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Nonexistence of Kneser solution for third order nonlinear neutral delay differential equations

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Abstract. The achieved sufficient conditions for nonexistence of so-called Kneser solutions are based on the new comparison principles, which help us decrease the problem of the wavering between the third and first-order equations. Examples are given to prove the significance of new theorems.

1. Introduction

The purpose of this work, we are concerned with third order nonlinear neutral delay differential equations of the form

$$\left(c_2(\ell)\big(\big(c_1(\ell)z'(\ell)\big)'\big)^{\gamma}\right)' + q(\ell)y^{\beta}(m(\ell)) = 0, \tag{1}$$

where $z(\ell) = y(\ell) + p(\ell)y(k(\ell))$. Further, assume the hypotheses are tacitly supposed to hold:

(A_1) $\gamma,\ \beta$ is a quotient of odd positive integers, $c_1,\ c_2 \in C(I, \mathbb{R}^+)$ take $I = [0, \infty)$ $p(\ell), q(\ell) > 0$, $0 \le p(\ell) \le p_0 < \infty$ and q does not vanish identically;

(A_2) $m,\ k \in C^1(I, \mathbb{R}^+)$, $m(\ell) < \ell$, $k'(\ell) \ge k_0 > 0$ and $\lim_{\ell\to\infty} k(\ell) = \lim_{\ell\to\infty} m(\ell) = \infty$.

Moreover,

$$C_1[\ell_0, \ell] = \infty, \quad M_2[\ell_0, \ell] = \infty \ as\ \ell \to \infty, \tag{2}$$

where

$$M_1[\ell_0, \ell] = \int_{\ell_0}^{\ell} c_2^{-1/\gamma}(s)ds, \quad M_2[\ell_0, \ell] = \int_{\ell_0}^{\ell} c_1^{-1}(s)ds.$$

By a solution to (1), we mean a function $y(\ell)$ in $C^2[T_y,\infty)$ for which $\big(c_1(\ell)z'(\ell)\big)'$, $c_2(\ell)\big(\big(c_1(\ell)z'(\ell)\big)'\big)^\gamma$ is in $C^1[T_y,\infty)$ and (1) is satisfied on some interval $[T_y,\infty)$, where $T_y \geq \ell_0$. We consider only solutions $y(\ell)$ for which $\sup\{|y(\ell)| : \ell \geq T\} > 0$ for all $T \geq T_y$. A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative on $[T_y,\infty)$ and otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Kiguradze and Kondratev [8], we say that (1) has *property A* (almost oscillatory) if any solution y of (1) is either oscillatory or satisfes $\lim_{\ell\to\infty} y(\ell) = 0$.

In recent years, a great deal of research has been done on numerous aspects of differential equations of third and higher order. These equations appear in the study of entry-flow phenomenon, a problem of hydrodynamics, in mathematical theory of thyroid-pituitary interaction, gravity driven flows and three-layer beams. Recently, a great deal of interest in oscillatory properties of neutral functional differential equations has been shown, we refer the reader to [2, 3, 4, 10, 9, 5, 6, 11] and the references cited therein.

Despite there, to the best of authors knowledge, here is nothing known regarding oscillation of all solutions of (1) by comparison principle method with $p(\ell) \geq 0$. So the aim of this paper to fill this gap by establishing various sufficient conditions for eliminating Kneser solutions of (1) and enable us also to eliminate some conditions imposed in the cited papers on the coefficients of (1). As usual, all occurring functional inequalities are considered to support eventually, that is, they are satisfied for all ℓ large enough.

2. Nonexistence Kneser Solution of (1)

For our further reference, let us denote

$$\begin{aligned} Q(\ell) &= \min\{q(\ell), q(k(\ell))\}, \quad Q^*(\ell) = \min\{q(m^{-1}(\ell)), q(m^{-1}(k(\ell)))\}, \\ M[\ell_0,\ell] &= \int_{\ell_0}^{\ell} \frac{M_1[\ell_0,s]}{a_1(s)} ds. \end{aligned}$$

Lemma 2.1 *Assume $a \geq 0$, $b \geq 0$, $\beta \geq 1$. Then*

$$(a+b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta). \tag{3}$$

Lemma 2.2 *Assume $a \geq 0$, $b \geq 0$, $0 < \beta \leq 1$. Then*

$$(a+b)^\beta \leq a^\beta + b^\beta. \tag{4}$$

Lemma 2.3 *Assume that (A_1), (A_2) holds and let $z(\ell)$ be an eventually positive solution of (1), then*

$$z(\ell) \in \mathcal{N}_0 \Longleftrightarrow \{z'(\ell) > 0,\ z(\ell)(\big(c_1(\ell)z'(\ell)\big)') < 0,\ z(\ell)(c_2(\ell)\big(\big(c_1(\ell)z'(\ell)\big)'\big)^\gamma) > 0\}$$

or

$$z(\ell) \in \mathcal{N}_2 \Longleftrightarrow \{z'(\ell) > 0,\ z(\ell)(\big(c_1(\ell)z'(\ell)\big)') > 0,\ z(\ell)(c_2(\ell)\big(\big(c_1(\ell)z'(\ell)\big)'\big)^\gamma) > 0\}.$$

A positive solution $y(\ell)$ whose corresponding function $z(\ell)$ of (1) is said to be Kneser solution if $z(\ell) \in \mathcal{N}_0$. We say that (1) has *property A* if its every positive solution $y(\ell) \in \mathcal{N}_0$, and $\lim_{\ell\to\infty} y(\ell) = 0$.

Theorem 2.4 *Let $k(\ell) \le \ell$, $k \circ m = m \circ k$ and $\beta \ge 1$. If there exists a $\zeta(\ell) \in C(I, \mathbb{R}^+)$, $m(\ell) < \zeta(\ell)$ and $k^{-1}(\zeta(\ell) < \ell$, k^{-1} is inverse function of k, such that the first-order delay differential equation*

$$w'(\ell) + \frac{k_0}{k_0 + p_0^\beta} \frac{Q(\ell)}{2^{\beta-1}} \Big(M[\zeta(\ell),\, m(\ell)]\Big)^{\gamma/\beta} w(k^{-1}(\zeta(\ell))) = 0, \tag{5}$$

is oscillatory, then $\mathcal{N}_0 = \emptyset$.

Proof. Assume that (1) has a non-oscillatory solution $x(\ell)$. Without loss of generality, we can suppose that $x(\ell)$ is eventually positive.

$$z(\ell) > 0, \quad z'(\ell) < 0, \quad \text{and} \quad \big(c_1(\ell) z'(\ell)\big)' > 0 \quad on \quad I.$$

Then from (A_1) and (A_2) the corresponding function $z(\ell)$ satisfies

$$\begin{aligned} z^\beta(m(\ell)) &= (y(m(\ell)) + p(m(\ell))\, y(k(m(\ell))))^\beta \\ &\le (y(m(\ell)) + p_0\, y(k(m(\ell))))^\beta \\ &\le 2^{\beta-1}\Big(y^\beta(m(\ell)) + p_0^\beta\, y(m(k(\ell)))^\beta\Big). \end{aligned} \tag{6}$$

In view of (1) and (A_2), we get

$$\Big(c_2(\ell)\big(\big(c_1(\ell) z'(\ell)\big)'\big)^\gamma\Big)' + q(\ell) y^\beta(m(\ell)) = 0, \tag{7}$$

and moreover taking (A_1) and (A_2) into account, we have

$$\begin{aligned} 0 &= \frac{p_0^\beta}{k'(\ell)} \Big(c_2(\ell)\big(\big(c_1(\ell) z'(k(\ell))\big)'\big)^\gamma\Big)' + p_0^\beta\, q(\ell)\, y^\beta(m(k(\ell)) \\ &\ge \frac{p_0^\beta}{k_0} \Big(c_2(\ell)\big(\big(c_1(\ell) z'(k(\ell))\big)'\big)^\gamma\Big)' + p_0^\beta\, q(\ell)\, y^\beta(m(k(\ell)). \end{aligned} \tag{8}$$

Combining (7), (8) and Lemma 2.1 we are led to,

$$\begin{aligned} \Big(c_2(\ell)\big(\big(c_1(\ell) z'(\ell)\big)'\big)^\gamma\Big)' &+ \frac{p_0^\beta}{k_0} \Big(c_2(\ell)\big(\big(c_1(\ell) z'(k(\ell))\big)'\big)^\gamma\Big)' \\ &+ q(\ell) y^\beta(m(\ell)) + p_0^\beta q(\ell) y^\beta(m(k(\ell))) \le 0, \end{aligned}$$

that is

$$\Big[c_2(\ell)\big(\big(c_1(\ell) z'(\ell)\big)'\big)^\gamma + \frac{p_0^\beta}{k_0} c_2(\ell)\big(\big(c_1(\ell) z'(k(\ell))\big)'\big)^\gamma\Big]' + \frac{Q(\ell)}{2^{\beta-1}} z^\beta(m(\ell)) \le 0. \tag{9}$$

Since $\Big(c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma\Big)' \leq 0$, $c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma$ is nondecreasing. Then we have

$$\begin{aligned} c_1(\ell)z'(\ell) \geq c_1(\ell)z'(\ell) - c_1(\ell_0)z'(\ell_0) &= \int_{\ell_0}^{\ell} \frac{c_2^{1/\gamma}(s)\big(c_1(s)z'(s)\big)'}{c_2^{1/\gamma}(s)} ds \\ &\geq m_2^{1/\gamma}(\ell)\big(c_1(\ell)z'(\ell)\big)' M_1[\ell_0, \ell]. \end{aligned}$$

Again integrate, we get

$$\begin{aligned} z(\ell) \geq m_2^{1/\gamma}(\ell)\big(c_1(\ell)z'(\ell)\big)' \int_{\ell_0}^{\ell} \frac{M_1[\ell_0, s]}{c_1(s)} ds \\ = m_2^{1/\gamma}(\ell)\big(c_1(\ell)z'(\ell)\big)' M[\ell_0, \ell]. \end{aligned}$$

That is,

$$z^\gamma(m(\ell)) \geq c_2(\ell)\big((c_1(\ell)z'(\zeta(\ell)))'\big)^\gamma \Big(M[\zeta(\ell), m(\ell)]\Big)^{1/\beta}. \tag{10}$$

Combining (10) together with (9), we get that $y(\ell)$ is a positive solution of

$$\begin{aligned} &\Big[c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma + \frac{p_0^\beta}{k_0} c_2(\ell)\big((c_1(\ell)z'(k(\ell)))'\big)^\gamma\Big]' \\ &\quad + \frac{Q(\ell)}{2^{\beta-1}} c_2(\ell)\big((c_1(\ell)z'(\zeta(\ell)))'\big)^\gamma \Big(M[\zeta(\ell), m(\ell)]\Big)^{1/\beta} \leq 0. \end{aligned} \tag{11}$$

Let us denote

$$w(\ell) := c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma + \frac{p_0^\beta}{k_0} c_2(\ell)\big((c_1(\ell)z'(k(\ell)))'\big)^\gamma. \tag{12}$$

Since $z(\ell)$ decreasing and $k(\ell) \leq \ell$ that

$$w(\ell) \leq c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma \left(\frac{k_0 + p_0^\beta}{k_0}\right).$$

that is

$$c_2(\ell)\big((c_1(\ell)z'(\zeta(\ell)))'\big)^\gamma \geq \left(\frac{k_0 + p_0^\beta}{k_0}\right) w(k^{-1}(\zeta(\ell))).$$

Substituting above inequalities into (11), we have

$$w'(\ell) + \frac{k_0}{k_0 + p_0^\beta} \frac{Q(\ell)}{2^{\beta-1}} \Big(M[\zeta(\ell), m(\ell)]\Big)^{\gamma/\beta} w(k^{-1}(\zeta(\ell))) \leq 0. \tag{13}$$

It follows from Theorem 1 in [13] we get that $w(\ell)$ is a positive solution of (13), which contradicts the fact that this inequality does not have positive solutions. Thus, $\mathcal{N}_0 = \emptyset$.

Corollary 2.5 *Let $k(\ell) \leq t$, $k \circ m = m \circ k$ and $\beta \geq 1$. If there exists a function $\zeta(\ell) \in C(I, \mathbb{R}^+)$, $m(\ell) < \zeta(\ell)$ and $k^{-1}(\zeta(\ell)) < t$, k^{-1} is inverse function of k, such that*

$$\liminf_{t\to\infty} \int_{k^{-1}(\zeta(\ell))}^{t} Q(s)\, M^{\gamma/\beta}[\zeta(s),\, m(s)]\, ds > \frac{k_0 + p_0^\beta}{k_0\, e}, \tag{14}$$

then $\mathcal{N}_0 = \emptyset$.

Proof. In view of [[15], Theorem 2.1.1], the associated equation (13) also has a positive solution, which contradicts the oscillatory behavior of (13).

Theorem 2.6 *Let $k(\ell) \leq t$, $k \circ m = m \circ k$ and $\beta \geq 1$. If $\psi(\ell) \in C(I, \mathbb{R}^+)$, $\psi(\ell) < \ell$ and $m(\ell) < k(\psi(\ell))$ such that*

$$\limsup_{\ell\to\infty} M^{\gamma/\beta}[k(\psi(\ell)),\, m(\ell)] \int_{\psi(\ell)}^{\ell} Q(s)\, ds > 2^{\beta-1} \frac{k_0 + p_0^\beta}{k_0}, \tag{15}$$

then $\mathcal{N}_0 = \emptyset$.

Proof. Proceeding as in the proof of Theorem 2.4, we obtain (2.3). Integrating this inequality from $\psi(\ell)$ to t and using the fact that z is decreasing, we see that

$$\begin{aligned} c_2(\ell)\big((c_1(\ell)z'(\psi(\ell)))'\big)^\gamma &+ \frac{p_0^\beta}{k_0} c_2(\ell)\big((c_1(\ell)z'(\psi(\ell)))'\big)^\gamma \\ &\geq c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma + \frac{p_0^\beta}{k_0} c_2(\ell)\big((c_1(\ell)z'(k(\ell)))'\big)^\gamma \\ &\quad + \frac{1}{2^{\beta-1}} \int_{\psi(\ell)}^{t} Q(s) z^\beta(m(s)) ds \\ &\geq \frac{z^\beta(m(\ell))}{2^{\beta-1}} \int_{\psi(\ell)}^{t} Q(s) ds \end{aligned} \tag{16}$$

Since $k(\psi(\ell)) < k(\ell)$ and $c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma$ is nonincreasing, we have

$$c_2(\ell)\big((c_1(\ell)z'(\ell))'\big)^\gamma \left(\frac{k_0 + p_0^\beta}{k_0}\right) \geq \frac{z^\beta(m(\ell))}{2^{\beta-1}} \int_{\psi(\ell)}^{t} Q(s) ds. \tag{17}$$

that is,

$$2^{\beta-1} \frac{k_0 + p_0^\beta}{k_0} \geq M^{\gamma/\beta}[k(\psi(\ell)),\, m(\ell)] \int_{\psi(\ell)}^{\ell} Q(s)\, ds, \tag{18}$$

Taking the *limsup* on both sides of the above inequality, we obtain a contradiction to (15).

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