{cases} \alpha_{B}Y^{i}(x)dx + \sigma_{B}Y^{i}(x)dW^{i}_{B}(x), P\left(Y^{i}\left(\frac{1.3}{h^{i}}\right) = 1.0\right) = 1, \ 0 \le \frac{h}{h^{i}} < \frac{1.3}{h^{i}} \\ \beta_{T}\left(\alpha_{T} - Y^{i}(x)\right)dx + \sigma_{T}dW^{i}_{T}(x), P\left(Y^{i}\left(\frac{1.3}{h^{i}}\right) = 1.0\right) = 1, \ \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(14)

The diffusion processes defined by Eqs. (12-14) have the explicit probability density functions [15]. Therefore, we can define the trajectories of the diameter mean,  $d_k(h|d^i, h^i)$ , variance,  $w_k(h|d^i, h^i)$  and p-quantile development,  $q_k(h, p|d^i, h^i)$ , for all model (k=1, 2, 3), respectively:

$$d_{1}(h|d^{i},h^{i}) = \begin{cases} d^{i} \exp\left(\mu'_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},\mu_{0},0.0\right) + \frac{1}{2}\nu'_{g}\left(\frac{h}{h^{i}}|\sigma_{B},0.0\right)\right), & 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ d^{i}\mu_{V}\left(\frac{h}{h^{i}}|\alpha_{T},\beta_{T},0.0,1.0\right), & \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \\ & w_{1}\left(h|d^{i},h^{i}\right) = \end{cases}$$
(15)

$$\begin{cases} \left(d^{i}\right)^{2} \exp\left(2\mu'_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},\mu_{0},0.0\right) + \nu'_{g}\left(\frac{h}{h^{i}}|\sigma_{B},0.0\right)\right) \left(\exp\left(\nu'_{g}\left(\frac{h}{h^{i}}|\sigma_{B},0.0\right)\right) - 1\right), 0 \leq \frac{h}{h^{i}} \leq \frac{1.3}{h^{i}} \\ \left(d^{i}\right)^{2} \nu_{V}\left(\frac{h}{h^{i}}|\beta_{T},\sigma_{T},1.0\right), \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(16)

$$q_{1}(h,p|d^{i},h^{i}) = \begin{cases} d^{i} L \Phi_{p}^{-1} \left( \mu_{g}' \left( \frac{h}{h^{i}} | \alpha_{B}, \sigma_{B}, \mu_{0}, 0.0 \right); \nu_{g}' \left( \frac{h}{h^{i}} | \sigma_{B}, 0.0 \right) \right), 0 \leq \frac{h}{h^{i}} \leq \frac{1.3}{h^{i}} \\ d^{i} \Phi_{p}^{-1} (\mu_{V}(x | \alpha_{T}, \beta_{T}, 0.0, 1.0); \nu_{V} \left( \frac{h}{h^{i}} | \beta_{T}, \sigma_{T}, 1.0) \right), \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(17)

$$d_{2}(h|d^{i},h^{i}) = \begin{cases} d^{i} \exp\left(\mu_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},1.0,\frac{1.3}{h^{i}}\right) + \frac{1}{2}\nu_{g}\left(\frac{h}{h^{i}}|\sigma_{B},\frac{1.3}{h^{i}}\right)\right), & 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ d^{i}\mu_{V}\left(\frac{h}{h^{i}}|\alpha_{T},\beta_{T},0.0,1.0\right), & \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \\ & w_{i}\left(h|d^{i},h^{i}\right) = \end{cases}$$
(18)

$$\begin{cases} \left(d^{i}\right)^{2} \exp\left(2\mu_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},1.0,\frac{1.3}{h^{i}}\right) + \nu_{g}\left(\frac{h}{h^{i}}\left|\sigma_{B},\frac{1.3}{h^{i}}\right)\right) \left(\exp\left(\nu_{g}\left(\frac{h}{h^{i}}\left|\sigma_{B},\frac{1.3}{h^{i}}\right)\right) - 1\right), 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ \left(d^{i}\right)^{2} \nu_{V}\left(\frac{h}{h^{i}}\left|\beta_{T},\sigma_{T},1.0\right),\frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(19)

$$q_{2}(h,p|d^{i},h^{i}) = \begin{cases} d^{i} L \Phi_{p}^{-1} \left( \mu_{g} \left( \frac{h}{h^{i}} | \alpha_{B}, \sigma_{B}, 1.0, \frac{1.3}{h^{i}} \right); \nu_{g} \left( \frac{h}{h^{i}} | \sigma_{B}, \frac{1.3}{h^{i}} \right) \right), 0 \leq \frac{h}{h^{i}} \leq \frac{1.3}{h^{i}} \\ d^{i} \Phi_{p}^{-1} (\mu_{V}(x|\alpha_{T},\beta_{T}, 0.0, 1.0); \nu_{V} \left( \frac{h}{h^{i}} | \beta_{T}, \sigma_{T}, 1.0) \right), \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(20)

$$d_{3}(h|d^{i},h^{i}) = \begin{cases} d^{i} \exp\left(\mu_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},1.0,\frac{1.3}{h^{i}}\right) + \frac{1}{2}\nu_{g}\left(\frac{h}{h^{i}}\left|\sigma_{B},\frac{1.3}{h^{i}}\right)\right), & 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ d^{i}\mu_{V}\left(\frac{h}{h^{i}}|\alpha_{T},\beta_{T},1.0,\frac{1.3}{h^{i}}\right), & \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \\ & \cdots \\ (h|d^{i},h^{i}) \end{cases}$$
(21)

$$\begin{cases} \left(d^{i}\right)^{2} \exp\left(2\mu_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},1.0,\frac{1.3}{h^{i}}\right) + \nu_{g}\left(\frac{h}{h^{i}}|\sigma_{B},\frac{1.3}{h^{i}}\right)\right) \left(\exp\left(\nu_{g}\left(\frac{h}{h^{i}}|\sigma_{B},\frac{1.3}{h^{i}}\right)\right) - 1\right), 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ \left(d^{i}\right)^{2} \nu_{V}\left(\frac{h}{h^{i}}|\beta_{T},\sigma_{T},\frac{1.3}{h^{i}}\right), \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(22)

$$q_{3}(h,p|d^{i},h^{i}) = \begin{cases} d^{i}L\phi_{p}^{-1}\left(\mu_{g}\left(\frac{h}{h^{i}}|\alpha_{B},\sigma_{B},1.0,\frac{1.3}{h^{i}}\right);v_{g}\left(\frac{h}{h^{i}}|\sigma_{B},\frac{1.3}{h^{i}}\right)\right), 0 \le \frac{h}{h^{i}} \le \frac{1.3}{h^{i}} \\ d^{i}\phi_{p}^{-1}(\mu_{V}(x|\alpha_{T},\beta_{T},1.0,\frac{1.3}{h^{i}});v_{V}(\frac{h}{h^{i}}|\beta_{T},\sigma_{T},\frac{1.3}{h^{i}})), \frac{h}{h^{i}} > \frac{1.3}{h^{i}} \end{cases}$$
(23)

where  $\Phi_p^{-1}(*,*)$  is the inverse normal distribution, and  $L\Phi_p^{-1}(*,*)$  is the inverse lognormal distribution.

#### 2.2. Data

The diameter of each birch tree was remeasured every 2 m, starting from the diameter on the butt, i.e., 0, 1, 1.3, 3, 5, etc. Stump heights were not measured, and a constant height of 0.0 m was assumed in the analysis. There was a total of 333 sample stems, and all section measurements consist of 4,228 points. For testing and validation purposes the complete observed dataset was randomly divided into two datasets. 230 stems (2,931 measurements) were selected for model fitting, and the remaining dataset of 103 stems (1,297 measurements) were used for model validation. The relative diameter against the relative heights of all stems are presented in Figures 1 and 2.

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#### 3. Results and discussion

Complex inventories of forests require accurate estimates of the total and merchantable volume of each tree. For several decades, forest science focused much attention on modelling individual tree height, taper and volume as functions of individual tree attributes. Traditionally, the relationships between predictor variables (diameter at breast height, height) and tree biomass, stem taper and stem volume have been described based on regression models [8].

An observed dataset consists of longitudinal measurements of a continuous diameter growth process. Longitudinal measurements are collected through a series of repeated observations of the same stem and have two features that complicate their statistical analysis: a) within-individual stem correlation, and b) extremely high variability between measurements for different stems.

For all developed models the fixed effect parameters, and random effects were estimated by the maximum likelihood and approximated maximum likelihood techniques [15], using a segmented conditional probability density functions. All results on parameter estimation are implemented using mathematical software Maple [21]. The parameter estimators are presented in Table 1.

Evaluation and comparison of stem taper models fitted for birch tree species in Lithuania was performed using the analysis of the residuals and the following fit statistics: root mean square error (RMSE), mean bias (B), mean absolute bias (AB), and coefficient of determination (R2). This research also examined the applicability of new developed stem taper equations in accurately estimating stem volume (m3). In general, SDE taper models prove for at least 95% of the variation in diameter outside bark for the estimation dataset, and 93% of the variation in diameter outside bark for the validation dataset. For volume predictions all models showed an insignificant bias. All three models produced percent root-mean-square error values of 11.60% to 13.83% for the diameter predictions. The best results showed Model 1, where the diameter at the butt is lognormally distributed with unknown parameters.

Foresters stem taper equations use for estimating stem diameter at any particular heights, height to specified diameters, and volumes of various productions of stem bole. Basically, stem volume is calculated by mathematical integration. All new developed stochastic stem taper models examined in previous section are plotted in Figure 3. Three stems from the validation dataset were used for illustrating stochastic modeling techniques.

Figure 4 shows the 5% and 95% quantiles taper trajectories with the mean trend and the observed dataset for three randomly selected stems corresponding to large, medium, and small tree. This figure illustrates how well the observed datasets are covered by the 5% and 95% quantiles area. The quantile trajectories can be used to distinguish abnormal tree stems and to shape a way to predict a range of values while having a certain amount of confidence in that range [20].

Model	Parameters									
	$a_{\rm B}$	$\sigma_{ m B}$	ατ	$\beta_{T}$	$\sigma_{ ext{T}}$	μο	$\sigma_0$			
M1	-4.7638	0.3058	1.0048	2.0150	0.1759	0.2748	0.0091			
IVII	(0.0603)	(0.0101)	(0.0100)	(0.0404)	(0.0030)	(0.0006)	(0.0004)			
м2	3.6504	0.4380	1.1306	1.7760	0.1712					
1012	(0.1033)	(0.0145)	(0.0111)	(0.0312)	(0.0027)	-	-			
M3	3.6504	0.4380	-12.0531	0.0718	0.1443					
	(0.0165)	(0.0145)	(2.9271)	(0.0165)	(0.0021)	-	-			

Table 1. Estimation of fixed effect parameters	(standard						
errors) for all used models.							

<b>Table 2.</b> Comparison indexes for all fitted taper models
--

Model	B (%)	AB (%)	RMSE (%)	R <sup>2</sup>	B (%)	AB (%)	RMSE (%)	R <sup>2</sup>
Estimation dataset					Validation dataset			
M1	-0.2936	1.3432	1.9075	0.9662	-0.2856	1.4981	2.1142	0.9553
	(-1.79)	(8.21)	(11.66)		(-1.79)	(9.39)	(13.19)	
M2	-0.2714	1.2755	1.9527	0.9646	-0.4906	1.5795	2.3082	0.9468
	(-1.66)	(7.79)	(11.93)		(-3.07)	(9.90)	(14.47)	
M3	-0.0517	1.6588	2.2624	0.9524	-0.5070	1.9641	2.6387	0.9304
	(-0.32)	(10.14)	(13.83)		(-3.18)	(12.31)	(16.54)	







**Figure 3.** The development of diameter against height. Large tree is shown in black, medium tree in blue, and small tree in red. Observed dataset in circles.



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**Figure 4.** The development of the 5% and 95% quantiles and mean trajectories of diameter against height. Large tree is shown in black, medium tree in blue, and small tree in red. Observed dataset in circles. Quantiles in dashed lines. Mean in solid line.

#### 4. Conclusions

The segmented SDE models with estimated joint point 1.3 m provided a better quantification of stem taper when compared to traditionally used regression models. Based on the goodness-of-fit statistical measures and the statistical analysis of the residuals the best performance showed Model 1. It may be noted that SDE segmented models have many other potential applications in forestry for other species.

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# Acknowledgments

To conclude, I would like to express my gratitude to Prof. Dr. Petrauskas E. and the Institute of Forest Management and Wood Sciences at Vytautas Magnus University for placing valuable empirical data at my disposal.