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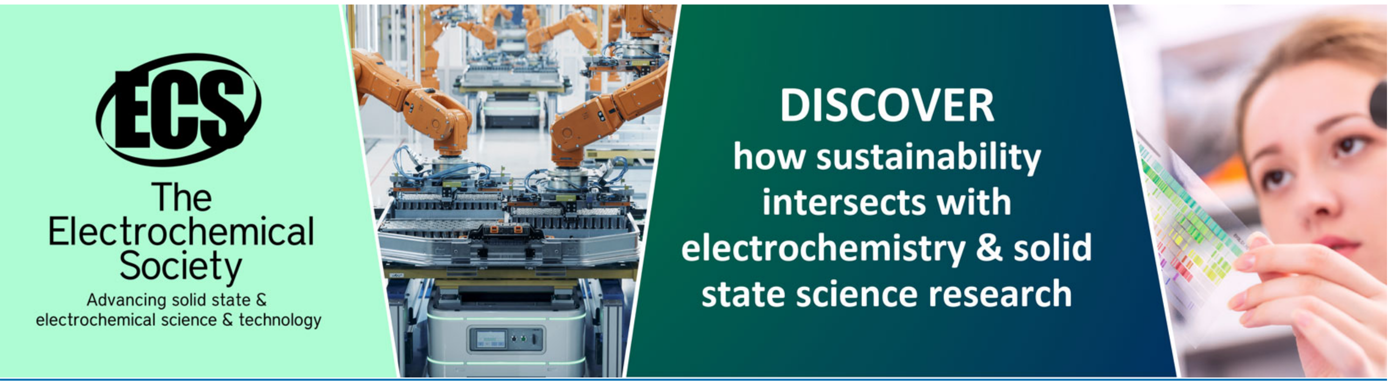

Dynamics of a Discrete-time Predator-Prey Model of Leslie type with Predator Harvesting

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Abstract. In the present work, we attempt to analyze the discrete form of Leslie type predator-prey system with the extension of nonlinear type harvesting of predators. The proposed model is obtained with the help of theory of piecewise constant argument for differential equations. We give the local stability properties of all possible non-negative fixed points. Also, our study reveals that the discrete system admits two bifurcations which are flip and Neimark-Sacker by making the use of center manifold argument and bifurcation theory, where the harvesting parameter is varied in order to take place of such bifurcations. Some simulations are carried out to depicts the obtained analytical results such as bifurcation plots and phase portraits. Also, it can be confirmed from numerical simulations that the considered system exhibits chaotic behavior for smaller values of harvesting parameter and becomes stable for larger values of same parameter. The largest Lyapunov exponents are plotted to show the sensitivity of chaotic regime.

1. Introduction

In ecology mathematical models plays a vital role in describing the interacting between the populations. With this connection Lotka [1] and Volterra [2], who first modeled the interaction between two species (predators and prey) in 1970's. Since then the area of mathematical ecology has become popular among researchers, for more details see [3–5]. Generally, there are two different types of time domains are used while modeling such interactions, namely, continuous [6–8] and discrete [9–12] time models. In particular, several exploratory works have suggested for the small size population of predator-prey models, thus the discrete version is more appropriate when compared to continuous ones [13]. It is worth mentioning that the model of discrete time brings more complexity than those of the continuous model.

Leslie-Gower(LG) model [14] is considered as an important mechanism of interaction between prey and predator, where the predator reduction rate is reciprocal to the per capita availability of its likely food. Recently many researcher have been proposed this type of model by implementing Allee effect [15], prey harvesting [8], prey refuge [16] etc., while modified LG model with various types of functional have been reported in [6, 17, 18]. Due to the importance of fishery, wildlife management and forestry; harvesting is one of emerging topic in ecology [19]. Many authors have been studied the dynamical behaviour of population under the influence of harvesting rate, see for prey harvesting [5, 20–22], and predator harvesting [23–25]. The LG type model with extension of nonlinear type harvesting in predators have been investigated in [7]. The modified LG type model with extension of nonlinear type harvesting with diffusion have been investigated by Singh et al. in [3], where they showed that the diffusion term can induce Turing instability and spatially inhomogeneous periodic solutions. The modified LG model given in [4] exhibits various types of bifurcations which are Hopf-Andronov, saddle-node, transcritical and Bogdanov-Takens(BT) bifurcations by varying the model parameter values.

The authors in [26] have been considered a certain continuous-time model in [4] implemented with constant-yield type of harvesting, where the model admits various kinds of bifurcations, such as unstable and unstable type BT bifurcations of codimension 2, 3, saddle-node and Hopf bifurcation when the harvesting parameter vary. In [12], the discrete-time model of linear harvesting in predator have been considered and derived the conditions for flip and Hopf bifurcations by implementing steps as in [27]. Also the proposed system exhibits various behaviors such as higher periodic orbits, stable invariant circles, and some chaotic attractors. Such a similar studies can also be found in [10, 28]. The continuous system with hunting cooperation of predator in the classical Lotka-Volterra model has considered in [29]. Then this model is discretized by using the method of piecewise constant argument is obtained in [30], where the model undergoes period-halving and NS bifurcations. Recently, the problem of dynamical behaviors of the discrete type predator-prey systems were found in [11,31,32]. Since from the motivation of aforementioned work, here our primary work of this article is to study the dynamical behaviour of the discrete version of predator harvested LG type model which is obtained by method of piecewise constant argument.

A LG type continuous time predator-prey model with harvesting in predator is studied in [23], which is of the form

$$\begin{cases} \dot{X} = & r_1 X\left(1-\dfrac{X}{K}\right) - aXY, \\ \dot{Y} = & r_2 Y\left(1-\dfrac{Y}{bX}\right) - \dfrac{qEY}{m_1E+m_2Y}, \end{cases} \tag{1}$$

with $X(T) \geq 0$ and $Y(T) \geq 0$ which represents the population biomass of prey and predator at some time T. All the model parameters K, r_1, r_2, a, b, q, E, m_1, m_2 are assumed only positive values and its biological meanings are given as follows; K represents environmental carrying capacity of the prey; r_1 and r_2 are the intrinsic growth rate of prey and predators; a is the maximal predator per capita consumption rate; bX represents the prey-dependent carrying capacity for the predators, where b be the conversion factor of prey into predators; q and E represent catch-ability coefficient and harvesting effort of predators on prey, respectively, and m_1, m_2 are constants.

The rest of article is arranged as follows. We construct the discrete-time system in Section 2. In Section 3, the non-negative fixed points are derived and studied the local stability properties. The required conditions for the system to admits flip and Neimark-Sacker bifurcation were discussed by adopting center manifold theorem and normal form theory in Section 4. Section 5 presents some simulations numerically to ensure the obtained results. In section 6, conclusion is drawn.

2. The formulation of discrete-time system

Let us make the system (1) in simple form by making use of the variable transforms $t \longrightarrow T$, $x \longrightarrow \frac{X}{K}$ and $y \longrightarrow aY$. More preciously the resulting simplified system is given by:

$$\begin{cases} \dot{x} = & r_1 x(1-x) - xy, \\ \dot{y} = & y\left(r_2 - \dfrac{dy}{x}\right) - \dfrac{ey}{g+y}, \end{cases} \tag{2}$$

where $d = \frac{r_2}{abK}$, $e = \frac{aqE}{m_2}$ and $g = \frac{am_1E}{m_2}$, $x(t)$ and $y(t)$ are non-negative. The corresponding discrete version of system (2) is obtained by the method of piecewise constant argument, assuming $t \in [n, n+1)$, $n = 0, 1, 2, \cdots.$, thus the system (2) takes the form

$$\begin{cases} x(n+1) = & x(n)\exp\left[r_1(1-x(n)) - y(n)\right], \\ y(n+1) = & y(n)\exp\left[r_2 - \dfrac{dy(n)}{x(n)} - \dfrac{e}{g+y(n)}\right], \end{cases} \tag{3}$$

where $x(n)$ denotes the prey densities and and $y(n)$ denotes the predators densities in some time instant n, the parameters r_1, r_2, d e and $g > 0$ are assumed to be positive. Thus the aim of this current article is to analyze the local stability of all existing fixed points and possible bifurcation behavior of the discrete system (3) in the region $R_+ = \{(x, y)|x > 0, y > 0\}$.

3. Fixed points and local stability

The solutions of below non-linear equations are the fixed fixed points of the system (3):

$$\begin{cases} r_1(1-x) - y & = 0, \\ r_2 - \dfrac{dy}{x} - \dfrac{e}{g+y} & = 0. \end{cases} \tag{4}$$

In biological point of view, we need only the non-negative roots of equations (4). Thus, the existence of all possible fixed points of system (3) can be given in the below proposition:

Proposition 1 *The fixed point $(1, 0)$ is always exists and (x_1, y_1) is the interior fixed point of (3) with $y_1 = r_1(1 - x_1)$ and*

$$\left(dr_1^2 + r_1r_2\right)x_1^2 + \left(e - gr_2 - 2dr_1^2 - dgr_1 - r_2r_1\right)x_1 + d\left(r_1^2 + gr_1\right) = 0, \tag{5}$$

where x_1 is the positive root of (5). By Descartes rule of sign, equation (5) has almost two positive root, if $e < gr_2 - 2dr_1^2 - dgr_1 - r_1r_2$. So (x_1, y_1) exists if the following conditions holds:

$$0 < x_1 < 1, \ e < gr_2 + 2dr_1^2 + dgr_1 + r_1r_2. \tag{6}$$

Hereafter, we assume for the system (3), (x_1, y_1) be the non-negative interior fixed point, that is, (6) always holds.

3.1. Local Stability

The stability matrix of system (3) at any fixed point (x, y) is given by

$$J(x,y) = \begin{bmatrix} (1-r_1x)\exp[r_1(1-x)-y] & -x\exp[r_1(1-x)-y] \\ \left(\dfrac{dy^2}{x^2}\right)\exp\left[r_2 - \dfrac{dy}{x} - \dfrac{e}{g+y}\right] & \left(1 - \dfrac{dy}{x} + \dfrac{ey}{(g+y)^2}\right)\exp\left[r_2 - \dfrac{dy}{x} - \dfrac{e}{g+y}\right] \end{bmatrix}$$

Note we use modulus value of the eigenvalues of $J(x, y)$ to study the local dynamics. Thus, the local stability properties of the non-negative fixed points $(1, 0)$ and (x_1, y_1) are summarized in the following conclusions:

Proposition 2 *$J(x, y)$ at $(1, 0)$ has eigenvalues $\lambda_1 = 1 - r_1$, $\lambda_2 = \exp\left[r_2 - \frac{e}{g}\right]$, then $(1, 0)$ is*

(i) a sink if $r_1 < 2$ and $gr_2 < e$;

(ii) a source if $r_1 > 2$ and $gr_2 > e$;

(iii) a saddle point if $r_1 < 2$ (resp. $r_1 > 2$) and $gr_2 > e$ (resp. $gr_2 < e$);

(iv) a non-hyperbolic if $r_1 = 2$ *or* $gr_2 = e$.

Next, for the linearized system of (3) whose characteristic equation at the non-negative fixed point (x_1, y_1) is given by

$$\lambda^2 - p_1(x_1, y_1)\lambda + p_2(x_1, y_1) = 0, \tag{7}$$

where

$$p_1(x_1, y_1) = 2 - r_1x_1 - \frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2},$$
$$p_2(x_1, y_1) = 1 - \frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2} - r_1x_1 + r_1dy_1 - \frac{er_1x_1y_1}{(g+y_1)^2} + \frac{dy_1{}^2}{x_1}.$$

Proposition 3 *The eigenvalues of* $J(x_1, y_1)$ *are* $\lambda_1 = \frac{1}{2}(p_1 - \sqrt{p_1^2 - 4p_2})$, *and* $\lambda_2 = \frac{1}{2}(p_1 + \sqrt{p_1^2 - 4p_2})$, *then* (x_1, y_1) *is*

(i) stable if $p_2 < 1$ *and* $|p_1| < p_2 + 1$;
(ii) unstable if $p_2 > 1$ *and* $|p_1| > p_2 + 1$ *or* $|p_1| < p_2 + 1$;
(iii) a saddle if $0 < |p_1| + p_2 + 1 < 2|p_1|$;
(iv) a non-hyperbolic if $|p_1| = |p_2 + 1|$ *or* $p_2 = 1$ *and* $|p_1| \leq 2$.

4. Bifurcation analysis

We have studied the stability properties of (x_1, y_1) previously. In the following, we choose the harvesting parameter e as the bifurcation parameter to analyze two different bifurcations (flip and Neimark-Sacker) of system (3) at the non-negative fixed point (x_1, y_1). Here, we utilized the center manifold theorem and normal form theory as in [27, 33], the existence conditions for the system (3) to admits both bifurcations are derived. It should be noted that for other parameters even the bifurcations can hold .

The condition for the existence of flip bifurcation is one of the eigenvalues of $J(x_1, y_1)$ is -1 and other is neither 1 nor -1. Assume one of the eigenvalue is -1, them from (7) we have

$$4 - 2r_1x_1 - \frac{2ey_1}{x_1} + \frac{2ey_1}{(g+y_1)^2} + r_1dy_1 - \frac{er_1x_1y_1}{(g+y_1)^2} + \frac{dy_1{}^2}{x_1} = 0,$$

which provides $e = \dfrac{(2r_1x_1^2 + 2dy_1 - r_1dx_1y_1 - dy_1^2 - 4x_1)(g+y_1)^2}{x_1y_1(2 - r_1x_1)} = e_f(say)$, is necessary for the flip bifurcation.

Let $\Delta_F = \{(r_1, r_2, d, e, g) : e = e_f, r_1 > 0, r_2 > 0, d > 0, e > 0, g > 0\}$, at $e = e_f$ the system (3) submits flip bifurcation at (x_1, y_1), when $e = e_f$ changes in the neighborhood of Δ_F.

Next to admits NS bifurcation of system (3) the complex conjugate eigenvalues of the characteristic equation at (x_1, y_1) should have modulus value one, for this it is necessary to satisfy:

$$(p_1(x_1, y_1))^2 - 4p_2(x_1, y_1) < 0 \quad \text{and} \quad p_2(x_1, y_1) = 1, \tag{8}$$

which provides

$$A(e) = \left(2 - r_1x_1 - \frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2}\right)^2$$
$$- 4\left(1 - \frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2} - r_1x_1 + r_1dy_1 - \frac{er_1x_1y_1}{(g+y_1)^2} + \frac{dy_1{}^2}{x_1}\right)^2 < 0,$$
$$B(e) = -\frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2} - r_1x_1 + r_1dy_1 - \frac{er_1x_1y_1}{(g+y_1)^2} + \frac{dy_1{}^2}{x_1}$$

Now $B(e) = 0$ gives $e = \dfrac{(dy_1 + r_1x_1^2 - r_1dx_1y_1 - dy_1^2)(g+y_1)^2}{x_1y_1(1 - r_1x_1)} = e_h(say)$.

Let $\Delta_{NS} = \{(r_1, r_2, d, e, g) : e = e_h, A(e_h) < 0, r_1 > 0, r_2 > 0, d > 0, e > 0, g > 0\}$, the fixed point (x_1, y_1) can arise NS bifurcation at $e = e_h$ when it changes in the neighborhood of Δ_F.

4.1. Flip Bifurcation

Now we investigate the possible flip bifurcation of the model (3) at (x_1, y_1). Since $(r_1, r_2, d, e_f, g) \in \Delta_F$ on giving a perturbation $|e_1| \ll 1$ of e_f, then perturbation of model (3) is described as

$$\begin{cases} x_{n+1} = x_n \exp\left[r_1(1-x_n) - y_n\right], \\ y_{n+1} = y_n \exp\left[r_2 - \dfrac{dy_n}{x_n} - \dfrac{(e_f+e_1)}{g+y_n}\right]. \end{cases} \tag{9}$$

Next on shifting (x_1, y_1) to origin of (3) by using the transform $w_n = x_n - x_1$ and $z_n = y_n - y_1$. We have

$$\begin{cases} w_{n+1} = & \beta_1 w_n + \beta_2 z_n + \beta_3 e_1 + \beta_4 w_n^2 + \beta_5 z_n^2 + \beta_6 {e_1}^2 \\ & +\beta_7 w_n z_n + \beta_8 w_n e_1 + \beta_9 z_n g^* + o((|w_n| + |z_n| + |e_1|)^2), \\ z_{n+1} = & \delta_1 w_n + \delta_2 z_n + \delta_3 e_1 + \delta_4 w_n^2 + \delta_5 z_n^2 + \delta_6 {e_1}^2 \\ & +\delta_7 w_n z_n + \delta_8 w_n e_1 + \delta_9 z_n g^* + o((|w_n| + |z_n| + |e_1|)^2), \end{cases} \tag{10}$$

where

$$\begin{aligned} &\beta_1 = 1 - r_1 x_1,\ \beta_2 = -x_1,\ \beta_3 = \beta_6 = \beta_8 = \beta_9 = 0,\ \beta_4 = -r_1 + \tfrac{r_1^2 x_1}{2},\ \beta_5 = \tfrac{x_1}{2}, \\ &\beta_7 = -1 + r_1 x_1,\ \delta_1 = \tfrac{d{y_1}^2}{{x_1}^2},\ \delta_2 = 1 - \tfrac{dy_1}{x_1} + \tfrac{ey_1}{(g+y_1)^2},\ \delta_3 = -\tfrac{y_1}{g+y_1},\ \delta_4 = -\tfrac{d{y_1}^2}{{x_1}^3} + \tfrac{d^2{y_1}^3}{2{x_1}^4}, \\ &\delta_5 = -\tfrac{d}{x_1} + \tfrac{d^2 y_1}{2{x_1}^2} + \tfrac{e}{(g+y_1)^2} - \tfrac{edy_1}{x_1(g+y_1)^2} + \tfrac{e^2 y_1}{2(g+y_1)^4} - \tfrac{ey_1}{(g+y_1)^3}, \delta_6 = \tfrac{y_1}{2(g+y_1)^2}, \\ &\delta_7 = \tfrac{2dy_1}{{x_1}^2} - \tfrac{d^2{y_1}^2}{{x_1}^3} + \tfrac{edy_1{}^2}{{x_1}^2(g+y_1)^2},\ \delta_8 = -\tfrac{d{y_1}^2}{{x_1}^2(g+y_1)},\ \delta_9 = \tfrac{y_1 d}{x_1(g+y_1)} - \tfrac{g}{(g+y_1)^2} - \tfrac{ey_1}{(g+y_1)^3}. \end{aligned}$$

Let us assume that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3 - r_1 x_1 - \frac{dy_1}{x_1} + \frac{ey_1}{(g+y_1)^2}$ for the matrix J with $|\lambda_1| = 1$, $|\lambda_2| \neq 1$.

Next, we construct the non-singular matrix L as follows:

$$L = \begin{pmatrix} \beta_2 & \beta_2 \\ -1-\beta_1 & \lambda_2 - \beta_1 \end{pmatrix},$$

and use the translation $\begin{pmatrix} w_n \\ z_n \end{pmatrix} = L \begin{pmatrix} W_n \\ Z_n \end{pmatrix}$, then (10) can be written as:

$$\begin{cases} W_{n+1} = -W_n + F_1(w_n, z_n, e_1) + o((|w_n| + |z_n| + |e_1|)^2), \\ Z_{n+1} = \lambda_2 Z_n + (w_n, z_n, e_1) + o((|w_n| + |z_n| + |e_1|)^2), \end{cases} \tag{11}$$

where

$$\begin{aligned} F_1(w_n, z_n, e_1) &= M_1 e_1 + M_2 w_n^2 + M_3 z_n^2 + M_4 {e_1}^2 + M_5 w_n z_n + M_6 w_n e_1 + M_7 z_n e_1, \\ F_2(w_n, z_n, \alpha^*) &= N_1 \alpha^* + N_2 w_n^2 + N_3 z_n^2 + N_4 {e_1}^2 + N_5 w_n z_n + N_6 w_n e_1 + N_7 z_n e_1, \end{aligned}$$

and

$$\begin{aligned} &M_1 = \tfrac{(\lambda_2-\beta_1)\beta_3 - \beta_2\delta_3}{\beta_2(1+\lambda_2)}, && M_2 = \tfrac{(\lambda_2-\beta_1)\beta_4 - \beta_2\delta_4}{\beta_2(1+\lambda_2)}, && M_3 = \tfrac{(\lambda_2-\beta_1)\beta_5 - \beta_2\delta_5}{\beta_2(1+\lambda_2)}, && M_4 = \tfrac{(\lambda_2-\beta_1)\beta_6 - \beta_2\delta_6}{\beta_2(1+\lambda_2)}, \\ &M_5 = \tfrac{(\lambda_2-\beta_1)\beta_7 - \beta_2\delta_7}{\beta_2(1+\lambda_2)}, && M_6 = \tfrac{(\lambda_2-\beta_1)\beta_8 - \beta_2\delta_8}{\beta_2(1+\lambda_2)}, && M_7 = \tfrac{(\lambda_2-\beta_1)\beta_9 - \beta_2\delta_9}{\beta_2(1+\lambda_2)}, && N_1 = \tfrac{(1+\beta_1)\beta_3 + \beta_2\delta_3}{\beta_2(1+\lambda_2)}, \\ &N_2 = \tfrac{(1+\beta_1)\beta_4 + \beta_2\delta_4}{\beta_2(1+\lambda_2)}, && N_3 = \tfrac{(1+\beta_1)\beta_5 + \beta_2\delta_5}{\beta_2(1+\lambda_2)}, && N_4 = \tfrac{(1+\beta_1)\beta_6 + \beta_2\delta_6}{\beta_2(1+\lambda_2)}, && N_5 = \tfrac{(1+\beta_1)\beta_7 + \beta_2\delta_7}{\beta_2(1+\lambda_2)}, \\ &N_6 = \tfrac{(1+\beta_1)\beta_8 + \beta_2\delta_8}{\beta_2(1+\lambda_2)}, && N_7 = \tfrac{(1+\beta_1)\beta_9 + \beta_2\delta_9}{\beta_2(1+\lambda_2)}. \end{aligned}$$

Now, let us assume G^c be the center manifold(CM), then by using the CM theorem we approximate the CM G^c of (11) at the origin, for small changes in $e_1 = 0$ can be given by

$$G^c(0,0) = \{(W_n, Z_n) : Z_n = h(W_n, e_1)\}$$

$$= \left\{ (W_n, Z_n) : Z_n = c_1 e_1 + c_2 W_n^2 + c_3 e_1 W_n + c_4 {e_1}^2 + o((|W_n| + e_1)^2) \right\}. \qquad (12)$$

On substituting (11) on both sides of $Z_n = h(W_n, e_1)$, we have

$$\begin{aligned} \lambda_2 Z_n + F_2(w_n, z_n, e_1) =& c_1 e_1 + c_2(-W_n + F_1(w_n, z_n, e_1))^2 + c_3 e_1(-W_n + F_1(w_n, z_n, e_1)) \\ &+ c_4 {e_1}^2 + o((|W_n| + e_1)^2), \end{aligned}$$

where

$$\begin{aligned} w_n =& \beta_2(W_n + Z_n) = \beta_2(W_n + h(W_n, e_1)), \\ z_n =& (-1 - \beta_1)W_n + (\lambda_2 - \beta_1)Z_n = (-1 - \beta_1)W_n + (\lambda_2 - \beta_1)h(W_n, e_1) \\ c_1 =& \frac{N_1}{1 - \lambda_2},\ c_2 = \frac{1}{1 - \lambda_2}\left[N_2\beta_2^2 + N_3(1 + \beta_1)^2 - N_5\beta_2(1 + \beta_1)\right] \\ c_3 =& \frac{1}{1 + \lambda_2}\left[-2c_2 M_1 - N_6\beta_2 + N_7(1 + \beta_1) - 2c_1 N_2 \beta_2^2\right., \\ & \left.+ 2c_1 N_3(1 + \beta_1)(\lambda_2 - \beta_1) - c_1 N_5 \beta_2(\delta_2 - \beta_1)\right] \\ c_4 =& \frac{1}{1 - \lambda_2}\left[c_1^2 N_2 \beta_2^2 + c_1^2 N_3(\lambda_2 - \beta_1)^2 + c_1^2 N_5 \beta_2(\lambda_2 - \beta_1) + c_1 N_6 \beta_2\right., \\ & \left.+ c_1 N_7(\lambda_2 - \beta_1) - c_2 M_1^2 + N_4 - M_1 c_3\right]. \end{aligned}$$

Accordingly, on the CM G^c at origin we have

$$\begin{aligned} w_n^2 =& \beta_2^2(W_n^2 + 2W_n Z_n + Z_n^2), \\ w_n z_n =& -\beta_2(1 + \beta_1)W_n^2 + \beta_2(\delta_2 - \beta_1)W_n Z_n + \beta_2(\lambda_2 - \beta_1)Z_n^2, \\ z_n^2 =& (1 + \beta_1)^2 W_n^2 - 2(1 + \beta_1)(\lambda_2 - \beta_1)W_n Z_n + (\lambda_2 - \beta_1)^2 Z_n^2, \end{aligned}$$

where

$$\begin{aligned} W_n Z_n =& c_1 e_1 W_n + c_2 W_n^3 + c_3 e_1 W_n^2 + c_4 {e_1}^2 W_n + o((|W_n| + |e_1|)^3), \\ Z_n^2 =& c_1^2 {e_1}^2 + 2c_1 c_2 e_1 W_n^2 + 2c_1 c_4 {e_1}^3 + 0((|W_n| + |e_1|)^3). \end{aligned}$$

Moreover, the map confined to the CM $G^c(0,0)$ has takes the form

$$\begin{aligned} H^*(W_n) =& -W_n + F_1(w_n, z_n, e_1) \\ =& -W_n + d_1 e_1 + d_2 W_n^2 + d_3 W_n g^* + d_4 {e_1}^2 + d_5 W_n^2 e_1 \\ &+ d_6 W_n {e_1}^2 + d_7 W_n^3 + d_8 {e_1}^3 + o((|W_n| + |e_1|)^3), \end{aligned}$$

where

$$\begin{aligned} d_1 =& M_1, d_2 = M_2\beta_2^2 + M_3(1 + \beta_1)^2 - M_5\beta_2(1 + \beta_1), \\ d_3 =& 2c_1 M_2 \beta_2^2 - 2c_1 M_3(1 + \beta_1)(\lambda_2 - \beta_1) + c_1 M_5 \beta_2(\delta_2 - \beta_1) + M_6\beta_2 - M_7(1 + \beta_1), \\ d_4 =& c_1^2 M_2 \beta_2^2 + c_1^2 M_3(\lambda_2 - \beta_1)^2 + M_4 + c_1^2 M_5 \beta_2(\lambda_2 - \beta_1) + c_1 M_6 \beta_2 + c_1 M_7(\lambda_2 - \beta_1), \\ d_5 =& 2c_3 M_2 \beta_2^2 + 2c_1 c_2 M_2 \beta_2^2 - 2c_3 M_3(1 + \beta_1)(\lambda_2 - \beta_1) + 2c_1 c_2 M_3(\lambda_2 - \beta_1)^2 \\ &+ c_3 M_5 \beta_2(\delta_2 - \beta_1) + 2c_1 c_2 M_5 \beta_2(\lambda_2 - \beta_1) + c_2 M_6 \beta_2 + c_2 M_7(\lambda_2 - \beta_1), \\ d_6 =& 2c_4 M_2 \beta_2^2 + 2c_1 c_3 M_2 \beta_2^2 - 2c_4 M_3(1 + \beta_1)(\lambda_2 - \beta_1) + 2c_1 c_3 M_3(\lambda_2 - \beta_1)^2 \\ &+ c_4 M_5 \beta_2(\delta_2 - \beta_1) + 2c_1 c_3 M_5 \beta_2(\lambda_2 - \beta_1) + c_3 M_6 \beta_2 + c_3 M_7(\lambda_2 - \beta_1), \\ d_7 =& 2c_2 M_2 \beta_2^2 - 2c_2 M_3(1 + \beta_1)(\lambda_2 - \beta_1) + c_2 M_5 \beta_2(\delta_2 - \beta_1), \\ d_8 =& 2c_1 c_4 M_2 \beta_2^2 + 2c_1 c_4 M_3(\lambda_2 - \beta_1)^2 + 2c_1 c_4 M_5 \beta_2(\lambda_2 - \beta_1) + c_4 M_6 \beta_2 + c_4 M_7(\lambda_2 - \beta_1). \end{aligned}$$

Finally from [33], we define α_1 and α_2 as follows

$$\alpha_1 = \left(H^*_{W_n e_1} + \frac{1}{2} H^*_{e_1} H^*_{W_n W_n} \right) |_{(W_n, e_1) = (0,0)} = d_3 + d_1 d_2, \tag{13}$$

$$\alpha_2 = \left(\frac{1}{6} H^*_{W_n W_n W_n} + \left(\frac{1}{2} H^*_{W_n W_n} \right)^2 \right) |_{(W_n, e_1) = (0,0)} = d_7 + d_2^2. \tag{14}$$

Therefore, we have got the following findings about flip-bifurcation from the aforementioned study.

Theorem 1 *If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, the model (3) exhibits a flip bifurcation at (x_1, y_1) while changing the parameter e nearby e_1. Moreover, if $\alpha_2 > 0$(or $\alpha_2 < 0$) then the existing period two orbits from (x_1, y_1) are stable (or unstable).*

4.2. Neimark-Sacker(NS) Bifurcation

Now we analyze the properties of possible NS bifurcation around (x_1, y_1) for model (3) if suppose (8) holds for some e_h. Given a perturbation $|e_2| \ll 1$ of e_h, then perturbation of model (3) is described as

$$\begin{cases} x_{n+1} = x_n \exp\left[r_1(1 - x_n) - y_n \right], \\ y_{n+1} = y_n \exp\left[r_2 - \dfrac{d y_n}{x_n} - \dfrac{(e_h + e_2)}{g + y_n} \right]. \end{cases} \tag{15}$$

Let us use the transform $w_n = x_n - x_1$, $z_n = y_n - y_1$ and shift (x_1, y_1) to $(0, 0)$ the system (15) takes the form

$$\begin{cases} w_{n+1} = (w_n + x_1) \exp\left[r_1(1 - (w_n + x_1)) - (z_n + y_1) \right] - x_1, \\ z_{n+1} = (z_n + y_1) \exp\left[r_2 - \dfrac{d(z_n + y_1)}{(w_n + x_1)} - \dfrac{(e_h + e_2)}{g + (z_n + y_1)} \right] - y_1. \end{cases} \tag{16}$$

Then the Taylor expansion of (16) at the origin up to order three, that is

$$\begin{cases} w_{n+1} = \rho_1 w_n + \rho_2 z_n + \rho_3 w_n^2 + \rho_4 w_n z_n + \rho_5 z_n^2 + \rho_6 w_n^3 \\ \qquad + \rho_7 w_n^2 z_n + \rho_8 w_n v_n^2 + \rho_9 v_n^3 + O((|w_n| + |z_n|)^3), \\ z_{n+1} = \eta_1 w_n + \eta_2 z_n + \eta_3 w_n^2 + \eta_4 w_n z_n + \eta_5 z_n^2 + \eta_6 w_n^3 \\ \qquad + \eta_7 w_n^2 z_n + \eta_8 w_n v_n^2 + \eta_9 v_n^3 + O((|w_n| + |z_n|)^3), \end{cases} \tag{17}$$

where

$$\begin{aligned}
&\rho_1 = 1 - r_1 x_1,\ \rho_2 = -x_1,\ \rho_3 = -r_1 + \tfrac{r_1^2 x_1}{2},\ \rho_4 = r_1 x_1 - 1,\ \rho_5 = \tfrac{x_1}{2}, \\
&\rho_6 = \tfrac{r_1^2}{2} - \tfrac{r_1^3 x_1}{6}, \rho_7 = 2r_1 - r_1^2 x_1, \rho_8 = 1 - r_1 x_1, \rho_9 = -\tfrac{x_1}{6}, \eta_1 = \tfrac{d y_1}{{x_1}^2}, \\
&\eta_2 = 1 - \tfrac{d y_1}{x_1} + \tfrac{e y_1}{(e+y_1)^2}, \eta_3 = -\tfrac{d {y_1}^2}{{x_1}^3} + \tfrac{d^2 {y_1}^3}{2{x_1}^4}, \eta_4 = \tfrac{2 d y_1}{{x_1}^2} - \tfrac{d^2 {y_1}^2}{{x_1}^3} + \tfrac{e d {y_1}^2}{{x_1}^2 (g+y_1)^2}, \\
&\eta_5 = -\tfrac{d}{x_1} + \tfrac{d^2 y_1}{2{x_1}^2} + \tfrac{e g}{(g+y_1)^3} - \tfrac{e d y_1}{x_1 (g+y_1)^2} + \tfrac{e^2 y_1}{2(g+y_1)^4}, \eta_6 = \tfrac{d {y_1}^2}{{x_1}^4} - \tfrac{2e^2 {y_1}^3}{3{x_1}^5} - \tfrac{d^2 {y_1}^3}{3{x_1}^5} + \tfrac{d^3 {y_1}^4}{6{x_1}^6}, \\
&\eta_7 = -\tfrac{2 e y_1}{{x_1}^3} + \tfrac{5 e^2 {y_1}^2}{2{x_1}^4} - \tfrac{d^3 {y_1}^3}{2{x_1}^5} - \tfrac{e d {y_1}^2}{{x_1}^3 (g+y_1)^2} + \tfrac{e d^2 {y_1}^3}{2{x_1}^4 (g+y_1)^2}, \\
&\eta_8 = \tfrac{d}{{x_1}^2} - \tfrac{2e^2 y_1}{{x_1}^3} + \tfrac{e d y_1}{{x_1}^2 (g+y_1)^2} + \tfrac{d^3 {y_1}^2}{2{x_1}^4} + \tfrac{e d g y_1}{{x_1}^2 (g+y_1)^3} - \tfrac{e d^2 {y_1}^2}{{x_1}^3 (g+y_1)^2} + \tfrac{e^2 d {y_1}^2}{2{x_1}^2 (g+y_1)^4}, \\
&\eta_9 = \tfrac{d^2}{2{x_1}^2} - \tfrac{e g}{(g+y_1)^4} - \tfrac{2 g d}{3 x_1 (g+y_1)^2} + \tfrac{2 g d y_1}{3 x_1 (g+y_1)^3} - \tfrac{d^3 y_1}{6{x_1}^3}, \\
&\qquad - \tfrac{e d g}{3 x_1 (g+y_1)^3} + \tfrac{e d^2 y_1}{2{x_1}^2 (g+y_1)^2} - \tfrac{e^2 d y_1}{2 x_1 (g+y_1)^4} + \tfrac{e^2 g}{3(g+y_1)^5} + \tfrac{e^3 y_1}{6(g+y_1)^6}.
\end{aligned}$$

The characteristic polynomial equation associated with linearized sytem of (14) at the origin, can be given as follows

$$\lambda^2 + q_1(e_2)\lambda + q_2(e_2) = 0, \tag{18}$$

where

$$q_1(e_2) = (r_1x_1 - 1)\Omega_1 - \left(1 - \frac{dy_1}{x_1} + \frac{(e_h+e_2)y_1}{(g+y_1)^2}\right)\Omega_2,$$
$$q_2(e_2) = \Omega_1\Omega_2\left[(1-r_1x_1)\left(1 - \frac{dy_1}{x_1} + \frac{(e_h+e_2)y_1}{(g+y_1)^2}\right) + \frac{dy_1{}^2}{x_1}\right],$$

with $\Omega_1 = \exp[r_1(1-x_1) - y_1]$ and $\Omega_2 = \exp\left[r_2 - \frac{dy_1}{x_1} - \frac{(e_h+e_2)}{g+y_1}\right]$.

Now the roots of equation (16) are pair of complex conjugates.

$$\lambda_{1,2} = \frac{1}{2}\left[-q_1(e_2) \pm i\sqrt{4q_2(e_2) - (q_1(e_2))^2}\right].$$

Since $(r_1, r_2, d, e_h, g) \in \Delta_{NS}$, we have $|\lambda_{1,2}| = \sqrt{q_2(e_2)}$ and

$$\begin{aligned}\frac{d|\lambda_{1,2}|}{de_2} =& \frac{1}{2\sqrt{q_2(0)}}\left\{(1-r_1x_1)\left[\frac{y_1}{(g+y_1)^2}\right] - \frac{1}{g+y_1}\right.\\ &\left.\times\left[(1-r_1x_1)\left(1-\frac{dy_1}{x_1} + \frac{e_hy_1}{(g+y_1)^2}\right) + \frac{dy_1{}^2}{x_1}\right]\right\} < 0.\end{aligned} \tag{19}$$

Further, we assume that $q_1(0) = -2 + r_1x_1 + \frac{dy_1}{x_1} - \frac{e_hy_1}{(g+y_1)^2} \neq 0. -1$ and from (8) implies $q_1(0) = \pm 2, 0, -1$, which says $\lambda_1^k, \lambda_2^k \neq 1$ for $k = 1, 2, 3, 4$, when $e_2 = 0$. We only require $q_1(0) \neq 0, 1$, which we can attain if it satisfies

$$r_1x_1 + \frac{dy_1}{x_1} - \frac{e_hy_1}{(g+y_1)^2} \neq 2, 3. \tag{20}$$

Letting $e_2 = 0$, $\xi = -\frac{q_1(0)}{2}$, $\theta = \frac{\sqrt{4q_2(0) - q_1^2(0)}}{2}$, we construct the non-singular matrix

$$L = \begin{pmatrix} \rho_2 & 0 \\ \xi - \rho_1 & \theta \end{pmatrix},$$

and use the translation $\begin{pmatrix} w_n \\ z_n \end{pmatrix} = L\begin{pmatrix} W_n \\ Z_n \end{pmatrix}$, thus the model (15) takes the form:

$$\begin{cases} W_{n+1} = \xi W_n + \theta Z_n + Q(W_n, Z_n) + o((|W_n| + |Z_n|)^3), \\ Z_{n+1} = -\theta W_n + \xi Z_n + R(W_n, Z_n) + o((|W_n| + |Z_n|)^3), \end{cases} \tag{21}$$

where

$$\begin{aligned}Q(W_n, Z_n) =& \frac{1}{\rho_2}\left[\ \{\rho_3\rho_2{}^2 + \rho_4\rho_2(\xi-\rho_1) + \rho_5(\xi-\rho_1)^2\}W_n^2 + \{\rho_4\rho_2\theta + 2\theta\rho_5(\xi-\rho_1)\}W_nZ_n\right.\\ &+ \rho_5\theta^2Z_n^2 + \{\rho_6\rho_2{}^3 + \rho_7\rho_2{}^2(\xi-\rho_1) + \rho_8\rho_2(\xi-\rho_1)^2 + \rho_9(\xi-\rho_1)^3\}W_n^3\\ &+ \{\rho_7\rho_2{}^2 + 2\theta\rho_8\rho_2(\xi-\rho_1) + 3\theta\rho_9(\xi-\rho_1)^2\}W_n^2Z_n\\ &\left.+ \{\theta^2\rho_8\rho_2 + 3\theta^2\rho_9(\xi-\rho_1)\}W_nZ_n^2 + \theta^3\rho_9Z_n^3\ \right],\\ R(W_n, Z_n) =& \frac{1}{\rho_2\theta}\left[\ \{\rho_2{}^2(\rho_3(\rho_1-\xi) + \rho_2\eta_3) + \rho_2(\xi-\rho_1)(\rho_4(\rho_1-\xi) + \rho_2\eta_4)\right.\\ &+ (\xi-\rho_1)^2(\rho_5(\rho_1-\xi) + \rho_2\eta_5)\}W_n^2 + \{\theta\rho_2(\rho_4(\rho_1-\xi) + \rho_2\eta_4)\\ &+ 2\theta(\xi-\rho_1)(\rho_5(\rho_1-\xi) + \rho_2\eta_5)\}W_nZ_n + \theta^2\{\rho_5(\rho_1-\xi) + \rho_2\eta_5\}Z_n^2\\ &+ \{\rho_2{}^3(\rho_6(\rho_1-\xi) + \rho_2\eta_6) + \rho_2{}^2(\xi-\rho_1)(\rho_7(\rho_1-\xi) + \rho_2\eta_7)\end{aligned}$$

$$
\begin{aligned}
&+ \rho_2(\xi-\rho_1)^2(\rho_8(\rho_1-\xi)+\rho_2\eta_8)(\xi-\rho_1)^3(\rho_9(\rho_1-\xi)+\rho_2\eta_9)\}W_n^3\\
&+ \{\theta{\rho_2}^2(\rho_7(\rho_1-\xi)+\rho_2\eta_7)+2\theta\rho_2(\xi-\rho_1)(\rho_8(\rho_1-\xi)+\rho_2\eta_8)\\
&+ 3\theta(\xi-\rho_1)^2(\rho_9(\rho_1-\xi)+\rho_2\eta_9)\}W_n^2Z_n+\{\theta^2\rho_2(\rho_8(\rho_1-\xi)+\rho_2\eta_8)\\
&+ 3\theta^2(\xi-\rho_1)(\rho_9(\rho_1-\xi)+\rho_2\eta_9)\}W_nZ_n^2+\theta^3(\rho_9(\rho_1-\xi)+\rho_2\eta_9)Z_n^3\,].
\end{aligned}
$$

Next, we require the non zero quantity a^*, to ensure (15) submits NS bifurcation.

$$
a^* = -Re\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right]-\frac{1}{2}|\xi_{11}|^2-|\xi_{02}|^2+Re(\bar{\lambda}\xi_{21}), \tag{22}
$$

where

$$
\begin{aligned}
\xi_{20} =& \frac{1}{8}[(Q_{W_nW_n}-Q_{Z_nZ_n}+2R_{W_nZ_n})+i(R_{W_nW_n}-R_{Z_nZ_n}-2Q_{W_nZ_n})],\\
\xi_{11} =& \frac{1}{4}\left[(Q_{W_nW_n}+Q_{Z_nZ_n})+i(R_{W_nW_n}+R_{Z_nZ_n})\right],\\
\xi_{02} =& \frac{1}{8}[(Q_{W_nW_n}-Q_{Z_nZ_n}-2R_{W_nZ_n})+i(R_{W_nW_n}-R_{Z_nZ_n}+2Q_{W_nZ_n})],\\
\xi_{21} =& \frac{1}{16}[(Q_{W_nW_nW_n}+Q_{W_nZ_nZ_n}+R_{W_nW_nZ_n}+R_{Z_nZ_nZ_n})\\
&+i(R_{W_nW_nW_n}+R_{W_nZ_nZ_n}-Q_{W_nW_nZ_n}-Q_{Z_nZ_nZ_n})].
\end{aligned}
$$

Finally, from [27], we can state the following findings:

Theorem 2 *If (17), (18) holds and the quantity a^* is non zero then the model (13) admits NS bifurcation at (x_1, y_1) when e_h changes in the neighbourhood of Δ_{NS}. Additionally, the quantity $a^* < 0$ (or resp. $a^* > 0$) then the stable (or resp. unstable) invariant closed curve from (x_1, y_1) starts bifurcates.*

5. Numerical Results

In the following section, we perform some simulations for the system (3) around (x_1, y_1) to ensure the obtained results, especially time series $x\ vs\ t$, phase portraits $x\ vs\ y$, one parameter bifurcation plots $x\ vs\ e$ along with its corresponding Largest Lyapunov exponents(LLE) $LLE\ vs\ e$ has plotted. The real parameters taken for simulations are given in two cases:

Case (i): Fixing $r_1 = 1$, $r_2 = 4$, $d = 2.4$, $g = 0.5$ and varying $e \in (0, 0.6]$, here we discuss the flip-bifurcation of system (3). On solving equations in (5) we have got a non-negative fixed point $(x_1, y_1) = (0.4020017560834107, 0.5979982439165893)$ at $e = 0.472004$ for system (3) and the characteristic equation has two eigenvalues $\lambda_1 = -1$, $\lambda_2 = -0.7380025864207043$. Hence, $(r_1, r_2, d, e, g) = (1, 4, 2.4, 0.472004, 0.5) \in \Delta_F$. Further, by computing (13) and (14), we can obtain that $\alpha_1 = 6.071963985676632 > 0$, $\alpha_2 = 13.93471200238389 > 0$. The flip bifurcation happens and attracting period two orbits start obtaining from (x_1, y_1), which illustrated in Theorem 1.

From Fig. 1a & 2a, we see for $e = 0.48 > 0.472004$ the (x_1, y_1) is stable and there is orbit of period-doubling starts on decreasing e from 0.472004. Furthermore, period-2, 4, 8 orbits for the values $e = 0.4, 0.28, 0.24$, can be shown in time plot Fig. 1b, 1c, 1d and phase portrait 2b, 2c, 2d, respectively. Moreover, irregular chaotic orbits is emerged with further decreasing of e is shown in Fig. 1e,1f, 2e and 2f . It states the system (3) becomes stable by reverse period-doubling phenomena.

Case (ii): Fixing $r_1 = 1.22$, $r_2 = 2.27$, $d = 0.9$, $g = 0.5$ and varying $e \in (0, 1.2]$, here the simulation results about NS bifurcation are discussed for the system (3). We have got $(x_1, y_1) = (0.44228649124664565, 0.6804104806790923)$ at $e = 1.04519$ and its characteristic

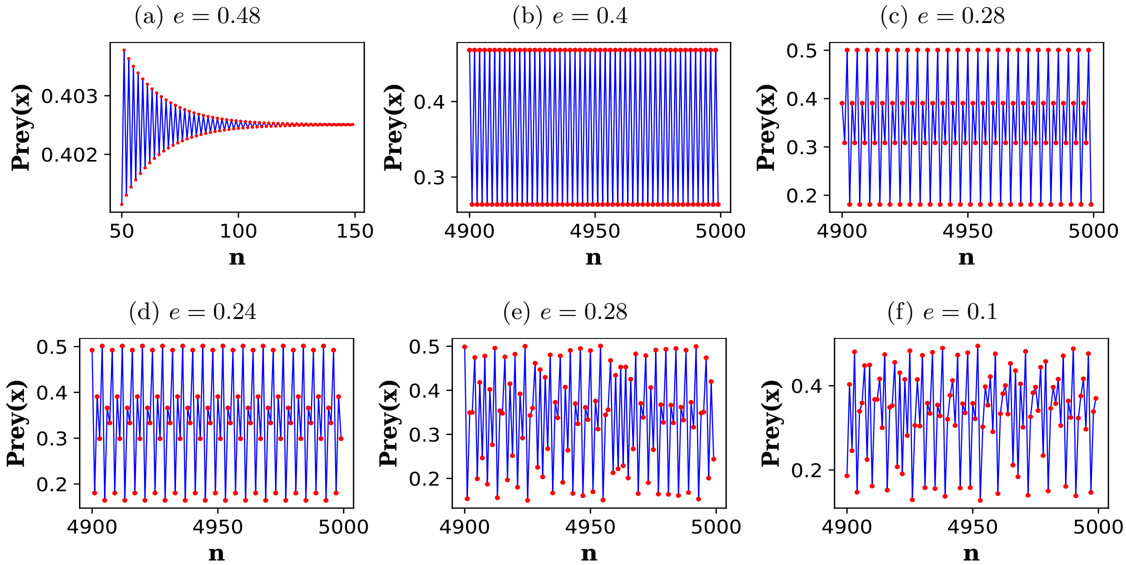

Figure 1: Time plot of prey for the system (3) with fixed values $r_1 = 1$, $r_2 = 4$, $d = 2.4$, $g = 0.5$ and different values of e: (a) $e = 0.48$, (b) $e = 0.4$, (c) $e = 0.28$, (d) $e = 0.24$, (e) $e = 0.2$ and (f) $e = 0.1$.

equation (18) has two eigenvalues $\lambda_{1,2} = 0.29312215830655564 \pm 0.9560750593527182i$ with $|\lambda_{1,2}| = 1$. Hence, $(r_1, r_2, d, e, g) = (1.22, 2.27, 0.9, 1.04519, 0.5) \in \Delta_{NS}$ and the quantity $a^* = 0.2809906013261425 > 0$, its results are discussed in Theorem 2.

And it is shown in Fig. 3a & 4a for $e > 1.04519$ the (x_1, y_1) of (3) is stable and the stability loss occurs at $e = 1.04519$, and when e decreases from 1.04519, that an invariant circle appears is shown in Fig. 3b & 4b. Further decreasing e, the irregular orbits emerges like period-9 orbit in Fig. 3c & 4c, four invariant circle like orbit in 3d & 4d, period-34 orbit in Fig. 3e & 4e and chaotic attractor in Fig. 3f & 4f at $e = 0.55, 0.348, 0.343, 0.3$, respectively.

Finally, the clear view of various complex behaviors of the system (3) is shown by the bifurcation plot. Thus, for the value $e \in (0, 0.6]$ the occurrence of flip bifurcation is shown in Fig. 5a and for the value $e \in (0, 1.2]$ the NS bifurcation is shown in Fig. 5c with fixed parameters as in both cases discussed above. Moreover, the system (3) shows chaotic behaviour for smaller e values. The emerge of chaotic behaviour is verified by its largest Lyapunov exponent, which is plotted in 5b & 5d.

6. Conclusion

In the present article, the qualitative analysis of discrete version predator-prey system of Leslie-Gower type with non-linear harvesting in predator is studied in detailed manner. We showed that, as harvesting parameter e increase or decrease in the Δ_F and Δ_{NS}, the system (3) can exhibits bifurcation (flip or Neimark-Sacker) behaviour around the unique positive fixed point algebraically. The simulations has performed numerically to ensure our obtained results, which includes cascade of reverse period-doubling, 4 attracting invariant cycles, period-9,34 and unpredictable behaviour like chaotic attractor, has showed the harvesting parameter e plays very important role in complex behaviour of system (3). Finally, it is proven that the system (3) in choice of choosing parameters and initial conditions had more sensitive.

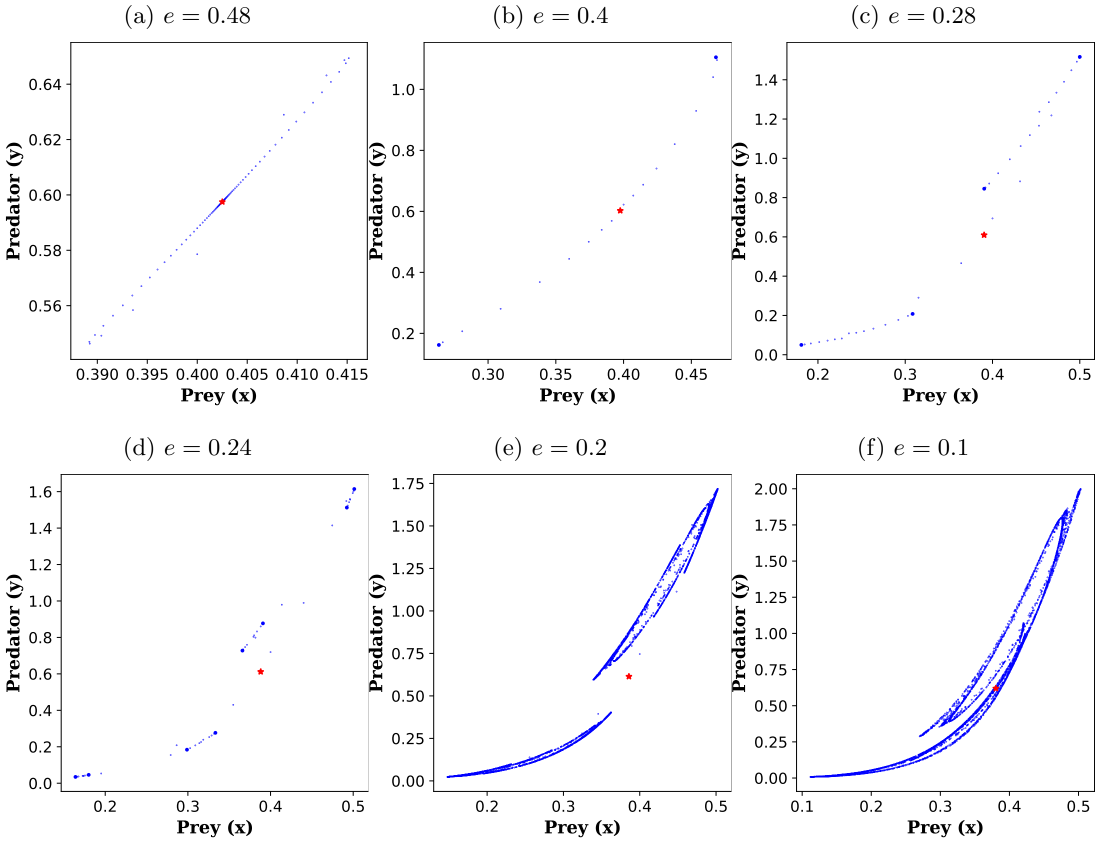

Figure 2: Phase portrait of system (3) with fixed values $r_1 = 1$, $r_2 = 4$, $d = 2.4$, $g = 0.5$ and different values of e: (a) $e = 0.48$, (b) $e = 0.4$, (c) $e = 0.28$, (d) $e = 0.24$, (e) $e = 0.2$ and (f) $e = 0.1$.

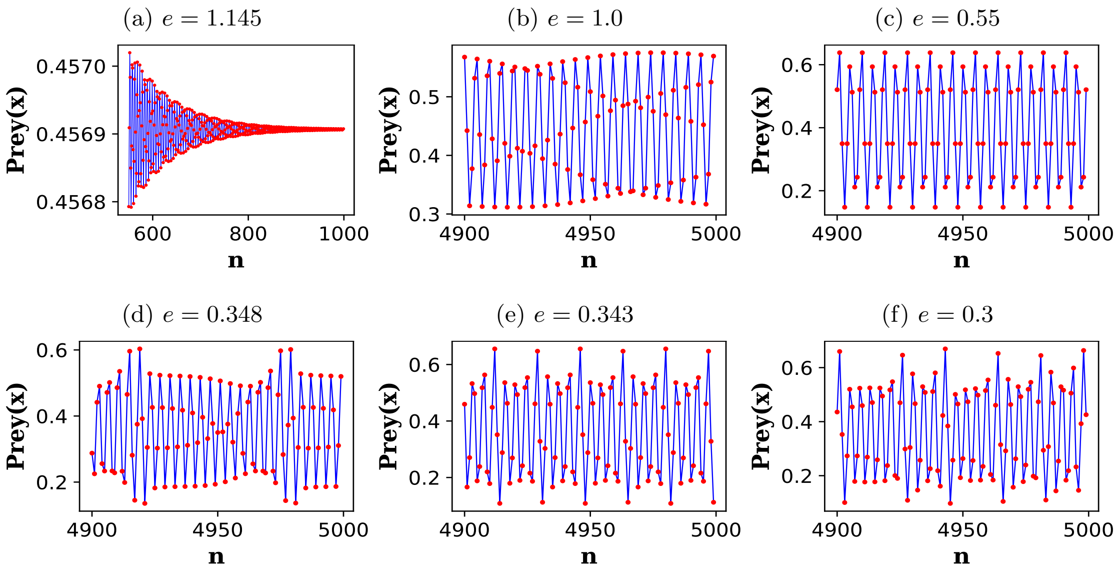

Figure 3: Time plot of prey for the system (3) with fixed values $r_1 = 1$, $r_2 = 4$, $d = 2.4$, $g = 0.5$ and different values of e: (a) $e = 1.145$, (b) $e = 1.0$, (c) $e = 0.55$, (d) $e = 0.348$, (e) $e = 0.343$ and (f) $e = 0.3$.

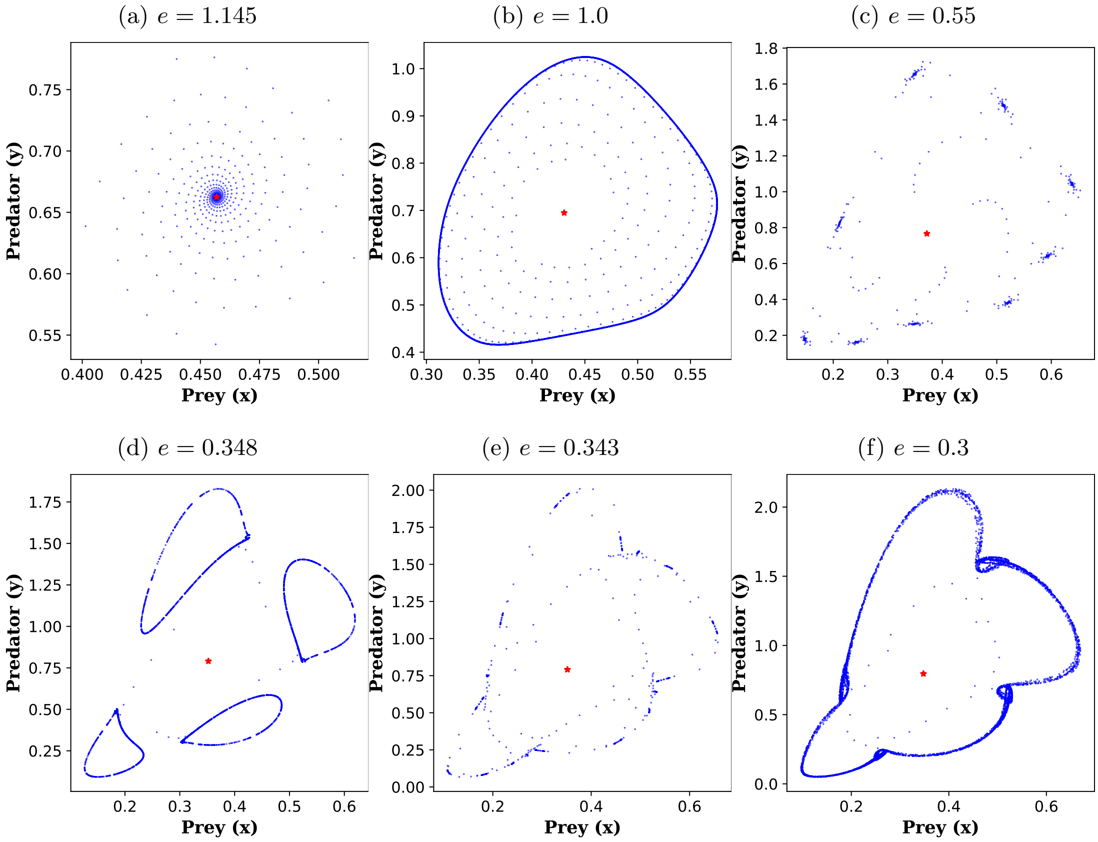

Figure 4: Phase portrait of system (3) with fixed parameters $r_1 = 1$, $r_2 = 4$, $d = 2.4$, $g = 0.5$ and different values of e: (a) $e = 1.145$, (b) $e = 1.0$, (c) $e = 0.55$, (d) $e = 0.348$, (e) $e = 0.343$ and (f) $e = 0.3$.

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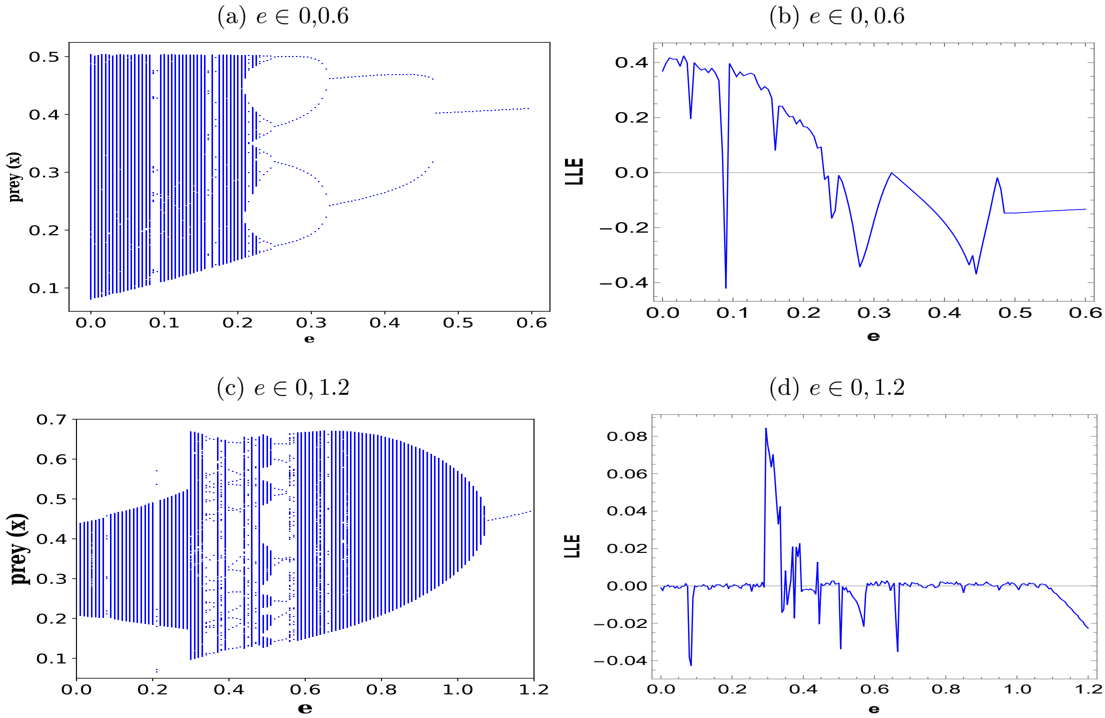

Figure 5: (a), (c) Flip and Neimark-Sacker bifurcation diagram of system (3) in (e, x) plane and its largest Lyapunov exponent in (b), (d), correspondingly

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