Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: J Thipcha and S Pinjai 2021 *J. Phys.: Conf. Ser.* **1850** 012116

Exponential Stability for the Neutral-type Inertial BAM Neural Networks with Time-varying Delays

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Abstract. In this paper, the global exponential stability for the neutral-type inertial bidirectional association memory neural networks with time-varying delays is considered . In our study, the lower and upper bounds of the activation functions are allowed to be either positive, negative or zero. By constructing new and improved Lyapunov-Krasovskii functional and introducing free-weighting matrices, a new and improved delay-dependent the neutral-type inertial bidirectional association memory neural networks with time-varying delays is derived in the form of linear matrix.

1. Introduction

A class of neural networks related to bidirectional associative memory (BAM) has been introduced by Kosko [9]. This model generalized the single-layer autoassociative Hebbian correlator to a two-layer pattern matched heteroassociative circuit. It is important that it produces many nice properties owing to the special structure of connection weights and its practice applications in storing paired patterns through both directional forward and backward directions. In [20, 24, 25, 26], several sufficient condition on the global asymptotic stability and global exponential stability of BAM neural networks with time-varying delays have been derived.

In general, time delay is often unavoidable in the communication and response of neurons due to the finite switching speed of amplifiers in the electronic implementation of analog neural network and communication time, see [14, 15, 17, 23]. It is well known that time delays might cause instability, divergence behavior and oscillations of neural networks. Therefore, several researchers have focused on the study of stability analysis of BAM neural networks with either constant delays or time-varying delays, see [4, 6, 18, 19].

Besides, it is usual that the time delay arises not only in system states but also in the derivatives of system states. Systems containing the information of past state derivatives are called neutral-type delay systems. Recently, the study of the stability analysis of neutral-type neural networks and the BAM neural network has been widely admitted interest from several researchers, see [1, 7, 13, 16, 21].

Moreover, the inertial neural network has been first introduced by Babcock and Westervelt [2]. This network is described by second-order differential equations. It should be pointed out that inertial neural networks are much more difficult to analyze their dynamical behaviors. Thus, the problem of stability analysis of the inertial neural networks has been attention, see [3, 5, 8, 11, 12, 22, 27]. In [10], the sufficient conditions are derived to ensure the global exponential stability of delayed inertial neural networks by constructing suitable Lyapunov Krasovskii functionals with the derivative of double integral terms. Manivannan R et al. [13] proposed stability analysis of interval time-varying delayed neural networks including neutral time-delay and leakage delay. By proposing a suitable Lyapunov Krasovskii functionals with the auxiliary function-based integral inequality and reciprocally convex approach, several sufficient conditions of interval time-varying delayed neural networks including neutral time-delay and leakage delay is obtained.

Motivated by the above discussions, in this letter, we investigate the problem of neutral-type inertial BAM neural networks with time-varying delays. Firstly, a variable transformation is given to simplify the stability analysis of neutral-type inertial BAM neural networks. Finally, the global exponential stability for the neutral-type inertial BAM neural networks with time-varying delays is ensured by using suitable Lyapunov Krasovskii functionals candidates, free weighting matrices, and convex combination approach within the lower and upper bounds of the activation functions are allowed to be either positive, negative or zero and without derivative time-varying delays.

2. Model description and preliminaries

In this paper, we consider the following neutral type BAM inertial Neural Networks with time varying delays

$$\begin{array}{l} \ddot{\tilde{u}}_i(t) = -\alpha_i\dot{\tilde{u}}_i(t) - a_i^{(1)}\tilde{u}_i(t) + \sum_{j=1}^m b_{ij}^{(1)}\tilde{f}_j(\tilde{v}_j(t)) + \sum_{j=1}^m c_{ij}^{(1)}\tilde{f}_j(\tilde{v}_j(t-\tau(t))) + \sum_{j=1}^m d_{ij}^{(1)}\dot{\tilde{u}}_j(t-\eta(t))) + I_i \\ \ddot{\tilde{v}}_j(t) = -\beta_j\dot{\tilde{v}}_j(t) - a_j^{(2)}\tilde{v}_j(t) + \sum_{i=1}^n b_{ji}^{(2)}\tilde{g}_i(\tilde{u}_i(t)) + \sum_{i=1}^n c_{ji}^{(2)}\tilde{g}_i(\tilde{u}_i(t-\sigma(t))) + \sum_{i=1}^n d_{ji}^{(2)}\dot{\tilde{v}}_i(t-\rho(t))) + J_j \end{array} \tag{1}$$

for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$, where second derivative is called an inertial term of system (1), $\tilde{u}$ and $\tilde{v}$ denote the states variable of the i^{th} neuron and the j^{th} neuron at the time t, $\alpha_i > 0$, $\beta_j > 0$, $a_i^{(1)}$ and $a_j^{(2)}$ are constants, $b_{ij}^{(1)}, c_{ij}^{(1)}, d_{ij}^{(1)}, b_{ji}^{(2)}, c_{ji}^{(2)}$ and $d_{ji}^{(2)}$ denote the connection weights at the time t, I_i and J_j are the external inputs, $\tilde{f}_j(\cdot)$ and $\tilde{g}_i(\cdot)$ denotes the activation function of the i^{th} neuron and the j^{th} neuron at the time t, $\tau(t), \sigma(t)$ are the time-varying delays, and $\eta(t), \rho(t)$ are the neutral-type time-varying delays. Let variable transformation:

$$\begin{array}{l} \tilde{x}_i(t) = \dot{\tilde{u}}_i(t) + \epsilon_i\tilde{u}_i(t) \\ \tilde{y}_j(t) = \dot{\tilde{v}}_j(t) + \bar{\epsilon}_j\tilde{v}_j(t), \end{array}$$

then system (1) can be rewritten as

$$\begin{array}{l} \dot{\tilde{u}}(t) = -\Xi_1\tilde{u}(t) + \tilde{x}(t), \\ \dot{\tilde{x}}(t) = -\Lambda_1\tilde{u}(t) - A_1\tilde{x}(t) + B_1\tilde{f}(\tilde{v}(t)) + C_1\tilde{f}(\tilde{v}(t-\tau(t))) + D_1\dot{\tilde{u}}(t-\eta(t)) + I, \\ \dot{\tilde{v}}(t) = -\Xi_2\tilde{v}(t) + \tilde{y}(t), \\ \dot{\tilde{y}}(t) = -\Lambda_2\tilde{v}(t) - A_2\tilde{y}(t) + B_2\tilde{g}(\tilde{u}(t)) + C_2\tilde{g}(\tilde{u}(t-\sigma(t))) + D_2\dot{\tilde{v}}(t-\rho(t)) + J \end{array} \tag{2}$$

where $A_1 = \text{diag}(\alpha_1-\epsilon_1, \ldots, \alpha_n-\epsilon_n), A_2 = \text{diag}(\beta_1-\bar{\epsilon}_1, \ldots, \beta_m-\bar{\epsilon}_m), B_1 = (b_{ij}^{(1)})_{n\times m}, B_2 = (b_{ji}^{(2)})_{m\times n}, C_1 = (c_{ij}^{(1)})_{n\times m}, C_2 = (c_{ji}^{(2)})_{m\times n}, D_1 = (d_{ij}^{(1)})_{n\times m}, D_2 = (d_{ji}^{(2)})_{m\times n}, I = (I_1, \ldots, I_n)^T, J = (J_1, \ldots, J_m)^T$

Definition 2.1 *The point* (u^*, v^*, x^*, y^*) *with* $u^* = (u_1^*, \ldots, u_n^*)^T$, $v^* = (v_1^*, \ldots, v_m^*)^T$, $x^* = (x_1^*, \ldots, x_n^*)^T$, $y^* = (y_1^*, \ldots, y_m^*)^T$ *is called an equilibrium point of system* (2) *if*

$$\begin{array}{lcl} -\Xi_1u^* + x^* &=& 0, \\ -\Lambda_1u^* - A_1x^* + B_1f(v^*) + C_1f(v^*) + I &=& 0, \\ -\Xi_2v^* + y^* &=& 0, \\ -\Lambda_2v^* - A_2y^* + B_2g(u^*) + C_2g(u^*) + J &=& 0. \end{array} \tag{3}$$

Let (u^*, v^*, x^*, y^*) be the equilibrium point of system (2). For the purpose of simplicity, we can shift the intended equilibrium (u^*, v^*, x^*, y^*) to be the origin by taken the following transformation:

$u(\cdot) = \tilde{u}(\cdot) - u^*,\ v(\cdot) = \tilde{v}(\cdot) - v^*,\ x(\cdot) = \tilde{x}(\cdot) - x^*,\ y(\cdot) = \tilde{y}(\cdot) - y^*,\ g(u(\cdot)) = \tilde{g}(u(\cdot) + u^*) - \tilde{g}(u^*),\ f(v(\cdot)) = \tilde{f}(v(\cdot) + v^*) - \tilde{f}(v^*)$. Then the system (2) can be transformed to

$$\begin{aligned}
\dot{u}(t) &= -\Xi_1 u(t) + x(t),\\
\dot{x}(t) &= -\Lambda_1 u(t) - A_1 x(t) + B_1 f(v(t)) + C_1 f(v(t-\tau(t))) + D_1 \dot{u}(t-\eta(t)),\\
\dot{v}(t) &= -\Xi_2 v(t) + y(t),\\
\dot{y}(t) &= -\Lambda_2 v(t) - A_2 y(t) + B_2 g(u(t)) + C_2 g(u(t-\sigma(t))) + D_2 \dot{v}(t-\rho(t)).
\end{aligned} \tag{4}$$

We have the following assumptions.

Assumption 2.2 *The delay $\sigma(t), \tau(t), \eta(t)$ and $\rho(t)$ are bounded time-varying delays satisfying:*

$$\begin{aligned}
&(i) && 0 \le \sigma_1 \le \sigma(t) \le \sigma_2, \quad 0 \le \tau_1 \le \tau(t) \le \tau_2\\
&(ii) && 0 \le \eta(t) \le \eta, \quad \dot{\eta}(t) \le \eta_d,\\
&(iii) && 0 \le \rho(t) \le \rho, \quad \dot{\rho}(t) \le \rho_d, \quad \forall t \ge 0,
\end{aligned}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2, \eta, \rho, \eta_d$ and ρ_d are real constants.

Assumption 2.3 *The neuron activation functions $f_j(\cdot)$ and $g_i(\cdot)$ are continuous, bounded which satisfy the following conditions: $f_j(0) = 0,\quad g_i(0) = 0$ when $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ and for any $\beta, \eta \in \mathbb{R}$ and $\beta \ne \eta$, there exist real numbers F_j^-, F_j^+, G_i^- and G_i^+ such that*

$$\begin{aligned}
&(i) && F_j^- \le \frac{f_j(\beta) - f_j(\eta)}{\beta - \eta} \le F_j^+, && j = 1, 2, \ldots, m,\\
&(ii) && G_i^- \le \frac{g_i(\beta) - g_i(\eta)}{\beta - \eta} \le G_i^+, && i = 1, 2, \ldots, n.
\end{aligned}$$

Denote

$$\begin{aligned}
F_1 &= diag(F_1^-, F_2^-, \ldots, F_m^-),\ F_2 = diag(F_1^+, F_2^+, \ldots, F_m^+),\\
G_1 &= diag(G_1^-, G_2^-, \ldots, G_n^-),\ G_2 = diag(G_1^+, G_2^+, \ldots, G_n^+).
\end{aligned}$$

Moreover, we assume that the initial conditions of system (4) has the form

$$\begin{aligned}
&u_i(s) = \phi_i^{(1)}(s),\ \dot{u}_i(s) = \psi_i^{(1)}(s),\ x_i(s) = \epsilon_i \phi_i^{(1)}(s) + \psi_i^{(1)}(s) \cong \varphi_i^{(1)}(s),\\
&v_j(s) = \phi_j^{(2)}(s),\ \dot{v}_j(s) = \psi_j^{(2)}(s),\ y_j(s) = \bar{\epsilon}_j \phi_j^{(2)}(s) + \psi_j^{(2)}(s) \cong \varphi_j^{(2)}(s),\ s \in [-\gamma, 0],\ \gamma = \max\{\sigma_2, \tau_2\}.
\end{aligned} \tag{5}$$

Definition 2.4 *The trivial solution of system* (4) *is said to be globally exponentially stable if there exist constants $k > 0$ and $\rho \ge 1$ such that*

$$\|u(t)\|^2 + \|x(t)\|^2 + \|v(t)\|^2 + \|y(t)\|^2 \le \rho e^{-2kt}(\|\phi^{(1)}\|^2 + \|\varphi^{(1)}\|^2 + \|\phi^{(2)}\|^2 + \|\varphi^{(2)}\|^2),\ \forall t \ge 0,$$

where one denotes

$$\begin{aligned}
\|\phi^{(1)}\|^2 + \|\varphi^{(1)}\|^2 + \|\phi^{(2)}\|^2 + \|\varphi^{(2)}\|^2 \quad = \quad & \sup_{-\max\{\sigma_2,\tau_2\} \le s \le 0} \|\phi^{(1)}(s)\|^2 + \sup_{-\max\{\sigma_2,\tau_2\} \le s \le 0} \|\varphi^{(1)}(s)\|^2\\
&+ \sup_{-\max\{\sigma_2,\tau_2\} \le s \le 0} \|\phi^{(2)}(s)\|^2 + \sup_{-\max\{\sigma_2,\tau_2\} \le s \le 0} \|\varphi^{(2)}(s)\|^2.
\end{aligned}$$

Lemma 2.5 *[23] For any constant matrix $\mathcal{R} \in \mathbb{R}^{n \times n}$, $\mathcal{R} = \mathcal{R}^T > 0$, a scalar function $\tau(t)$ with $0 < \tau(t) \le \tau_M$ and vector function $\dot{x} : [-\tau_M, 0] \to \mathbb{R}^n$ such that the integration concerned is well defined, let*

$$\int_{t-\tau(t)}^{t} \dot{x}(s) ds = \mathcal{E}\varphi(t),$$

where $\mathcal{E} \in \mathbb{R}^{n\times k}$ *and* $\varphi(t) \in \mathbb{R}^k$*. Then the following inequality holds for matrix* $\mathcal{M} \in \mathbb{R}^{n\times k}$

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s)\mathcal{R}\dot{x}(s)ds \le \varphi^T(t)\Upsilon_1\varphi(t),$$

where $\Upsilon_1 := -\mathcal{E}^T\mathcal{M} - \mathcal{M}^T\mathcal{E} + \tau(t)\mathcal{M}^T\mathcal{R}^{-1}\mathcal{M}$.

Lemma 2.6 *[13] For a given matrix* $\mathcal{G} \in \mathbb{R}^{n\times n}$*, scalar function* $\rho_M > \rho_m > 0$*, vector function* $r : [\rho_m, \rho_M] \to \mathbb{R}^n$ *such that the subsequent relation holds:*

$$-(\rho_M - \rho_m)\int_{t-\rho_M}^{t-\rho_m} r^T(\alpha)\mathcal{G}r(\alpha)d\alpha \quad \le \quad -\left(\int_{t-\rho_M}^{t-\rho_m} r(\alpha)d\alpha\right)^T \mathcal{G}\left(\int_{t-\rho_M}^{t-\rho_m} r(\alpha)d\alpha\right).$$

Lemma 2.7 *[13] For a given matrix* $\mathcal{G} > 0$ *and differentiable function* $\{r(\alpha) \mid \alpha \in [a,b]$*, the subsequent relations hold:*

$$\begin{aligned}
\int_a^b \dot{r}^T(\alpha)\mathcal{G}\dot{r}(\alpha)d\alpha \quad &\ge \quad \frac{1}{b-a}\ell_{11}^T\mathcal{G}\ell_{11} + \frac{3}{b-a}\ell_{12}^T\mathcal{G}\ell_{12} + \frac{5}{b-a}\ell_{13}^T\mathcal{G}\ell_{13},\\
\int_a^b\int_a^\beta \dot{r}^T(\alpha)\mathcal{G}\dot{r}(\alpha)d\alpha d\beta \quad &\ge \quad 2\ell_{14}^T\mathcal{G}\ell_{14} + 4\ell_{15}^T\mathcal{G}\ell_{15},
\end{aligned}$$

where

$$\begin{aligned}
\ell_{11} &= r(b) - r(a),\\
\ell_{12} &= r(b) + r(a) - \tfrac{2}{(b-a)}\int_a^b r(\alpha)d\alpha,\\
\ell_{13} &= r(b) - r(a) + \tfrac{6}{b-a}\int_a^b r(\alpha)d\alpha - \tfrac{12}{(b-a)^2}\int_a^b\int_\beta^b r(\alpha)d\alpha d\beta,\\
\ell_{14} &= r(b) - \tfrac{1}{b-a}\int_a^b r(\alpha)d\alpha,\\
\ell_{15} &= r(b) + \tfrac{2}{b-a}\int_a^b r(\alpha)d\alpha - \tfrac{6}{(b-a)^2}\int_a^b\int_\beta^b r(\alpha)d\alpha d\beta.
\end{aligned}$$

3. Main result

For this point on, for simplicity of notations, we assume that $t_0 = 0$. For the sake of simplicity on matrices representation, we let

$$\begin{aligned}
\xi(t) \quad = \quad & \mathrm{col}\{u(t),\ u(t-\sigma_1),\ u(t-\sigma_2),\ u(t-\sigma(t)),\ \dot{u}(t),\ x(t),\ \dot{x}(t),\ g(u(t)),\ g(u(t-\sigma_1)),\\
& g(u(t-\sigma_2)),\ g(u(t-\sigma(t))),\ \frac{1}{\sigma_2}\int_{t-\sigma_2}^t u(s)ds,\ \int_{t-\sigma(t)}^{t-\sigma_1} u(s)ds,\ \int_{t-\sigma_2}^{t-\sigma(t)} u(s)ds,\\
& \frac{1}{\sigma_2^2}\int_{-\sigma_2}^0\int_{t+s}^t u(\theta)d\theta ds,\ \dot{u}(t-\eta(t)),\ v(t),\ v(t-\tau_1),\ v(t-\tau_2),\ \dot{v}(t),\ y(t),\ \dot{y}(t),\\
& f(v(t)),\ f(v(t-\tau_1)),\ f(v(t-\tau_2)),\ f(v(t-\tau(t))),\ \frac{1}{\tau_2}\int_{t-\tau_2}^t v(s)ds,\ \int_{t-\tau(t)}^{t-\tau_1} v(s)ds,\\
& \int_{t-\tau_2}^{t-\tau(t)} v(s)ds,\ \frac{1}{\tau_2^2}\int_{-\tau_2}^0\int_{t+s}^t v(\theta)d\theta ds,\ \dot{v}(t-\rho(t))\}
\end{aligned}$$

and let $e_i \in \mathbb{R}^{n\times 32n}$ $(i = 1, 2, \ldots, 32)$ be defined as blocks entry matrices, for example $e_2 = [0_n \quad I_n \quad \underbrace{0_n \quad \cdots \quad 0_n}_{30}]$ where I_n and 0_n are identity and zero matrices of dimension $n \times n$, respectively.

Then, $u(t) = e_1\xi(t)$, $u(t-\sigma_1) = e_2\xi(t), \ldots,$ and $\dot{v}(t-\rho(t)) = e_{32}\xi(t)$, respectively. In addition, the notations of several matrices are defined as follows:

$$\begin{aligned}
\mathcal{L}_1 &= \mathrm{col}\{e_1, e_8\},\ \mathcal{L}_2 = \mathrm{col}\{e_2, e_9\},\ \mathcal{L}_3 = \mathrm{col}\{e_3, e_{10}\},\ \mathcal{L}_4 = \mathrm{col}\{e_{17}, e_{24}\},\\
\mathcal{L}_5 &= \mathrm{col}\{e_{18}, e_{25}\},\ \mathcal{L}_6 = \mathrm{col}\{e_{19}, e_{26}\},\ \mathcal{L}_7 = \mathrm{col}\{e_1, e_5\},\ \mathcal{L}_8 = \mathrm{col}\{e_{17}, e_{21}\},\\
\sigma_s &= \sigma_2 - \sigma_1,\ \tau_s = \tau_2 - \tau_1.
\end{aligned}$$

Theorem 3.1 *For given positive constants $\sigma_1,\ \sigma_2, \tau_1,\ \tau_2,\ \eta, \rho, \eta_d$, and ρ_d, the system (4) is globally exponentially stable with convergent rate k which satisfying the Assumptions 1.1 and Assumptions 1.2 , if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1,\dots,4)$, $S_i \in \mathbb{R}^{n\times n}$ $(i = 1,\dots,6)$, $Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i3} & Q_{i4} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ $(i = 1,\dots,4)$ where $Q_{ij} \in \mathbb{R}^{n\times n}(j = 1,\dots,4)$, $\mathcal{U} = \begin{bmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ * & \mathcal{U}_{22} \end{bmatrix}$, $\mathcal{V} = \begin{bmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} \\ * & \mathcal{V}_{22} \end{bmatrix}$, $\mathcal{R} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ * & \mathcal{R}_{22} \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ * & \mathcal{Z}_{22} \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 & 0 \\ * & \mu_2 \end{bmatrix}$, $\nu = \begin{bmatrix} \nu_1 & 0 \\ * & \nu_2 \end{bmatrix}$, positive diagonal matrices $\Delta_i = \mathrm{diag}(\lambda_1^i, \lambda_2^i,\ \dots, \lambda_n^i)$ $(i = 1,2), \Delta_j = \mathrm{diag}(\lambda_1^j, \lambda_2^j,\ \dots, \lambda_m^j)$ $(j = 3,4)$, $N_i = \mathrm{diag}(n_1^i, n_2^i, \dots, n_n^i)$ $(i = 1,\dots,4)$, $M_i = \mathrm{diag}(m_1^i, m_2^i, \dots, m_m^i)$ $(i = 1,\dots,4)$, and any matrices $Y_i (i = 1,\dots,4)$, H_i $(i = 1,\dots,8)$ with appropriate dimensions which satisfy the following LMIs:*

$$\begin{bmatrix} e^{-2k\sigma_s}\mathcal{U}_{11} & e^{-2k\sigma_s}\mathcal{U}_{12} & e^{-2k\sigma_s}\mu_1 & 0 \\ * & e^{-2k\sigma_s}\mathcal{U}_{22} & 0 & e^{-2k\sigma_s}\mu_2 \\ * & * & e^{-2k\sigma_s}\mathcal{U}_{11} & e^{-2k\sigma_s}\mathcal{U}_{12} \\ * & * & * & e^{-2k\sigma_s}\mathcal{U}_{22} \end{bmatrix} \geq 0, \tag{6}$$

$$\begin{bmatrix} e^{-2k\tau_s}\mathcal{Z}_{11} & e^{-2k\tau_s}\mathcal{Z}_{12} & e^{-2k\tau_s}\nu_1 & 0 \\ * & e^{-2k\tau_s}\mathcal{Z}_{22} & 0 & e^{-2k\tau_s}\nu_2 \\ * & * & e^{-2k\tau_s}\mathcal{Z}_{11} & e^{-2k\tau_s}\mathcal{Z}_{12} \\ * & * & * & e^{-2k\tau_s}\mathcal{Z}_{22} \end{bmatrix} \geq 0, \tag{7}$$

$$\begin{bmatrix} \Pi & e^{-2k\sigma_s}Y_2^T & e^{-2k\tau_s}Y_4^T \\ * & -\sigma_s e^{-2k\sigma_s}S_4 & 0 \\ * & * & -\tau_s e^{-2k\tau_s}S_6 \end{bmatrix} < 0, \tag{8}$$

$$\begin{bmatrix} \Pi & e^{-2k\sigma_s}Y_2^T & e^{-2k\tau_s}Y_3^T \\ * & -\sigma_s e^{-2k\sigma_s}S_4 & 0 \\ * & * & -\tau_s e^{-2k\tau_s}S_6 \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} \Pi & e^{-2k\sigma_s}Y_1^T & e^{-2k\tau_s}Y_4^T \\ * & -\sigma_s e^{-2k\sigma_s}S_4 & 0 \\ * & * & -\tau_s e^{-2k\tau_s}S_6 \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \Pi & e^{-2k\sigma_s}Y_1^T & e^{-2k\tau_s}Y_3^T \\ * & -\sigma_s e^{-2k\sigma_s}S_4 & 0 \\ * & * & -\tau_s e^{-2k\tau_s}S_6 \end{bmatrix} < 0, \tag{11}$$

where

$$\Pi = \sum_{l=1}^{8} \Pi_l,$$

$$\begin{aligned} \Pi_1 &= 2ke_1^T P_1 e_1 + 2e_1^T P_1 e_5 + 2ke_6^T P_2 e_6 + 2e_6^T P_2 e_7 + 2ke_{17}^T P_3 e_{17} + 2e_{17}^T P_3 e_{21} \\ &\quad +2ke_{22}^T P_4 e_{22} + 2e_{22}^T P_4 e_{23}, \\ \Pi_2 &= 4k(e_8 - G_1 e_1)^T \Delta_1 e_1 + 2(e_8 - G_1 e_1)^T \Delta_1 e_5 + 4k(G_2 e_1 - e_8)^T \Delta_2 e_1 \\ &\quad +2(G_2 e_1 - e_8)^T \Delta_2 e_5 + 4k(e_{24} - F_1 e_{17})^T \Delta_3 e_{17} + 2(e_{24} - F_1 e_{17})^T \Delta_3 e_{21} \\ &\quad +4k(F_2 e_{17} - e_{24})^T \Delta_4 e_{17} + 2(F_2 e_{17} - e_{24})^T \Delta_4 e_{21}, \end{aligned}$$

$$
\begin{aligned}
\Pi_3 &= e_5^T S_1 e_5 - (1-\eta_d)e^{-2k\eta} e_{16}^T S_1 e_{16} + e_{21}^T S_2 e_{21} - (1-\rho_d)e^{-2k\rho} e_{32}^T S_2 e_{32},\\
\Pi_4 &= \mathcal{L}_1^T\left[Q_1+Q_2\right]\mathcal{L}_1 + \mathcal{L}_4^T\left[Q_3+Q_4\right]\mathcal{L}_4 - e^{-2k\sigma_1}\mathcal{L}_2^T Q_1\mathcal{L}_2 - e^{-2k\sigma_2}\mathcal{L}_3^T Q_2\mathcal{L}_3\\
&\quad -e^{-2k\tau_1}\mathcal{L}_5^T Q_3\mathcal{L}_5 - e^{-2k\tau_2}\mathcal{L}_6^T Q_4\mathcal{L}_6,\\
\Pi_5 &= \mathcal{L}_7^T(\sigma_2^T\mathcal{R}+\sigma_s^2\mathcal{U})\mathcal{L}_7 + \mathcal{L}_8^T(\tau_2^2\mathcal{V}+\tau_s^2\mathcal{Z})\mathcal{L}_8 - e^{-2k\sigma_2}\begin{bmatrix}\sigma_2 e_{12}\\ e_1-e_3\end{bmatrix}^T\begin{bmatrix}\mathcal{R}_{11} & \mathcal{R}_{12}\\ * & \mathcal{R}_{22}\end{bmatrix}\begin{bmatrix}\sigma_2 e_{12}\\ e_1-e_3\end{bmatrix}\\
&\quad -e^{-2k\tau_2}\begin{bmatrix}\tau_2 e_{28}\\ e_{17}-e_{19}\end{bmatrix}^T\begin{bmatrix}\mathcal{V}_{11} & \mathcal{V}_{12}\\ * & \mathcal{V}_{22}\end{bmatrix}\begin{bmatrix}\tau_2 e_{28}\\ e_{17}-e_{19}\end{bmatrix} - e^{-2k\sigma_s}\begin{bmatrix}e_{13}\\ e_2-e_4\\ e_{14}\\ e_4-e_3\end{bmatrix}^T\begin{bmatrix}\mathcal{U} & \mu\\ * & \mathcal{U}\end{bmatrix}\begin{bmatrix}e_{13}\\ e_2-e_4\\ e_{14}\\ e_4-e_3\end{bmatrix}\\
&\quad -e^{-2k\tau_s}\begin{bmatrix}e_{29}\\ e_{18}-e_{20}\\ e_{30}\\ e_{20}-e_{19}\end{bmatrix}^T\begin{bmatrix}\mathcal{Z} & \nu\\ * & \mathcal{Z}\end{bmatrix}\begin{bmatrix}e_{29}\\ e_{18}-e_{20}\\ e_{30}\\ e_{20}-e_{19}\end{bmatrix},\\
\Pi_6 &= e_5^T(\sigma_2 e^{-2k\sigma_2}S_3 + \sigma_s e^{-2k\sigma_s}S_4)e_5 + e_{21}^T(\tau_2 e^{-2k\tau_2}S_5 + \tau_s e^{-2k\tau_s}S_6)e_{21}\\
&\quad -\tfrac{1}{\sigma_2}e^{-2k\sigma_2}(e_1-e_3)^T S_3(e_1-e_3) - \tfrac{3}{\sigma_2}e^{-2k\sigma_2}(e_1+e_3-2e_{12})^T S_3(e_1+e_3-2e_{12})\\
&\quad -\tfrac{5}{\sigma_2}e^{-2k\sigma_2}(e_1-e_3+6e_{12}-12e_{15})^T S_3(e_1-e_3+6e_{12}-12e_{15})\\
&\quad -e^{-2k\tau_2}\{\tfrac{1}{\tau_2}(e_{17}-e_{19})^T S_5(e_{17}-e_{19}) + \tfrac{3}{\tau_2}(e_{17}+e_{19}-2e_{28})^T S_5(e_{17}+e_{19}-2e_{28})\\
&\quad +\tfrac{5}{\tau_2}(e_{17}-e_{19}+6e_{28}-12e_{31})^T S_5(e_{17}-e_{19}+6e_{28}-12e_{31})\}\\
&\quad +e^{-2k\sigma_s}[(e_4-e_2)^T Y_1 + Y_1^T(e_4-e_2) + (e_3-e_4)^T Y_2 + Y_2^T(e_3-e_4)]\\
&\quad +e^{-2k\tau_s}[(e_{20}-e_{18})^T Y_3 + Y_3^T(e_{20}-e_{18}) + (e_{19}-e_{20})^T Y_4 + Y_4^T(e_{19}-e_{20})],\\
\Pi_7 &= (2e_1^T H_1 + 2e_5^T H_2)(-\Xi_1 e_1 + e_6 - e_5) + (2e_6^T H_3 + 2e_7^T H_4)(-\Lambda_1 e_1 - A_1 e_6\\
&\quad +B_1 e_{24} + C_1 e_{27} + D_1 e_{16} - e_7) + (2e_{17}^T H_5 + 2e_{21}^T H_6)(-\Xi_2 e_{17} + e_{22} - e_{21})\\
&\quad +(2e_{17}^T H_7 + 2e_{21}^T H_8)(-\Lambda_2 e_{17} - A_2 e_{22} + B_2 e_8 + C_2 e_{11} + D_2 e_{32} - e_{23}),\\
\Pi_8 &= -2(e_8 - G_2 e_1)^T N_1(e_8 - G_1 e_1) - 2(e_{11} - G_2 e_4)^T N_2(e_{11} - G_1 e_4) - 2(e_8 - e_{11} - G_2\\
&\quad \times(e_1 - e_4))^T N_3(e_8 - e_{11} - G_1(e_1 - e_8)) - 2(e_{11} - e_{10} - G_2(e_4 - e_3))^T N_4(e_{11} - e_{10}\\
&\quad -G_1(e_4 - e_3)) - 2(e_{24} - F_2 e_{17})^T M_1(e_{24} - F_1 e_{17}) - 2(e_{27} - F_2 e_{20})^T M_2(e_{27} - F_1 e_{20})\\
&\quad -2(e_{24} - e_{27} - F_2(e_{17} - e_{20}))^T M_3(e_{24} - e_{27} - F_1(e_{17} - e_{20}))\\
&\quad -2(e_{27} - e_{26} - F_2(e_{20} - e_{19}))^T M_4(e_{27} - e_{26} - F_1(e_{20} - e_{19})).
\end{aligned}
$$

Proof Choose the Lyapunov-Krasovskii functional candidate for the system as follow

$$
V(t) = \sum_{l=1}^{6} V_l(t), \tag{12}
$$

where

$$
\begin{aligned}
V_1 &= u^T(t)P_1u(t) + x^T(t)P_2x(t) + v^T(t)P_3v(t) + y^T(t)P_4y(t),\\
V_2 &= 2\sum_{i=1}^{n}\Big[\lambda_{1i}e^{2kt}\int_0^{u_i(t)}(g_i(s) - G_i^- s)ds\Big] + 2\sum_{i=1}^{n}\Big[\lambda_{2i}e^{2kt}\int_0^{u_i(t)}(G_i^+ s - g_i(s))ds\Big]\\
&\quad +2\sum_{j=1}^{n}\Big[\lambda_{3j}e^{2kt}\int_0^{v_j(t)}(f_j(s) - F_i^- s)ds\Big] + 2\sum_{j=1}^{m}\Big[\lambda_{4j}e^{2kt}\int_0^{v_j(t)}(F_j^+ s - f_j(s))ds\Big],\\
V_3 &= \int_{t-\eta(t)}^{t} e^{2ks}\dot{u}^T(s)S_1\dot{u}(s)ds + \int_{t-\rho(t)}^{t} e^{2ks}\dot{v}^T(s)S_2\dot{u}(s)ds,\\
V_4 &= \int_{t-\sigma_1}^{t} e^{2ks}\begin{bmatrix}u(s)\\ g(u(s))\end{bmatrix}^T Q_1\begin{bmatrix}u(s)\\ g(u(s))\end{bmatrix}ds + \int_{t-\sigma_2}^{t} e^{2ks}\begin{bmatrix}u(s)\\ g(u(s))\end{bmatrix}^T Q_2\begin{bmatrix}u(s)\\ g(u(s))\end{bmatrix}ds\\
&\quad +\int_{t-\tau_1}^{t} e^{2ks}\begin{bmatrix}v(s)\\ f(v(s))\end{bmatrix}^T Q_3\begin{bmatrix}v(s)\\ f(v(s))\end{bmatrix}ds + \int_{t-\tau_2}^{t} e^{2ks}\begin{bmatrix}v(s)\\ f(v(s))\end{bmatrix}^T Q_4\begin{bmatrix}v(s)\\ f(v(s))\end{bmatrix}ds,
\end{aligned}
$$

$$
\begin{aligned}
V_5 &= \sigma_2 \int_{-\sigma_2}^{0}\int_{t+s}^{t} e^{2k\theta}\varsigma^T(\theta)\mathcal{R}\varsigma(\theta)d\theta ds + \sigma_s \int_{-\sigma_2}^{-\sigma_1}\int_{t+s}^{t} e^{2k\theta}\varsigma^T(\theta)\mathcal{U}\varsigma(\theta)d\theta ds \\
&\quad +\tau_2 \int_{-\tau_1}^{0}\int_{t+s}^{t} e^{2k\theta}\bar{\varsigma}^T(\theta)\mathcal{V}\bar{\varsigma}(\theta)d\theta ds + \tau_s \int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t} e^{2k\theta}\bar{\varsigma}^T(\theta)\mathcal{Z}\bar{\varsigma}(\theta)d\theta ds, \\
V_6 &= \int_{-\sigma_2}^{0}\int_{t+s}^{t} e^{2k\theta}\dot{u}^T(\theta)S_3\dot{u}(\theta)d\theta ds + \int_{-\sigma_2}^{-\sigma_1}\int_{t+s}^{t} e^{2k\theta}\dot{u}^T(\theta)S_4\dot{u}(\theta)d\theta ds \\
&\quad + \int_{-\tau_2}^{0}\int_{t+s}^{t} e^{2k\theta}\dot{v}^T(\theta)S_5\dot{v}(\theta)d\theta ds + \int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t} e^{2k\theta}\dot{v}^T(\theta)S_6\dot{v}(\theta)d\theta ds.
\end{aligned}
$$

The derivative of $V(t)$ in (12) along the trajectories of the system (4) is given by

$$
\begin{aligned}
\dot{V}_1 &= e^{2kt}\{2ku^T(t)P_1u(t) + 2u^T(t)P_1\dot{u}(t) + 2kx^T(t)P_2x(t) + 2x^T(t)P_2\dot{x}(t) + 2kv^T(t)P_3v(t) \\
&\quad +2v^T(t)P_3\dot{v}(t) + 2ky^T(t)P_4y(t) + 2y^T(t)P_4\dot{y}(t)\} \\
&= \mathrm{e}^{2kt}xi^T(t)\{2ke_1^TP_1e_1 + 2e_1^TP_1e_5 + 2ke_6^TP_2e_6 + 2e_6^TP_2e_7 + 2ke_{17}^TP_3e_{17} + 2e_{17}^TP_3e_{21} \\
&\quad +2ke_{22}^TP_4e_{22} + 2e_{22}^TP_4e_{23}\}\xi(t), \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_2 &\le e^{2kt}\{4k(g(u(t)) - G_1u(t))^T\Delta_1u(t) + 2(g(u(t)) - G_1u(t))^T\Delta_1\dot{u}(t) + 4k(G_2u(t) - g(u(t)))^T\Delta_2u(t) \\
&\quad +2(G_2u(t) - g(u(t)))^T\Delta_2\dot{u}(t) + 4k(f(v(t)) - F_1v(t))^T\Delta_3v(t) + 2(f(v(t)) - F_1v(t))^T\Delta_3\dot{v}(t) \\
&\quad +4k(F_2v(t) - f(v(t)))^T \times \Delta_4v(t) + 2(F_2v(t) - f(v(t)))^T\Delta_4\dot{v}(t)\} \\
&\le e^{2kt}\xi^T(t)\{4k(e_8 - G_1e_1)^T\Delta_1e_1 + 2(e_8 - G_1e_1)^T\Delta_1e_5 + 4k(G_2e_1 - e_8)^T\Delta_2e_1 + 2(G_2e_1 - e_8)^T \\
&\quad \times\Delta_2e_5 + 4k(e_{24} - F_1e_{17})^T\Delta_3e_{17} + 2(e_{24} - F_1e_{17})^T\Delta_3e_{21} + 4k(F_2e_{17} - e_{24})^T\Delta_4e_{17} \\
&\quad +2(F_2e_{17} - e_{24})^T\Delta_4e_{21}\}\xi(t), \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3 &\le e^{2kt}\{\dot{u}^T(t)S_1\dot{u}(t) - (1-\eta_d)e^{-2k\eta}\dot{u}^T(t-\eta(t))S_1\dot{u}(t-\eta(t)) + \dot{v}^T(t)S_2\dot{v}(t) \\
&\quad -(1-\rho_d)\dot{v}^T(t-\rho(t))S_2\dot{v}(t-\rho(t))\} \\
&\le e^{2kt}\xi^T(t)\{e_5^TS_1e_5 - (1-\eta_d)e^{-2k\eta}e_{16}^TS_1e_{16} + e_{21}^TS_2e_{21} - (1-\rho_d)e^{-2k\rho}e_{32}^TS_2e_{32}\}\xi(t), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_4 &\le e^{2kt}\Bigg\{\begin{bmatrix} u(t) \\ g(u(t)) \end{bmatrix}^T Q_1 \begin{bmatrix} u(t) \\ g(u(t)) \end{bmatrix} - e^{-2k\sigma_1}\begin{bmatrix} u(t-\sigma_1) \\ g(u(t-\sigma_1)) \end{bmatrix}^T Q_1 \begin{bmatrix} u(t-\sigma_1) \\ g(u(t-\sigma_1)) \end{bmatrix} \\
&\quad + \begin{bmatrix} u(t) \\ g(u(t)) \end{bmatrix}^T Q_2 \begin{bmatrix} u(t) \\ g(u(t)) \end{bmatrix} - e^{-2k\sigma_2}\begin{bmatrix} u(t-\sigma_2) \\ g(u(t-\sigma_2)) \end{bmatrix}^T Q_2 \begin{bmatrix} u(t-\sigma_2) \\ g(u(t-\sigma_2)) \end{bmatrix} \\
&\quad + \begin{bmatrix} v(t) \\ f(v(t)) \end{bmatrix}^T Q_3 \begin{bmatrix} v(t) \\ f(v(t)) \end{bmatrix} - e^{-2k\tau_1}\begin{bmatrix} v(t-\tau_1) \\ f(v(t-\tau_1)) \end{bmatrix}^T Q_3 \begin{bmatrix} v(t-\tau_1) \\ f(v(t-\tau_1)) \end{bmatrix} \\
&\quad + \begin{bmatrix} v(t) \\ f(v(t)) \end{bmatrix}^T Q_4 \begin{bmatrix} v(t) \\ f(v(t)) \end{bmatrix} - e^{-2k\tau_2}\begin{bmatrix} v(t-\tau_2) \\ f(v(t-\tau_2)) \end{bmatrix}^T Q_4 \begin{bmatrix} v(t-\tau_2) \\ f(v(t-\tau_2)) \end{bmatrix}\Bigg\} \\
&\le e^{2kt}\xi^T(t)\{\mathcal{L}_1^T[Q_1 + Q_2]\mathcal{L}_1 + \mathcal{L}_4^T[Q_3 + Q_4]\mathcal{L}_4 - e^{-2k\sigma_1}\mathcal{L}_2^TQ_1\mathcal{L}_2 - e^{-2k\sigma_2}\mathcal{L}_3^TQ_2\mathcal{L}_3 \\
&\quad -e^{-2k\tau_1}\mathcal{L}_5^TQ_3\mathcal{L}_5 - e^{-2k\tau_2}\mathcal{L}_6^TQ_4\mathcal{L}_6\}\xi(t), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_5 &\le e^{2kt}\{\varsigma^T(t)(\sigma_2^2\mathcal{R} + \sigma_s^2\mathcal{U})\varsigma(t) + \bar{\varsigma}^T(t)(\tau_2^2\mathcal{V} + \tau_s^2\mathcal{Z})\bar{\varsigma}(t) - \sigma_2e^{-2k\sigma_2}\int_{t-\sigma_2}^{t}\varsigma^T(s)\mathcal{R}\varsigma(s)ds \\
&\quad -\tau_2e^{-2k\tau_2}\int_{t-\tau_2}^{t}\bar{\varsigma}^T(s)\mathcal{V}\bar{\varsigma}(s)ds - \sigma_se^{-2k\sigma_s}\int_{t-\sigma_2}^{t-\sigma_1}\varsigma^T(s)\mathcal{U}\varsigma(s)ds \\
&\quad -\tau_se^{-2k\tau_s}\int_{t-\tau_2}^{t-\tau_1}\bar{\varsigma}^T(s)\mathcal{Z}\bar{\varsigma}(s)ds\}. \qquad (17)
\end{aligned}
$$

By the use of Lemma 2.5 ,we have

$$\begin{aligned} -\sigma_2 e^{-2k\sigma_2}\int_{t-\sigma_2}^{t}\varsigma^T(s)\mathcal{R}\varsigma(s)ds &\le -e^{-2k\sigma_2}\begin{bmatrix} \int_{t-\sigma_2}^{t}u(s)ds \\ u(t)-u(t-\sigma_2)\end{bmatrix}^T\begin{bmatrix}\mathcal{R}_{11} & \mathcal{R}_{12}\\ * & \mathcal{R}_{22}\end{bmatrix} \\ &\quad \times\begin{bmatrix} \int_{t-\sigma_2}^{t}u(s)ds \\ u(t)-u(t-\sigma_2)\end{bmatrix}, \end{aligned} \tag{18}$$

$$\begin{aligned} -\tau_2 e^{-2k\tau_2}\int_{t-\tau_2}^{t}\bar{\varsigma}^T(s)\mathcal{V}\bar{\varsigma}(s)ds &\le -e^{-2k\tau_2}\begin{bmatrix} \int_{t-\tau_2}^{t}v(s)ds \\ v(t)-v(t-\tau_2)\end{bmatrix}^T\begin{bmatrix}\mathcal{V}_{11} & \mathcal{V}_{12}\\ * & \mathcal{V}_{22}\end{bmatrix} \\ &\quad \times\begin{bmatrix} \int_{t-\tau_2}^{t}v(s)ds \\ v(t)-v(t-\tau_2)\end{bmatrix}. \end{aligned} \tag{19}$$

Utilizing the same idea in [15], if (6) and (7) are holds, then some integral terms can obtain

$$\begin{aligned} -\sigma_s e^{-2k\sigma_s}\int_{t-\sigma_2}^{t-\sigma_1}\varsigma^T(s)\mathcal{U}\varsigma(s)ds &\le \sigma_s e^{-2k\sigma_s}\Big(-\int_{t-\sigma(t)}^{t-\sigma_1}\varsigma^T(s)\mathcal{U}\varsigma(s)ds-\int_{t-\sigma_2}^{t-\sigma(t)}\varsigma^T(s)\mathcal{U}\varsigma(s)ds\Big) \\ &\le e^{-2k\sigma_s}\begin{bmatrix} \int_{t-\sigma(t)}^{t-\sigma_1}\varsigma(s)ds \\ \int_{t-\sigma_2}^{t-\sigma(t)}\varsigma(s)ds\end{bmatrix}^T\begin{bmatrix}\mathcal{U} & \mu\\ * & \mathcal{U}\end{bmatrix}\begin{bmatrix} \int_{t-\sigma(t)}^{t-\sigma_1}\varsigma(s)ds \\ \int_{t-\sigma_2}^{t-\sigma(t)}\varsigma(s)ds\end{bmatrix}, \end{aligned} \tag{20}$$

$$\begin{aligned} -\tau_s e^{-2k\tau_s}\int_{t-\tau_2}^{t-\tau_1}\bar{\varsigma}^T(s)\mathcal{Z}\bar{\varsigma}(s)ds &\le \tau_s e^{-2k\tau_s}\Big(-\int_{t-\tau(t)}^{t-\tau_1}\bar{\varsigma}^T(s)\mathcal{Z}\bar{\varsigma}(s)ds-\int_{t-\tau_2}^{t-\tau(t)}\bar{\varsigma}^T(s)\mathcal{Z}\bar{\varsigma}(s)ds\Big) \\ &\le -e^{-2k\tau_s}\begin{bmatrix} \int_{t-\tau(t)}^{t-\tau_1}\bar{\varsigma}(s)ds \\ \int_{t-\tau_2}^{t-\tau(t)}\bar{\varsigma}(s)ds\end{bmatrix}^T\begin{bmatrix}\mathcal{Z} & \nu\\ * & \mathcal{Z}\end{bmatrix}\begin{bmatrix} \int_{t-\tau(t)}^{t-\tau_1}\bar{\varsigma}(s)ds \\ \int_{t-\tau_2}^{t-\tau(t)}\bar{\varsigma}(s)ds\end{bmatrix}, \end{aligned} \tag{21}$$

$$\begin{aligned} \dot{V}_6 \le\ & e^{2kt}\{\dot{u}^T(t)(\sigma_2 e^{-2k\sigma_2}S_3+\sigma_s e^{-2k\sigma_s}S_4)\dot{u}(t)+\dot{v}^T(t)(\tau_2 e^{-2k\tau_2}S_5+\tau_s e^{-2k\tau_s}S_6)\dot{v}(t) \\ & -e^{-2k\sigma_2}\int_{t-\sigma_2}^{t}\dot{u}^T(s)S_3\dot{u}(s)ds-e^{-2k\tau_2}\int_{t-\tau_2}^{t}\dot{v}^T(s)S_5\dot{v}(s)ds \\ & -e^{-2k\sigma_s}\int_{t-\sigma_2}^{t-\sigma_1}\dot{u}^T(s)S_4\dot{u}(s)ds-e^{-2k\tau_2}\int_{t-\tau_2}^{t-\tau_1}\dot{v}^T(s)S_6\dot{v}(s)ds\}. \end{aligned}$$

By the use of Lemma 2.5 and Lemma 2.7 ,we have

$$\begin{aligned} \dot{V}_6 \le\ & e^{2kt}\xi^T(t)\{e_5^T(\sigma_2 e^{-2k\sigma_2}S_3+\sigma_s e^{-2k\sigma_s}S_4)e_5)+\ e_{21}^T(\tau_2 e^{-2k\tau_2}S_5+\tau_s e^{-2k\tau_s}S_6)e_{21} \\ & +e^{-2k\sigma_2}\Big(-\frac{1}{\sigma_2}(e_1-e_3)^TS_3(e_1-e_3)-\frac{3}{\sigma_2}(e_1+e_3-2e_{12})^TS_3(e_1+e_3-2e_{12}) \\ & -\frac{5}{\sigma_2}(e_1-e_3+6e_{12}-12e_{15})^TS_3(e_1-e_3+6e_{12}-12e_{15})\Big)+e^{-2k\tau_2}\Big(-\frac{1}{\tau_2}(e_{17}-e_{19})^TS_5 \\ & \times(e_{17}-e_{19})-\frac{3}{\tau_2}(e_{17}+e_{19}-2e_{28})^TS_5(e_{17}+e_{19}-2e_{28})-\frac{5}{\tau_2}(e_{17}-e_{19}+6e_{28}-12e_{31})^T \\ & \times S_5(e_{17}-e_{19}+6e_{28}-12e_{31})\Big)+e^{-2k\sigma_s}[(e_4-e_2)^TY_1+Y_1^T(e_4-e_2) \\ & +(\sigma(t)-\sigma_1)Y_1^TS_4^{-1}Y_1+(e_3-e_4)^TY_2+Y_2^T(e_3-e_4)+(\sigma_2-\sigma(t))Y_2^TS_4^{-1}Y_2] \\ & +e^{-2k\tau_s}[(e_{20}-e_{18})^TY_3+Y_3^T(e_{20}-e_{18}+(\tau(t)-\tau_1)Y_3^TS_6^{-1}Y_3) \\ & +(e_{19}-e_{20})^TY_4+Y_4^T(e_{19}-e_{20})+(\tau_2-\tau(t))Y_4^TS_6^{-1}Y_4]\}\xi(t). \end{aligned} \tag{22}$$

In order to reduce the conservatism, we add the following zero equations: for any matrices $H_i (i = 1, 2, \ldots, 8)$ with appropriate dimension

$$\begin{array}{lcl}\left[2u^T(t)H_1 + 2\dot{u}^T(t)H_2\right][-\Xi_1 u(t) + x(t) - \dot{u}(t)] & = & 0,\\ \left[2x^T(t)H_3 + 2\dot{x}^T(t)H_4\right][-\Lambda_1 u(t) - A_1 x(t) + B_1 f(v(t)) + C_1 f(v(t-\tau(t))) & & \\ +D_1\dot{u}(t-\eta(t)) - \dot{x}(t)] & = & 0,\\ \left[2x^T(t)H_5 + 2\dot{x}^T(t)H_6\right][-\Xi_2 v(t) + y(t) - \dot{v}(t)] & = & 0,\\ \left[2x^T(t)H_7 + 2\dot{x}^T(t)H_8\right][-\Lambda_2 v(t) - A_2 y(t) + B_2 g(u(t)) + C_2 g(u(t-\sigma(t))) & & \\ +D_2\dot{v}(t-\rho(t)) - \dot{y}(t)] & = & 0.\end{array} \tag{23}$$

By Assumption 2.3, the following inequalities hold

$$\begin{array}{lcl}-2[g(u(t)) - G_2u(t)]^T N_1 [g(u(t)) - G_1u(t)] & \geq & 0,\\ -2[g(u(t-\sigma(t))) - G_2u(t-\sigma(t))]^T N_2 [g(u(t-\sigma(t))) - G_1u(t-\sigma(t))] & \geq & 0,\\ -2[g(u(t)) - g(u(t-\sigma(t))) - G_2(u(t) - u(t-\sigma(t)))]^T N_3 & & \\ \times [g(u(t)) - g(u(t-\sigma(t))) - G_1(u(t) - u(t-\sigma(t)))] & \geq & 0,\\ -2\,[g(u(t-\sigma(t))) - g(u(t-\sigma_2)) - G_2(u(t-\sigma(t)) - u(t-\sigma_2))]^T N_4 & & \\ [g(u(t-\sigma(t))) - g(u(t-\sigma_2)) - G_1(u(t-\sigma(t)) - u(t-\sigma_2))] & \geq & 0,\\ -2[f(v(t)) - F_2v(t)]^T M_1 [f(v(t)) - F_1v(t)] & \geq & 0,\\ -2[f(v(t-\tau(t))) - F_2v(t-\tau(t))]^T M_2 [f(v(t-\tau(t))) - F_1v(t-\tau(t))] & \geq & 0,\\ -2[f(v(t)) - f(v(t-\tau(t))) - F_2(v(t) - v(t-\tau(t)))]^T M_3 & & \\ \times [f(v(t)) - f(v(t-\tau(t))) - F_1(v(t) - v(t-\tau(t)))] & \geq & 0,\\ -2\,[f(v(t-\tau(t))) - f(v(t-\tau_2)) - F_2(v(t-\tau(t)) - v(t-\tau_2))]^T M_4 & & \\ [f(v(t-\tau(t))) - f(v(t-\tau_2)) - F_1(v(t-\tau(t)) - v(t-\tau_2))] & \geq & 0.\end{array} \tag{24}$$

Using inequalities (13) – (25), we obtain

$$\dot{V}(t) \leq e^{2kt}\xi(t)^T \Pi_0 \xi(t),$$

where

$$\begin{array}{rcl}\Pi_0 & = & \Pi + (\sigma_2 - \sigma(t))e^{-2k\sigma_s}Y_2^T S_4^{-1} Y_2 + (\sigma(t) - \sigma_1)e^{-2k\sigma_s}Y_1^T S_4^{-1} Y_1\\ & & +(\tau_2 - \tau(t))e^{-2k\tau_s}Y_4^T S_6^{-1} Y_4 + (\tau(t) - \tau_1)e^{-2k\tau_s}Y_3^T S_6^{-1} Y_3.\end{array}$$

Since $(\sigma_2 - \sigma(t))e^{-2k\sigma_s}Y_2^T S_4^{-1} Y_2 + (\sigma(t) - \sigma_1)e^{-2k\sigma_s}Y_1^T S_4^{-1} Y_1 + (\tau_2 - \tau(t))e^{-2k\tau_s}Y_4^T S_6^{-1} Y_4 + (\tau(t) - \tau_1)e^{-2k\tau_s}Y_3^T S_6^{-1} Y_3$ is a convex combination of matrices $Y_2^T S_4^{-1} Y_2, Y_1^T S_4^{-1} Y_1, Y_4^T S_6^{-1} Y_4$ and $Y_3^T S_6^{-1} Y_3$ of $\sigma(t)$ and $\tau(t)$, respectively, Π_0 can be handled by four corresponding boundary LMIs:

$$\Pi + (\sigma_2 - \sigma_1)e^{-2k\sigma_s}Y_2^T S_4^{-1} Y_2 + (\tau_2 - \tau_1)e^{-2k\tau_s}Y_4^T S_6^{-1} Y_4 < 0 \tag{25}$$

$$\Pi + (\sigma_2 - \sigma_1)e^{-2k\sigma_s}Y_2^T S_4^{-1} Y_2 + (\tau_2 - \tau_1)e^{-2k\tau_s}Y_3^T S_6^{-1} Y_3 < 0 \tag{26}$$

$$\Pi + (\sigma_2 - \sigma_1)e^{-2k\sigma_s}Y_1^T S_4^{-1} Y_1 + (\tau_2 - \tau_1)e^{-2k\tau_s}Y_4^T S_6^{-1} Y_4 < 0 \tag{27}$$

$$\Pi + (\sigma_2 - \sigma_1)e^{-2k\sigma_s}Y_1^T S_4^{-1} Y_1 + (\tau_2 - \tau_1)e^{-2k\tau_s}Y_3^T S_6^{-1} Y_3 < 0. \tag{28}$$

Using Schur Complement, (25) – (28) are equivalent to (8) – (11). For showing the convergence rate, we have for all and we obtain

$$V(0) = \chi_1\|\phi^{(1)}\|^2 + \chi_2\|\varphi^{(1)}\|^2 + \chi_3\|\phi^{(2)}\|^2 + \chi_4\|\varphi^{(2)}\|^2.$$

On the other hand, we have

$$\begin{array}{rcl}V(t) & \geq & e^{2kt}\{\lambda_m(P_1)\|u(t)\|^2 + \lambda_m(P_2)\|x(t)\|^2 + \lambda_m(P_3)\|v(t)\|^2\\ & & +\lambda_m(P_4)\|y(t)\|^2\}.\end{array}$$

Therefore,

$$\|u(t)\|^2 + \|x(t)\|^2 + \|v(t)\|^2 + \|y(t)\|^2 \leq \gamma e^{-2kt}\{\|\phi^{(1)}\|^2 + \|\varphi^{(1)}\|^2 + \|\phi^{(2)}\|^2 + \|\varphi^{(2)}\|^2\},$$

where $\gamma = \max(\chi_1, \chi_2, \chi_3, \chi_4)/\min(\lambda_m(P_1), \lambda_m(P_2), \lambda_m(P_3), \lambda_m(P_4)) \geq 1$. Therefore, the system (4) is globally exponentially stable with the convergent rate $k > 0$.

4. Conclusions

In this paper, we have investigated exponential stability problem for the neutral-type inertial BAM neural networks with time-varying delays. By constructing a new and improved Lyapunov-Krasovskii function containing some new augmented terms and using convex combination technique, a delay-dependent exponential stability criterion for the neutral-type inertial BAM neural networks with time-varying delays has been formulated in terms of LMIs.

Acknowledgments

This work was jointly supported by the research grant fund of Mathematics Program, Faculty of Science, Maejo University, Thailand.

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