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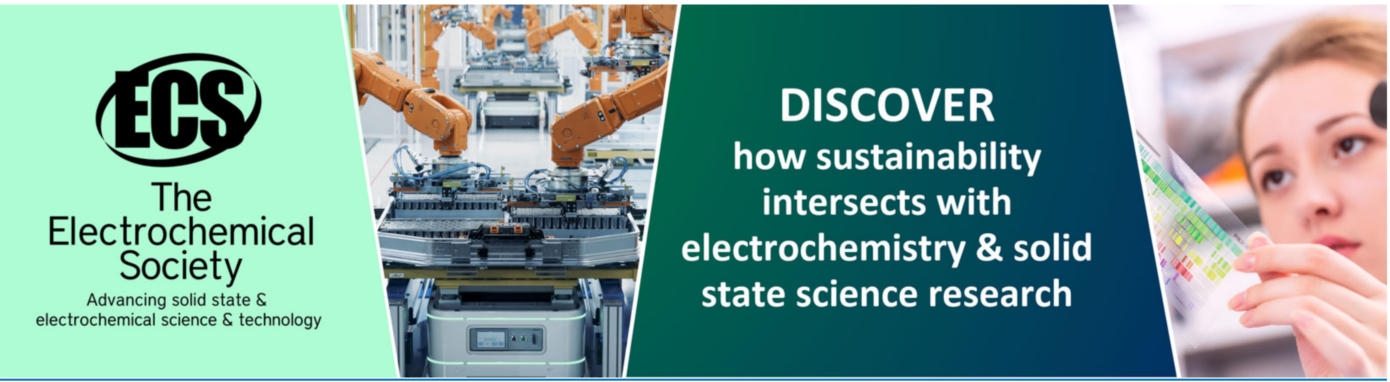

Certain Special Self-Similar Solutions of Khokhlov - Zabolotskaya - Kuznetsov Equation, Generalized Burgers Equation in (1+2)-dimensions

B Mayil Vaganan, B Balasakthi, P Jeengo Deepa and C P Nila Punnahai

Department of Applied Mathematics and Statistics, Madurai Kamaraj University, Madurai-625021, INDIA

E-mail: vkbmv66@gmail.com, balasakthi2017@gmail.com, deepajeengo@gmail.com, nilapunnahai@gmail.com

Abstract. Certain special forms of self-similar solutions which satisfy Khokhlov-Zabolotskaya-Kuznetsov equations or a generalized Burgers equation in (1+2)-dimensions are reported.

1. Introduction

The Khokhlov - Zabolotskaya- Kuznetsov (KZK) equation [1, 2] is

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\delta}{2}\frac{\partial^2 u}{\partial x^2}\right)+\epsilon\frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

describes a plethora of phenomena such as

- the quasi-one-dimensional propagation of a signal in a nonlinear, homogeneous, isentropic medium
- the nonlinear propagation of a finite-amplitude sound beam pulse in the thermo-viscous medium
- the long waves in ferromagnetic media

In addition, the KZK equation is used in lithotripsy in which kidney stones are broken with the help of ultrasound waves.

The scheme of this paper is as follows: In Section 2, some special exact, analytic travelling wave solutions of the Khokhlov - Zabolotskaya - Kuznetsov equation (1) is reported; one of these solutions tends to a constant limit as $t \to \infty$.

2. A Special Travelling Waves of Khokhlov - Zabolotskaya - Kuznetsov Equation

We shall proceed to obtain a special travelling wave solutions [3] of (1). For if an ansatz

$$u(x,y,t)=v(r,s), \quad r(x,t)=x-c_1t,\ s(x,t)=y-c_2t, \tag{2}$$

where c_1 and c_2 are wave speeds, is inserted into the KZK equation (1) then the partial differential equation (PDE) for $v(r,s)$ may be written in the conservation form

$$\frac{\partial}{\partial r}\left(-c_1 v_r + v v_r - \frac{\delta}{2} v_{rr}\right) + \frac{\partial}{\partial s}\left(v_s - c_2 v_r\right) = 0. \tag{3}$$

We shall solve (3) by the method of equation-splitting [4]; for, we write

$$-c_1 v_r + v v_r - \frac{\delta}{2} v_{rr} = A(s), \tag{4}$$

$$v_s - c_2 v_r = A_1(r). \tag{5}$$

Integrating (4) with respect to r, we get

$$-c_1 v + \frac{1}{2} v^2 - \frac{\delta}{2} v_r = A(s) r + A_2(s), \tag{6}$$

where $A_2(s)$ is the function of integration. With $A(s) = A_2(s) = 0$, (6) simplifies to a Riccati equation

$$-c_1 v + \frac{1}{2} v^2 - \frac{\delta}{2} v_r = 0. \tag{7}$$

It is easily verified that

$$v = \frac{1}{\frac{1}{2c_1} + B(s) e^{\frac{2c_1}{\delta} r}}, \tag{8}$$

with $B_(s)$ a function of s solves (7). Now differentiating (8) with respect to r and s, and inserting into (5), with $A_1(r) = 0$, we have

$$-B'(s) + \frac{2c_1 c_2}{\delta} B(s) = 0 \tag{9}$$

The general solution of (9) is

$$B(s) = c e^{\frac{2c_1 c_2}{\delta} s}, \tag{10}$$

where c is an arbitrary constant. Substituting (10) in (8), we thus find that

$$v(r,s) = \frac{1}{\frac{1}{2c_1} + e^{\frac{2c_1}{\delta}(r + c_2 s)}} \tag{11}$$

Equations (2) and (11) lead to the following travelling wave solution of the KZK equation (2):

$$u(x,y,t) = \frac{1}{\frac{1}{2c_1} + e^{\frac{2c_1}{\delta}(x + c_2 y - (c_1 + c_2) t)}}. \tag{12}$$

It is evident from (12) that $u(x,y,t)$ tends to 'twice the wave speed' $2c_1$ as $t \to \infty$.

The very form of the solution (12) of KHZ equation (2) suggests that we seek an ansatz in the form

$$u = \phi(z) \quad z = ax + by - ct. \tag{13}$$

In view of $\frac{\partial}{\partial x}$ as $\frac{\partial}{\partial z}\frac{\partial z}{\partial x}$, equation (1) becomes

$$\frac{\partial}{\partial z}\frac{\partial z}{\partial x}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\delta}{2}\frac{\partial^2 u}{\partial x^2}\right)+\epsilon\frac{\partial^2 u}{\partial y^2} \quad = \quad 0. \tag{14}$$

Inserting (13), equation (14) becomes

$$a\frac{\partial}{\partial z}\left(-c\phi_z+a\phi\phi_z-\frac{\delta}{2}a^2\phi_{zz}\right)+\epsilon b^2\phi_{zz} \quad = \quad 0 \tag{15}$$

Integrating equation (15) once with respect to z, we get

$$H\phi'+B\phi\phi'+A\phi'' \quad = \quad p,\ H=ac-\epsilon b^2,\ B=-a^2,\ A=\delta a^3/2, \tag{16}$$

where p is an integration constant.

Now we shall proceed to find some special solution of equation (16).

2.1. Case - 1: p=0

It is easy to ascertain that

$$\phi_{known}(z)=q-(H/B)z, \tag{17}$$

where q is a free constant, is a solution of (16) with $p=0$, namely

$$H\phi'+B\phi\phi'+A\phi'' \quad = \quad 0. \tag{18}$$

In order to obtain a solution which is more general than (17), we write

$$\phi(z)=q-(H/B)z+w(z). \tag{19}$$

Substitution of (19) in to (18) leads to

$$(qB-Hz+Bw(z))w'(z)+Aw''(z)=0. \tag{20}$$

An integration of (20) with respect to z yields

$$w'(z)=\frac{2\sqrt{AH}}{2c^*\sqrt{AH}e^{\frac{q^2B^2}{2AH}}+B\sqrt{2\pi}\mathrm{Erfi}[\frac{Hz-qB}{\sqrt{2AH}}]}e^{\frac{Hz^2}{2A}-\frac{qBz}{A}+\frac{q^2B^2}{2AH}}, \tag{21}$$

where c^* is an arbitrary constant. Thus (19) becomes

$$\phi(z)=q-(H/B)z+\int^z\frac{2\sqrt{AH}}{2c^*\sqrt{AH}e^{\frac{q^2B^2}{2AH}}+B\sqrt{2\pi}\mathrm{Erfi}[\frac{Hz-qB}{\sqrt{2AH}}]}e^{\frac{Hz^2}{2A}-\frac{qBz}{A}+\frac{q^2B^2}{2AH}}dz. \tag{22}$$

2.2. Case - 2: H=0

In this case, equation (16) reduces to

$$B\phi\phi'+A\phi'' \quad = \quad p, \tag{23}$$

The general solution of (23) is

$$\phi(z)=\frac{2\frac{BpAiryBiPrime[\frac{2^{2/3}(\frac{Bpz}{2A^2}-\frac{C_1B}{2A})}{(\frac{Bp}{A^2})^{2/3}}]}{2^{1/3}A(\frac{Bp}{A^2})^{2/3}}+2C_2\frac{BpAiryAiPrime[\frac{2^{2/3}(\frac{Bpz}{2A^2}-\frac{C_1B}{2A})}{(\frac{Bp}{A^2})^{2/3}}]}{2^{1/3}A(\frac{Bp}{A^2})^{2/3}}}{BAiryBi[\frac{2^{2/3}(\frac{Bpz}{2A^2}-\frac{C_1B}{2A})}{(\frac{Bp}{A^2})^{2/3}}]+C_2BAiryAi[\frac{2^{2/3}(\frac{Bpz}{2A^2}-\frac{C_1B}{2A})}{(\frac{Bp}{A^2})^{2/3}}]} \tag{24}$$

where C_i $(i = 1, 2)$ are arbitrary constants.

2.3. Case - 3: p=H=0

Equation (16) simply becomes

$$\frac{-\delta a}{2}\phi'' + \phi\phi' = 0 \tag{25}$$

After integration, with l a free constant, (25) yields a Riccati equation

$$\phi' + \frac{1}{\delta a}\phi^2 = \frac{-l}{\delta a}, \tag{26}$$

with the solution

$$\phi(z) = \sqrt{l}\,\tan[C - \frac{\sqrt{l}}{a\delta}z]. \tag{27}$$

In (27), C is an arbitrary constant.

3. Khokhlov - Zabolotskaya- Kuznetsov Equation with A Variable Coefficient

Now we consider Khokhlov - Zabolotskaya- Kuznetsov (KZK) equation with a variable coeffient

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\delta}{2}\frac{\partial^2 u}{\partial x^2}\right) + \epsilon_0 t^{-1/2}\frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

If we seek solutions in the form

$$u = t^{-\frac{1}{2}}v(r,s),\ r = t^{-\frac{1}{2}}x,\ s = t^{-\frac{1}{2}}y, \tag{2}$$

then the PDE governing $v(r, s)$ is

$$2v_r + rv_{rr} + sv_{rs} + \delta v_{rrr} - 2v_{rr}v - 2{v_r}^2 - 2\epsilon_0 v_{ss} = 0. \tag{3}$$

Again the transformation

$$v(r,s) = w(z),\ z = r - cs, \tag{4}$$

replaces (3) with

$$2w_z + zw_{zz} + \delta w_{zzz} - 2ww_{zz} - 2{w_z}^2 - 2\epsilon_0 c^2 w_{zz} = 0. \tag{5}$$

Equation (5) can be integrated to

$$w + zw' + \delta w'' - 2ww' - 2\epsilon_0 c^2 w' = \alpha, \tag{6}$$

where α is arbitrary constant. Further integration of (6) results in a Riccati equation

$$\delta w' = w^2 + 2\epsilon_0 c^2 w - zw + \alpha z + \beta, \tag{7}$$

where β is an arbitrary constant. A solution of (7) is easily found to be

$$w(z) = z - \alpha - 2c^2\epsilon_0, \tag{8}$$

provided that $\beta = -\alpha^2 - 2c^2\alpha\epsilon_0$.

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In view of (4) and (8) the corresponding solution of (5) is

$$v(r,s) = r - cs - \alpha - 2c^2\epsilon_0. \tag{9}$$

Thus, in turn, the solution of (1) is

$$u(x,y,t) = t^{-\frac{1}{2}}(x - cy) - \alpha - 2c^2\epsilon_0, \tag{10}$$

which is a decaying travelling wave propagating in the right direction with speed c.

4. Generalized (1+2)-dimensional Burgers Equation

Let us now investigate a generalized (1+2)-dimensional Burgers equation in the form (Edwards and Broadbridge [5])

$$\frac{\partial u}{\partial t} + xu\frac{\partial u}{\partial x} - \frac{\delta}{2}x^2\frac{\partial^2 u}{\partial x^2} + \epsilon y^2\frac{\partial^2 u}{\partial y^2} \quad = \quad 0, \tag{1}$$

for its self-similar solutions. We seek solutions of (1) in the form

$$u(x,y,t) \quad = \quad t^\alpha v(r,s),\ r(x,t) = t^\beta x^m,\ s(y,t) = t^\lambda y^n. \tag{2}$$

If we insert from (2), then equation (1) reduces to

$$\begin{aligned} &\alpha t^{-1}v + t^{-1}\beta r\frac{\partial v}{\partial r} + t^{-1}\lambda s\frac{\partial v}{\partial s} + t^\alpha vmr\frac{\partial v}{\partial r} \\ &-\frac{\delta}{2}rm(m-1)\frac{\partial v}{\partial r} - \frac{\delta}{2}r^2m^2\frac{\partial^2 v}{\partial r^2} + \epsilon sn(n-1)\frac{\partial v}{\partial s} + \epsilon s^2n^2\frac{\partial^2 v}{\partial s^2} \quad = \quad 0. \end{aligned} \tag{3}$$

Two cases arise, namely, $\alpha = 0$ and $\alpha = -1$. Let us consider the second case in detail. With $\alpha = -1$, equation (3) may be spilt into two equations:

$$(\beta + vm)\, r\frac{\partial v}{\partial r} + \lambda s\frac{\partial v}{\partial s} - v \quad = \quad 0, \tag{4}$$

$$\epsilon sn\left((n-1)\frac{\partial v}{\partial s} + sn\frac{\partial^2 v}{\partial s^2}\right) - \frac{\delta}{2}rm\left((m-1)\frac{\partial v}{\partial r} + rm\frac{\partial^2 v}{\partial r^2}\right) \quad = \quad 0. \tag{5}$$

Further transformation is

$$v(r,s) \quad = \quad V(R,S),\ R = \log r,\ S = \log s, \tag{6}$$

and the equation satisfied by $V(R,S)$ is

$$\epsilon n\,(nV_{SS} - V_S) + \frac{\delta}{2}m\,(V_R - mV_{RR}) \quad = \quad 0. \tag{7}$$

A solution of (7) is

$$V \quad = \quad \frac{-\beta}{\lambda m}S + \frac{1}{m}R, \tag{8}$$

and the corresponding solution of (5) is

$$v \quad = \quad \frac{-\beta}{\lambda m}\log s + \frac{1}{m}\log r. \tag{9}$$

Inserting from (9) into (2) we thus obtain a solution of (1)

$$u(x,y,t) \quad = \quad t^{-1}\left(\frac{-\beta}{\lambda m}\log(t^{\lambda}y^{n}) + \frac{1}{m}\log(t^{\beta}x^{m})\right), \tag{10}$$

provided that $2n\beta\epsilon + m\delta\lambda = 0$.

5. Conclusion

We report that for Khokhlov - Zabolotskaya- Kuznetsov equation (1.1) four solutions, namely, (2.11), (2.12) with (2.21), (2.12) with (2.23), (2.12) with (2.26).

And for the for Khokhlov - Zabolotskaya- Kuznetsov equation with a variable coefficient (3.1) we report the solution (3.10).

For the generalized Burgers equation in (1+2)-dimensions (4.1) the solution we report is (4.10).

It is noteworthy to mention the use of a symbolic manipulation program MATHEMATICA 3.0 [6] and MathLie [7] in order to facilitate the computation of certain solutions reported in this Paper.

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