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Quasi – injective Gamma acts

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Abstract. In this work we introduce the concept of quasi – injective gamma acts as a generalization of both injective and weakly injective gamma acts. In general we study the endomorphism set of gamma acts and certain types of gamma subacts which are used later. In the main part, we study basic properties of quasi – injective gamma acts and the effect of their endomorphism set to quasi – injective. We show that for any gamma act there is quasi – injective extension.

1. Introduction

A semigroup (S, $\star$) consist of a nonempty set S on which an associative operation $\star$ is defined. The concept of semigroup has been generalized to the nontion of gamma semigroups in [2]. Let S and Γ be two nonempty sets. S is called Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written $(s_1, \alpha, s_2) \mapsto s_1\alpha s_2$ such that $(s_1\alpha s_2)\beta s_3 = s_1\alpha(s_2\beta s_3)$ for all $s_1, s_2, s_3 \in S$ and $\alpha, \beta \in \Gamma$.

An element, denoted by, 0 in a Γ-semigroup S is called right (left) zero, if $0 = 0\alpha s$ ($0 = s\alpha 0$) for all $s \in S$ and $\alpha \in \Gamma$, and is called zero element if it is both right and left zero. An element, denoted by 1 in Γ-semigroup S is called Γ-identity, if $1\alpha s = s = s\alpha 1$ for all $s \in S$ and $\alpha \in \Gamma$.

For an arbitrary fixed element $\alpha \in \Gamma$. An element, denoted by 1_α in S is called α – identity if (1) $s\alpha 1_\alpha = s = 1_\alpha \alpha s$ (2) $s\beta 1_\alpha = 1_\alpha \beta s$ for all $s \in S$ and $\beta \in \Gamma \setminus \{\alpha\}$, then S is called a Γ-monoid (an α-monoid) having a Γ-identity (an α –identity) 1 (1_α) .

A nonempty subset A of a Γ-semigroup S is called right (left) Γ- ideal of S, if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$), where $X\Gamma Y = \{ x\alpha y \mid x \in X, y \in Y, \alpha \in \Gamma$ and $a \in A \}$ for any nonempty subset X and Y of S.

Let S and T be two Γ-semigroups. A mapping g : S → T is called Γ-homomorphism if $g(s_1\alpha s_2) = g(s_1)\alpha g(s_2)$ for all $s_1, s_2 \in S$ and $\alpha \in \Gamma$.

A Γ-semigroup S is called commutative if $a\alpha b = b\alpha a$ for all $\alpha \in \Gamma$ and a, b ∈ S. Let S be a Γ-semigroup and $\alpha \in \Gamma$ be an arbitrary fixed element in Γ. An element x in S is called α-idempotent, if $x\alpha x = x$.

In [3]. The authors introduced a generalization of a Γ - semigroup as follows. Let S be a Γ-semigroup. A nonempty set M is called a right gamma act over S ,denoted by S_Γ-act, if there is a mapping $M \times \Gamma \times S \to M$ written $(m, \alpha, s) \to m\alpha s$ such that $(m\alpha s_1)\beta s_2 = m\alpha(s_1\beta s_2)$ for all $s_1, s_2 \in S$, $\alpha, \beta \in \Gamma$ and m ∈ M. Similarly we can define a left gamma act.

Let S and T be two Γ-semigroups. A nonempty set M is called $(T - S)_\Gamma$-biact, if (1) M is a right S_Γ-act, (2) M is a left T_Γ-act, and (3) $t\alpha(m\beta s) = (t\alpha m)\beta s$, for all m ∈ M, $\alpha, \beta \in \Gamma$, t ∈ T, and s ∈ S,

A right S_Γ-act M is Γ – unitary (α – unitary, for arbitrary fixed $\alpha \in \Gamma$), if S has Γ-identity 1 (α- identity 1_α) such that $m = m\beta 1$ for all $\beta \in \Gamma$($m = m\alpha 1_\alpha$) A nonempty subset N of an S_Γ-act M is called S_Γ-subact, denoted by N ≤ M, if $N\Gamma S \subseteq N$ where $N\Gamma S = \{n\alpha s \mid n \in N, \alpha \in \Gamma, \text{ and } s \in S\}$. A nonempty subset N of a $(T\text{-}S)_\Gamma$-biact M is called $(T\text{-}S)_\Gamma$-subbiact of M if N is an S_Γ-subact of the right S_Γ-act M and the left T_Γ-act of M.

An element Θ in a right S_Γ-act M is called a fixed element if $\Theta = \Theta\alpha s$ for all $\alpha \in \Gamma$ and s ∈ S. If S has a zero element 0. Then $m\alpha 0$ is a fixed element in M for all m ∈ M and $\alpha \in \Gamma$. It is possible for S_Γ-act has more than one fixed elements. An S_Γ-act M is called Γ-centered, if it has a unique fixed element Θ, and in this case we say that Θ is the zero element of M, every S_Γ-subact N of M must contain Θ where S has zero 0. An S_Γ-act M is called simple S_Γ-act, if it contains no gamma subact other than Θ and M itself.

Let S be Γ-semigroup and $\{ M_i \mid i \in I \}$ be an arbitrary family of Γ-centered right S_Γ-acts. Then the Cartesian product of M_i which is denoted by $\prod_{i\in I} M_i$ has the structure of a right S_Γ-act componentwisely. $\prod_{i\in I} M_i$ is called the product of M_i , i ∈ I. The direct sum $\bigoplus_{i\in I} M_i$ of $\{M_i \mid i \in I\}$ is a subset of $\prod_{i\in I} M_i$ which contain all element $(m_i)_{i\in I} \in \prod_{i\in I} M_i$ such that the set $\{i \mid mi \neq \Theta_{M_i}\}$ is finite. $\bigoplus_{i\in I} M_i$ is an S_Γ-subact of $\prod_{i\in I} M_i$.

Let $\{ M_i \mid i \in I \}$ be an arbitrary family of Γ-centered right S_Γ-acts. Then the coproduct of M_i, denoted by $\coprod_{i\in I} M_i$ is the the disjoint union $\uplus_{i\in I} M_i$ of M_i . Clearly $\coprod_{i\in I} M_i$ is a right S_Γ-act.

Let S be Γ-semigroup and M and N two right S_Γ-acts. A mapping f : M → N is called S_Γ-homomorphism, if $f(m\alpha s) = f(m)\alpha s$ for all m ∈ M, s ∈ S and $\alpha \in \Gamma$. An S_Γ-homomorphism is called S_Γ-monomorphism (S_Γ-epimorphism, S_Γ- isomorphism) if it is injective (surjective , bijective). We say that two right S_Γ-acts M and N is isomorphic, if there exist an S_Γ-isomorphism between them and denoted by $M \cong N$.

Let M be a right S_Γ-act. An equivalence relation ρ on M is called S_Γ-congruence, if $(a\alpha s, b\alpha s) \in \rho$ for all $s \in S$, $\alpha \in \Gamma$, and $(a, b) \in \rho$. Then the set $M/\rho = \{ x\rho \mid x \in M \}$ is called the quotient of M by ρ, where $x\rho$ is the equivalent class of x under ρ. M/ have the structure of righ gamma act by the mapping $(x\rho)\alpha s \mapsto (x\alpha s)\rho$ for all $x \in M$, $\alpha \in \Gamma$, and $s \in S$. Let N be an S_Γ-subact of an S_Γ-act M. The equivalence relation ρ_N on M which is define by $\rho_N = \{(m, n) \in M \times M \mid m, n \in N \text{ or } m = n\}$ is an S_Γ-congruence on M, ρ_N is called Rees S_Γ-congruence of M with respect to N.

Nowadays, A. A. Mustafa in his Ph. D thesis studying the concept of injective gamma acts. An S_Γ-act M is called injective if given any S_Γ-monomorphism $f : A \to B$ where A and B are any two S_Γ-acts and for any S_Γ-homomorphism $g : A \to M$, there exists an S_Γ-homomorphism $\bar{g} : B \to M$ such that $\bar{g}f = g$. He explicit a lot of properties of injective gamma acts analogous to that in the module and act theory, but there are properties in module theory not hold in gamma act theory. The most famous characterization of injective module is so – called is Baer's condition, in fact the criterion is not true in gamma acts. So he introduced a generalization of injective gamma acts which satisfies Baer's condition and called weakly injective. Finally he proved that for any gamma act, there exists an injective extension gamma act called the injective envelope and it is unique up to isomorphism.

2. Endomorphism of gamma acts.

In this section we define the endomorphism set of gamma act, and we show the set of endomorphism becomes gamma semigroup under some condition, after this, we can consider it as gamma biact by considering the set of endomorphism is gamma semigroup.

Let $f : M \to N$ be S_Γ-homomorphism . Then we defined the kernel of f as follows $\ker(f) = \{(m_1, m_2) \in M \times M \mid f(m_1) = f(m_2)\}$, and ker(f) is S_Γ-congruence for any S_Γ-homomorphism in S_Γ-act M.

The set of all S_Γ-homomorphisms from M into N denoted by $\mathrm{Hom}_{S_\Gamma}(M, N)$. In particular if M = N then $\mathrm{Hom}_{S_\Gamma}(M, M)$ denoted by $\mathrm{End}_{S_\Gamma}(M)$ and is called the endomorphism set of M. It is well - known that in module theory the endomorphism set forms a ring with identity and in act theory it forms a monoid. In gamma acts we have the following. Let M be a right Γ – unitary S_Γ-act . Define a mapping

$$\mathrm{End}_{S_\Gamma}(M) \times \Gamma \times \mathrm{End}_{S_\Gamma}(M) \to \mathrm{End}_{S_\Gamma}(M)$$

$$\text{Which } (f, \gamma, g) \mapsto f\gamma g$$

$$\text{Where } f\gamma g\,(m) = f(g(m)\gamma 1) \text{ for all } m \in M$$

Then $\mathrm{End}_{S_\Gamma}(M)$ is Γ-monoid. But in case M is α-unitary S_Γ-act, then $\mathrm{End}_{S_\Gamma}(M)$ may not be Γ-monoid with respect the above mapping as in the following example

Example. (2.1). Let S = {a, b, c, d, e, f} and Γ = {α, β} and consider the multiplication tables as follows

β	a	b	c	d	e	f
a	d	e	f	a	b	c
b	f	c	d	b	a	e
c	e	d	b	c	f	a
d	a	b	c	d	e	f
e	c	f	a	e	d	b
f	b	a	e	f	c	d

α	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	a	e	f	c	d
c	c	f	a	e	d	b
d	d	e	f	a	b	c
e	e	d	b	c	f	a
f	f	c	d	b	a	e

Then S is Γ-semigroup and has α- identity a, we can consider S as an S_Γ-act, it is a matter of checking that each S_Γ-endomorphism of S is of the form x_α^l where x_α^l(m) = xαm for all m and s in S. But End_{S_Γ}(S) is not Γ-monoid, since c_α^l , $d_\alpha^l \in \mathrm{End}_{S_\Gamma}$(S) then $c_\alpha^l \beta d_\alpha^l$ is not in End_{S_Γ}(S), indeed

$$(c_\alpha^l \beta d_\alpha^l)(e\alpha f) = (c_\alpha^l \beta d_\alpha^l)(a) = c_\alpha^l(d_\alpha^l(a)\beta a) = c_\alpha^l(d\beta a) = c_\alpha^l(a) = c \text{ and}$$

$$(c_\alpha^l \beta d_\alpha^l)(e)\alpha f = c_\alpha^l(d_\alpha^l(e)a)\alpha f = c_\alpha^l(b\beta a)\alpha f = c_\alpha^l(f)\alpha f = b\alpha f = d \text{ which implies that}$$

$$(c_\alpha^l \beta d_\alpha^l)(e\alpha f) \neq (c_\alpha^l \beta d_\alpha^l)(e)\alpha f.$$

We consider conditions under which End_{S_Γ}(M) being -monoid for α-unitary S_Γ-act M

Definition. (2.2). An S_Γ-act M is called Γ-commute if the following condition is hold

$$m(s_1\beta s_2) = m\beta(s_1\alpha s_2) \text{ for all } s_1, s_2 \in S,\ \alpha, \beta \in \Gamma \text{ and } m \in M.$$

Examples. (2.3).

(1). Z as Z_N- act with usual addition and multiplication is N-commute.

(2). Let S = {$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ | a, b, c and d ∈ Z} and Γ = {$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ | a, and b ∈ Z}. Then S is S_Γ-act which is not Γ-commute under usual multiplication of matrices.

(3). Example (2.1) is not Γ – commute

The above definition depends only on elements in Γ which satisfies the condition of associative in S_Γ-act. Firstly, we have the following, if M is Γ-commute α- unitary, then End_{S_Γ}(M) is α - monoid. Indeed, let f, g ∈ End_{S_Γ}(M) and $\gamma \in \Gamma$ we will show fγg ∈ End_{S_Γ}(M), for each m ∈ M, $\alpha \in \Gamma$ and s ∈ S, (fγg)(mαs) = f(g(mαs)$\gamma 1_\alpha$) = f((g(m)αs)$\gamma 1_\alpha$) = f(g(m)α(s$\gamma 1_\alpha$)) = f(g(m)α(1_α γs)) = f(g(m)γ(1_α αs))) = f((g(m)$\gamma 1_\alpha$)αs) = (fγg)(m)αs which implies that the above multiplication * is absolute mapping which is belong to End_{S_Γ}(M). Secondly, we will show the associative law is hold. Let f, g, t ∈ End_{S_Γ}(M), β, γ ∈ Γ and for m ∈ M we have (fγ(gβt))(m) = f ((gβt)(m)$\gamma 1_\alpha$) = f (g(t(m)$\beta 1_\alpha$) = (fγg)(t(m)$\beta 1_\alpha$) = ((fγg)βt)(m) that is fγ(gβt) = (fγg)βt. Then End_{S_Γ}(M) is α-monoid with α–identity 1_α.

This preface the way to consider M as a gamma act over the α-monoid $\mathrm{End}_{S_\Gamma}(M)$ as follows:

For any right α-unitary Γ-commute S_Γ-act M, we define the mapping

$$\mathrm{End}_{S_\Gamma}(M) \times \Gamma \times M \to M \text{ defined by}$$
$$(f, \gamma, m) \mapsto f(m\gamma 1_\alpha) \text{ for } m \in M, \gamma \in \Gamma \text{ and } f \in \mathrm{End}_{S_\Gamma}(M)$$

To check the associative law, for f, g $\in \mathrm{End}_{S_\Gamma}(M)$, α, $\beta \in \Gamma$ and $m \in M$, $(f\alpha g)\beta m = (f\alpha g)(m\beta 1_\alpha) = f(g(m\beta 1)\alpha 1_\alpha) = f\alpha(g(m\beta 1_\alpha)) = f\alpha(g\beta m)$, then M is also α- unitary left $\mathrm{End}_{S_\Gamma}(M)_\Gamma$ – act . Now we have for any right S_Γ-act can consider M is $(\mathrm{End}_{S_\Gamma}(M) - S)_\Gamma$ – biact. Indeed, let $f \in \mathrm{End}_{S_\Gamma}(M)$, α, $\beta \in \Gamma$, $s \in S$, and $m \in M$, then $(f\alpha m)\beta s = f(m\alpha 1_\alpha)\beta s = f(m)\alpha(1_\alpha \beta s) = f(m)\beta(s\alpha 1_\alpha) = f(m\beta(s\alpha 1_\alpha)) = f((m\beta s)\alpha 1_\alpha) = f\alpha(m\beta s)$. Thus M is $(\mathrm{End}_{S_\Gamma}(M) - S)_\Gamma$ – biact. In case Γ-unitary S_Γ-act M, it is clear directly $\mathrm{End}_{S_\Gamma}(M)$ is Γ-monoid and hence M is $(\mathrm{End}_{S_\Gamma}(M) - S)_\Gamma$ – biact.

From now the word "S_Γ-act" means Γ-commute right α – unitary Γ-centered S_Γ-act.

3. Some types of gamma subacts

In this section we introduce the notions Γ-essential, Γ- meet S_Γ-subacts, and we investigate the relationship between them, we give definition of another classes S_Γ-subacts, as Γ-retract, and Γ-direct summand.

Let M be an S_Γ-act. An S_Γ-subact N of M is called Γ-essential (or Γ-large) in M, if for any S_Γ-homomorphism $g : M \to B$ (B is any S_Γ-act) such that $g|_N$ is S_Γ-monomorphism, then g is S_Γ-monomorphism itself. We denote this situation by $N \subseteq_{\Gamma-ess} M$ [1]

If this is a case, then M is called Γ-essential extension of N. In [1] give a characterization of above definitions as follows: An S_Γ-act M is Γ-essential extension of S_Γ-subact N if and only if for every S_Γ-congruence ρ on M such that $\rho \neq I_M$ implies that $\rho|_N \neq I_N$. And he proved if N_1, N_2 be two S_Γ-subacts of M with $N_1 \leq N_2$. Then $N_1 \subseteq_{\Gamma-ess} M$ if and only if $N_1 \subseteq_{\Gamma-ess} N_2$ and $N_2 \subseteq_{\Gamma-ess} M$.

Definition. (3.1). Let M be an S_Γ-act. An S_Γ-subact N of M is called Γ-direct summand in M, if there exist an S_Γ-subact L of M such that $M = N \cup L$ and $N \cap L = 0$.

Definition. (3.2). An S_Γ-subact N of an S_Γ-act M is called Γ-retract of M if there exists S_Γ-homomorphisms $f : M \to N$ and $g : N \to M$ such that $fg = 1_N$, we denote this notion by $M \underset{g}{\overset{f}{\subset}} N$, and f if is called Γ-retraction S_Γ-homomorphism

Example. (3.3). Let S = {1, a, b, 0} and Γ = {α, β} and consider the following tables as follows

α	1	a	b	0
1	1	a	b	0
a	a	a	b	0
b	b	b	a	0
0	0	0	0	0

β	1	a	b	0
1	1	b	a	0
a	b	b	a	0
b	a	a	b	0
0	0	0	0	0

Then S is right S_Γ-act, let N = {a, b}. Then N is a Γ-retract S_Γ-subact of S.

Any Γ-retract S_Γ-subact is Γ-direct summand, but the converse may not true as in the following example.

Example. (3.4). It is well – known that Z_6 is Z_N-act with multiplication mapping, then N = {$\bar{0}$, $\bar{3}$} is Γ-retract, but not Γ-direct summand.

Definition. (3.5). An S_Γ-act M is called Γ- completely reducible, if it is disjoint union of simple S_Γ-subacts, that mean M = $\bigcup_{i\in I} M_i$ and M_i is simple S_Γ-subact of M for all i ∈ I, and $M_i \cap M_j = 0$ for i ≠ j.

Proposition. (3.6). Let M be a Γ- completely reducible S_Γ-act. Then

1. If N is a S_Γ-subact of M, then N is Γ-direct summand.
2. Any nonzero S_Γ-subact of M contains a proper simple S_Γ-subact.

Proof. 1. Let N be an S_Γ-act of a Γ- completely reducible S_Γ-act M = $\bigcup_{i\in I} M_i$ where M_i is simple S_Γ-act for all i ∈ I. Then there exists a subset $I_\circ$ of I such that N = $\bigcup_{i\in I_\circ} M_i$ and hence M = $\bigcup_{i\in I} M_i$ = [$\bigcup_{i\in I_\circ} M_i$] ∪ [$\bigcup_{i\in I_\circ^c} M_i$] = N ∪ $\bigcup_{i\in I_\circ^c} M_i$

2. Let N be a proper nonzero S_Γ-subact of a Γ- completely reducible S_Γ-act M and let m (≠ 0) ∈ M with m ∉ N. Consider the following set

$$G = \{ L \leq M \mid m \notin L \}.$$

As an application of Zorn's lemma, G has a maximal element B say. By **(1)** B is a Γ-direct summand S_Γ-subact of M, that is M = B ⊎ C. We claim that C is a simple S_Γ-subact of M which contained in N. If C is not simple, then there is an S_Γ-subact D of M which contained in C, so C = D ⊎ E, hence M = B ⊎ (D ⊎ E). By maximally of B in G, m ∈ B ⋃ D and m ∈ B ⋃ E, then m ∈ E ⋂ D, which contradicts E ⋂ D = 0.

Proposition. (3.7). Let S be an α-moniod. Then the following statements are equivalent

1. Any α-unitary S_Γ-act M is Γ- completely reducible.
2. S is a Γ- completely reducible S_Γ-act.

Proof. (1) → (2) it is clear (2) → (1). Let S = $\biguplus_{i\in I} V_i$ where V_i is simple S_Γ-subact of S for all i ∈ I. Since M = $\cup_{m\in M}$ mαS, then M = $\cup_{m\in M}$ m($\biguplus_{i\in I} V_i$) = $\cup_{m\in M} \biguplus_{i\in I}(m\alpha V_i)$. It is enough show that $m\alpha V_i$ is simple S_Γ-subact of M for all $m \in M$ and $i \in I$. If $m\alpha V_i$ is not simple, then there is a nonzero S_Γ-subact L of $m\alpha V_i$, hence there exists a nonzero element $\bar{v} \in V_i$ such that $m\alpha\bar{v} \notin L$, but $m\alpha\bar{v} \in m\alpha V_i$. Consider the following subset T of V_i where $T = \{v \in V_i \mid m\alpha v \in L\}$. Then clearly that T is S_Γ-subact of V_i and $T \neq V_i$, since $\bar{v} \notin T$. Which contradicts the simplicity of V_i.

Corollary. (3.8). Let M be an S_Γ-act. Then M is a Γ- completely reducible if and only if any S_Γ-subact N of M is Γ-direct summand.

We will introduce another class of S_Γ-subacts which contains that of Γ-essential S_Γ-subacts.

Definition. (3.9). Let M be an S_Γ-act. An S_Γ-subact N of M is called Γ – meet large in M if for any nonzero S_Γ-subact L of M implies that $N \cap L$ is nonzero. If this is the case we use the notions Γ-⋂-large, and write $N \subseteq_{\Gamma-\cap-large} M$. If N is Γ-⋂-large in M, then M is called gamma meet extension of N.

The following propositions give properties of Γ-⋂-large S_Γ-subacts.

Proposition. (3.10). Let M be S_Γ-act, and N, T be two S_Γ-subacts of M. Then

(1). If $N, T \subseteq_{\Gamma-\cap-large} M$ then $N \cap T \subseteq_{\Gamma-\cap-large} M$, in particular the finite intersection of Γ-⋂-large S_Γ-subacts of M is Γ-⋂-large.

(2). Let $N \leq T \leq M$. Then $N \subseteq_{\Gamma-\cap-large} T \subseteq_{\Gamma-\cap-large} M$ if and only if $N \subseteq_{\Gamma-\cap-large} T \subseteq_{\Gamma-\cap-large} M$.

(3). Let $f \in \mathrm{Hom}_{S_\Gamma}(M, L)$ and B be a Γ-⋂-large in L. Then $f^{-1}(B)$ is Γ-⋂-large in M.

Proof. 1. Let A be a nonzero S_Γ-subact of M, then $A \cap N$ is nonzero S_Γ-subact of M and hence $(A \cap N) \cap T$ is nonzero S_Γ-subact of M.

2. Let $N \subseteq_{\Gamma-\cap-large} T \subseteq_{\Gamma-\cap-large} M$, and H a nonzero S_Γ-subact of M. Then $N \cap H = (N \cap T) \cap H = N \cap (T \cap H)$ is nonzero S_Γ-subact in M. Conversely, let H be any nonzero S_Γ-subact of M, then $N \cap H \subseteq T \cap H$ and $N \cap H$ is nonzero S_Γ-subact, implies that $T \cap H$ is nonzero S_Γ-subact of M.

3. without loss of generality we can assume that f is nonzero S_Γ-homomorphism. Let H be any S_Γ-subact of M with $f^{-1}(B) \cap H = 0_M$. Then $B \cap f(H) = 0_L$, but B is Γ-⋂-large in L, then $f(H) = 0_L$ and hence $H \subseteq f^{-1}(B)$.

Let M be an S_Γ-act and x, y, a, b ∈ M. Then the two elements x and y are`called drivable by the two elements a and b, if there exist elements $s_1, s_2 \in S$, and $\alpha_1, \alpha_2 \in \Gamma$ such that $x = a\alpha_1 s_1$ and $b\alpha_2 s_2 = y$

Theorem. (3.11). Let M be an S_Γ-act. If N is Γ-essential S_Γ-subact of M, then N is Γ-⋂-large in M.

Proof. Assume that N is a Γ-essential S_Γ-subact of M, and A a nonzero S_Γ-subact of M. For any distant elements a, b ∈ A. Define a relation ρ on M by ρ = $W_1 \cup I_M \cup W_2$ where W_1 = {(x, y) ∈ M × M | x and y are drivable by a and b} and W_2 = {(a, b), (b, a)}. Clearly that ρ is S_Γ-congruence on M, and since $\rho \neq I_M$. Then $\rho|_N \neq I_N$, there exists (m, n) ∈ $\rho|_N$ such that m, n ∈ N and m ≠ n. Thus either (m, n) ∈ W_1 or (m, n) ∈ W_2. Since A is an S_Γ-subact of M, then in either cases we have A ∩ N is nonzero

Corollary. (3.12). Let M be S_Γ-act and N be Γ-∩-large S_Γ-subact in M. Then for all $\alpha \in \Gamma$ and m ∈ M, $m^{-1}\alpha N = \{s \in S \mid m\alpha s \in N\}$ is Γ-∩-large S_Γ-subact in S.

Let M be S_Γ-act and $\alpha \in \Gamma$. We define the following relation on M by

Ψ_M^α = {(m, n) ∈ M × M | there exists Γ-∩-large S_Γ-subact H in S such that $m\alpha h = n\alpha h$ for all h ∈ H}.

Ψ_M^α is called α-singular relation on M. If M is a Γ-commute α-unitary, then Ψ_M^α is S_Γ-congruence on M. The proof of the following proposition is trivial, so we omitted.

Proposition. (3.13). Let M be and S_Γ-act and N be an S_Γ-subact of M. If $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$, then Ψ_N^α (= $(\Psi_M^\alpha)|_N$)= I_N.

For the converse of Proposition**.** (2.13) we have the following.

Proposition. (3.14). If N is Γ-essential S_Γ-subact of M and $\Psi_N^\alpha = I_N$ for some $\alpha \in \Gamma$, then Ψ_M^α = I_M.

Proof. Let M be an S_Γ-act and N be an S_Γ-subact of N with and $\Psi_N^\alpha = I_N$ for some $\alpha \in \Gamma$. Now if N is Γ-essential of M such that $\Psi_M^\alpha \neq I_M$ then we have a contradiction

Definition. (3.15). Let M be an S_Γ-act**.** An S_Γ-subact N of M is called Γ-invariant in M if f(N) ⊆ N for all f ∈ End_{S_Γ}(M).

The following proposition gives a characterization of Γ- invariant S_Γ-subact.

Proposition. (3.16). Let M be an S_Γ-act. The following statements are equivalent for an S_Γ-subact N of M.

(1). N is Γ-invariant of M.

(2). N is $(\mathrm{End}_{S_\Gamma}(M) - S)_\Gamma$ – subbiact.

Proof. (1) → (2). Let f ∈ End_{S_Γ}(M), n ∈ N, and $\beta \in \Gamma$. Then $f\beta n = f(n\beta 1_\alpha) \in N$. (2) → (1). Let f ∈ End_{S_Γ}(M) and n ∈ N then $f(n) = f(n\alpha 1_\alpha) = f\alpha n \in N$.

Lemma. (3.17). Let f and g ∈ $End_{S_\Gamma}(M)$ and f, g be coincide one some Γ-∩-large S_Γ-subact of M. If $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$, then f = g.

Proof. Let f, g : M → M be S_Γ-homomorphism with f(n) = g(n) for all n ∈ N and N is Γ-∩-large S_Γ-subact of M. Let m ∈ M then $m^{-1}\alpha N$ is Γ-∩-large S_Γ-subact of S_Γ-act S. By using **corollary (3.12)** and $f(m)\alpha s = f(m\alpha s) = g(m\alpha s) = g(m)\alpha s$ for all $s \in m^{-1}\alpha N$ imply that $(g(m), f(m)) \in \Psi_M^\alpha = I_M$ and hence g(m) = f(m).

4. Quasi – injective gamma act

In the following we investigate a generalization of injectivity which contains both injective and weakly – injective gamma act, we will study the most of their properties.

Definition. (4.1). Let M be an S_Γ-act. M is called quasi – injective, if for any S_Γ-subact N of M and any f ∈ $Hom_{S_\Gamma}(N, M)$, there exists an extension $\bar{f}$ ∈ $End_{S_\Gamma}(M)$ to f, that is $\bar{f}i_N = f$, where i_N is the inclusion mapping of N into M.

Example. (4.2).

1. It is clear from the definition that any injective gamma act is quasi – injective.
2. Z_6 is a quasi – injective Z_N- act with multiplication mapping, but not injective (we will see this later).
3. Q as Z_N- act with multiplication mapping is quasi – injective
4. Let S = {0 , a, b, 1} and Γ = {α, β}, Consider the multiplication tables.

α	1	a	b	0
1	1	a	b	0
a	a	a	a	0
b	b	b	b	0
0	0	0	0	0

β	1	a	b	0
1	0	0	0	0
a	0	0	0	0
b	0	0	0	0
0	0	0	0	0

Then S is quasi – injective S_Γ-act. {0}, {0, a}, {0, b}, {0, a, b}| and S itself are the only S_Γ-subacts of S

5. Z_4 is Z_N – act with multiplication mapping is quasi – injective.

Lemma. (4.3). Let M be an S_Γ-act. If M is Γ- invariant S_Γ-subact in E(M), then M is quasi – injective .

Proof. Let N be an S_Γ-subact of M and f ∈ $Hom_{S_\Gamma}(N, M)$. By injectivtly of E(M), there is $\bar{f}$ ∈ $End_{S_\Gamma}(E(M))$**,** but $\bar{f}(M) \subseteq M$, so $\bar{f}\,|_M$ is an extension of f.

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For the converse we have the following.

Proposition. (4.4). Let M be a quasi – injective S_Γ-act with $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$. Then M is Γ- invariant S_Γ-subact of E(M).

Proof. Let h ∈ End_{S_Γ}(E(M))**.** Since M is Γ-essential S_Γ-subact of E(M), then by theorem (2.11) M is Γ-∩-large in E(M), hence $h^{-1}(M) \cap M$ is Γ-∩-large S_Γ-subact of E(M), proposition. (2.10)**.** Let N = $h^{-1}(M) \cap M$, and define a : N ⟶ M by a(x) = h(x) for all x ∈ N. Quasi – injectivity of M implies that there exists b ∈ End_{S_Γ}(M) to a such that a(n) = b(n) for all n ∈ N. Now by injectivtly of E(M) there exists an extension c ∈ End_{S_Γ}(E(M)) such that c(x) = b (x) for all x ∈ M, hence c(n) = b(n) = a(n) = h(n) for all n ∈ N. Since $\Psi_M^\alpha = I_M$, then by proposition. (2.14) we get that $\Psi_{E(M)}^\alpha = I_{E(M)}$. By the help of lemma (2.17) we get that c = b and so h(M) = c(M) ⊆ M. This shows that M is Γ- invariant S_Γ-subact of E(M)

Theorem. (4.5). Let M be an S_Γ-act for which $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$. Then the following are equivalent.

1. M is quasi – injective.
2. $End_{S_\Gamma}(M) \cong End_{S_\Gamma}(E(M))$ as Γ-semigroup.

Proof. (2) ⟶ (1). We can consider M is $(End_{S_\Gamma}(E(M))\text{-}S)_\Gamma$ – subbiact of E(M), implies that M is Γ- invariant S_Γ-subact of E(M). Thus M is Quasi – injective, lemma (4.4) **(1) ⟶ (2).** Define φ : $End_{S_\Gamma}(M) \longrightarrow End_{S_\Gamma}(E(M))$ as follows : for each $\alpha \in End_{S_\Gamma}(M)$, by injectivity of E(M) there exists $\bar{\alpha} \in End_{S_\Gamma}(E(M))$, put $\varphi(\alpha) = \bar{\alpha}$. Let f, g ∈ End_{S_Γ}(M) and f = g. Then by injectivty of E(M) there exist $\bar{f}$ and $\bar{g} \in End_{S_\Gamma}(E(M))$ which is extension of f, g respectively. lemma (3.17) implies that $\bar{f} = \bar{g}$. For f, g ∈ End_{S_Γ}(M) and $\gamma \in \Gamma$, $(f\gamma g) = \overline{f\gamma g} = \bar{f}\gamma\bar{g}$ = $(f)\gamma\varphi(g)$, and hence φ is Γ-homomorphism. To show that φ is onto. Let g ∈ End_{S_Γ}(E(M). Then by proposition. (4.4) M is Γ – invariant in E(M), and hence $g|_M \in End_{S_\Gamma}(M)$ with $\varphi(g|_M) = g$. It is clear that φ is injective.

Proposition. (4.6). A Γ- retract of quasi – injective gamma act is quasi – injective.

Proof. Let M be a quasi – injective S_Γ-act and N be a Γ-retract of M. Then there exist an S_Γ-homomorphisms g : M ⟶ N and f : N ⟶ M with gf = I_N. Let L be an S_Γ-subact of N and $\alpha \in Hom_{S_\Gamma}$**(**L, N). By quasi – injectivtly of M there exists an extension $\bar{\alpha} \in End_{S_\Gamma}(M)$ of α. Define $\bar{\bar{\alpha}} \in End_{S_\Gamma}(N)$ by $\bar{\bar{\alpha}} = g\bar{\alpha}i_N$ where i_N is the inclusion mapping of N into M. This implies that $\bar{\bar{\alpha}}i_L = (g\bar{\alpha}i_N)i_L = g\bar{\alpha}(i_N i_L) = g(\bar{\alpha}i_L) = g(f\alpha) = (gf)\alpha = I_N\alpha = \alpha$

Corollary. (4.7). Let $\{M_i \mid i \in I\}$ be a family of S_Γ-acts. If $\prod_{i\in I} M_i$ is quasi – injective S_Γ-act, then M_i is quasi – injective for all i ∈ I.

Proof. It is straightforward since M_i is Γ- retract of $\prod_{i\in I} M_i$ for all i ∈ I

Corollary. (4.8). Any Γ-direct summand S_Γ-subact of quasi – injective S_Γ-act is quasi injective, and hence every Γ- completely reducible S_Γ-act is quasi - injective

It is well - known that every gamma act has a gamma injective envelope which is unique up to isomorphism. It is a natural question is arising, for every gamma act is there a quasi – injective envelope? Let M be a α – unitary S_Γ-act with H = End_{S_Γ}(E(M)). We define the following set

$$H \times \Gamma \times M = \{h\beta m \in E(M) \mid h\in H, \beta \in \Gamma\ m \in M\}.$$
$$= \{h(m\beta 1_\alpha) \in E(M) \mid h \in H, \beta \in \Gamma\ m \in M\}.$$

Clearly that M is Γ- invariant S_Γ-subact of H × Γ × M, by using lemma. (3.4) we have the following

Proposition. (4.9). Let M be an S_Γ-act. Then H × Γ × M is quasi – injective S_Γ-subact of E(M) which Γ- essential extension of M.

Let M be an S_Γ-act. In the following theorem we investigate an some condition under which the set H × Γ × M is the smallest quasi – injective S_Γ-act which contains M.

Theorem. (4.10). Let M be an S_Γ-act and $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$. Then H × Γ × M = $\bigcap_{B\in\Omega} B$ where Ω = {B ≤ E(M) | B is quasi – injective S_Γ-act contain M}.

Proof. Let P ∈ Ω. We will show that H × Γ × M ⊆ H × Γ × P ⊆ P. Let f ∈ H, p ∈ P, and $\beta \in$ Γ. M is Γ-⋂-large S_Γ-subact in E(M) and M ≤ P ≤ E(M), then P is Γ-⋂-large in E(M) by proposition(2.10) and 0 ≠ P ⋂ f^{-1}(P) is Γ-⋂-large S_Γ-subact in E(M). Consider the mapping : P ⋂ f^{-1}(P) ⟶ P which define by φ(x) = f(x) for all x ∈ P ⋂ f^{-1}(P). By quasi – injectivitly of P, there exists $\bar{\varphi} \in \mathrm{End}_{S_\Gamma}$(P) such that $\bar{\varphi}$ extension of φ. Injectivity of E(M) implies that there exists $\breve{\varphi}$ ∈ H which extends $\bar{\varphi}$, thus $\breve{\varphi}$(P) ≤ P and $\breve{\varphi}(x) = \bar{\varphi}(x) = \varphi(x)$ = f(x) for all x ∈ P ⋂ f^{-1}(P). lemma (3.17) implies that f = $\breve{\varphi}$ and hence fβp ∈ P

Let M be an S_Γ-act, m ∈ M and $\gamma \in \Gamma$. Consider the following subset of S × S denoted by R_s^γ(m) is defined by R_s^γ(m) = {(s, t) ∈ S × S | mγs = mγt}. R_s^γ(m) is called the right annihilator of m with respect to γ in S. Let M be an S_Γ-act. For each right Γ-ideal I of S. Define the set

$$\Omega(M) = \{f : I \longrightarrow M \mid \text{there are } m \in M \text{ and } \gamma \in \Gamma \text{ with } R_s^\gamma(m)\ |_I \subseteq \ker(f)\}, \text{ where } R_s^\gamma(m)\ |_I = R_s^\gamma(m) \cap (I \times I).$$

The following theorem gives a characterization of quasi – injective S_Γ-acts which analogist to that of quasi – injective modules.

Theorem. (4.11). An α- unitary S_Γ-act M is quasi – injective if and only if for any right Γ-ideal I of S, and S_Γ-homomorphism f : I ⟶ M with f ∈ Ω(M), there exists an extension S_Γ-homomorphism g : S ⟶ M of f.

Proof. Let I be right Γ-ideal of Γ-semigroup S, and f : I ⟶ M be an S_Γ-homomorphism such that $f \in \Omega(M)$. Then there exists $m \in M$, $\gamma \in \Gamma$ such that $R_s^\gamma(m)\ |_I \subseteq \ker(f)$. Put $N = m\gamma I$. Then N is an S_Γ-subact of M. Define the mapping $g : N \to M$ by $g(m\gamma i) = f(i)$ for all $m\gamma i \in N$. g is well-defined, for let $m\gamma i_1 = m\gamma i_2 \in N$ then $(i_1, i_2) \in R_s^\gamma(m) \cap I \times I \subseteq \ker(f)$ implies that $f(i_1) = f(i_2)$ and it is clear that g is an S_Γ-homomorphism. By quasi- injectivity of M, there exist extension $\bar{g} : M \to M$ of g. Define $\bar{f} : S \to M$ by $\bar{f}(s) = \bar{g}(m\gamma s)$ for all $s \in S$. Clearly $\bar{f}$ is an S_Γ-homomorphism and it is an extension to f. Conversely. Let N be an S_Γ-subact of M and $f : N \to M$ be an S_Γ-homomorphism. Define

$$\mho = \{(C, h) \mid N \leq C \leq M \text{ and } h : C \to M \text{ is an } S_\Gamma\text{-homomorphism with } h|_N = f\}.$$

Then ℧ is nonempty set and ordered the set ℧ by $(C_1, f_1) \leq (C_2, f_2)$ if and only if $C1 \leq C2$ and $f_2|_{C1} = f_1$. Zorn's lemma Implies that ℧ has a maximal element (C_0,f_0) say. If $C_0 = M$, then the proof is complete. Otherwise, let $m_0 \in M$ and $m_0 \notin C_0$. Consider the right Γ-ideal $I = \{s \in S \mid m_0\alpha s \in C_0\}$. Define $g : I \to M$ by $g(i) = f_0(m_0\alpha i)$ for all $i \in I$. Let $(s_1, s_2) \in R_s^\alpha(m_0) \cap I \times I$. Then $m_0\alpha s_1 = m_0\alpha s_2$ and hence $f_0(m_0\alpha s_1) = f_0(m_0\alpha s_2)$ so $g(s_1) = g(s_2)$. Thus $R_s^\alpha(m_0)\ |_I \subseteq \ker(g)$ and hence $g \in \Omega(M)$. The hypothesis implies that there exists $\bar{g} : S \to M$ such that $\bar{g}\ |_I = g$. Now define an S_Γ-homomorphism $\overline{f_0} : C_0 \cup m_0\alpha S \to M$ by $\overline{f_0}(c) = f_0(c)$ for all $c \in C_0$ and $\overline{f_0}(m_0\alpha s) = \bar{g}(s)$ for all $s \in S$. To show $\overline{f_0}$ is well-defined, let $c = m_0\alpha s \in C_0 \cap m_0\alpha S$, then $s \in I$ implies that $\overline{f_0}(m_0\alpha s) = \bar{g}(s) = g(s) = f_0(m_0\alpha s) = f_0(c) = \overline{f_0}(c)$, thus $\overline{f_0}$ is well-defined and it is an easy matter to see that $\overline{f_0}$ is S_Γ-homomorphism, so we have $(C_0,f_0) \leq (C_0 \cup m_0\alpha S, \overline{f_0})$ which contradicts the maximally of (C_0,f_0) in ℧

Corollary. (4.12). Every weakly - injective S_Γ-act is quasi- injective

The converse of above corollary may not be true in general as in the following example.

Example. (4.13). Let S = {1, a, b, 0} and Γ = { , β} and consider the following tables.

α	1	a	b	0
1	1	a	b	0
a	a	a	a	0
b	b	b	b	0
0	0	0	0	0

β	1	a	b	0
1	0	0	0	0
a	0	0	0	0
b	0	0	0	0
0	0	0	0	0

Then S is α – monoid. Let M = {z, x, c} and consider the mapping M × Γ × S ⟶ M which define by the tables

α	1	a	b	0
x	x	x	x	c
z	z	z	z	c
c	c	c	c	c

β	1	a	b	0
x	c	c	c	c
z	c	c	c	c
c	c	c	c	c

Then M is an α – unitary S_Γ-act. Let I = {0, a, b}. Then I is a right Γ – ideal of S. Consider the mappiong f : I ⟶ M which define by f(a) = z, f(b) = x, and f(0) = c. f is an S_Γ-homomorphism which has no extension, so M is not weakly - injective S_Γ-act, but it is easy to see M is quasi injective, since N_1 = {c}, N_2 = { c, x} and N_3 = {c, z} are the only nontrivial S_Γ-subacts of M, and any S_Γ-homomorphism form N_1, N_2, and N_3 to M has an extension.

For the converse, we consider the following concepts.

Definition. (4.14). Let M be an S_Γ-act, ∈ Γ. Then M is called

(1). α – faithful. If for all nonzero m ∈ M and t_1 , t_2 ∈ S, $m\alpha t_1 = m\alpha t_2$ implies that $t_1 = t_2$.

(2). Γ- faithful. If for all nonzero m ∈ M, $\beta \in \Gamma$ and $t_1, t_2 \in S$, $m\beta t_1 = m\beta t_1$ implies that $t_1 = t_2$.

(3). Strongly α - faithful. If there exists nonzero m ∈ M, $m\alpha t_1 = m\alpha t_1$ implies that $t_1 = t_2$.

Theorem. (4.15). Let M be a quasi – injective S_Γ-act. If M is strongly α –faithful for some α ∈ Γ, then M is weakly – injective.

Proof. Let I be a Γ – ideal of S and f : I ⟶ M be a nonzero S_Γ-homomorphism. Then there exists a nonzero h = f(i) ∈ M for some i ∈ I. Since M is strongly α –faithful, then $R_S^\alpha(h) = I_S$, so $R_S^\alpha(h)\mid_{\mathbf{I}} \subseteq \ker(f)$ and hence f ∈ Ω(M). By quasi – injectivity of M there exists an extension S_Γ-homomorphism g : S ⟶ M of f, **Theorem. (3.11)**

Proposition. (4.16). Let $\{M_i \mid i \in I\}$ be family of S_Γ-acts and $\coprod_{i\in I} M_i$ is quasi – injective. Then M_i is quasi – injective for all i ∈ I.

Proof. Let I be a right Γ-ideal of S and f : I ⟶ M_i an S_Γ-homomorphism such that f ∈ Ω(M), that is, there exists m ∈ M_i and $\gamma \in \Gamma$ such that $R_S^\gamma(m)\mid_{\mathbf{I}} \subseteq \ker(f)$. By definition of $\coprod_{i\in I} M_i$, m ∈ $\coprod_{i\in I} M_i$. Then quasi – injectivity of $\coprod_{i\in I} M_i$ implies that exists an S_Γ-homomorphism g : S ⟶ $\coprod_{i\in I} M_i$ such that $g\mid_I = i_{M_i} f$, where i_{Mi} is the injection mapping of M_i into $\coprod_{i\in I} M_i$. Consider the mapping h : S ⟶ M_i defined by h = $P_{M_i}g$, hence h is an extension of f, where P_{M_i} is the projection mapping from $\coprod_{i\in I} M_i$ onto M_i .

Proposition. (4.17). Let M be a quasi – injective S_Γ-act and $\Psi_M^\alpha = I_M$ for some $\alpha \in \Gamma$. If E(M) = $\uplus_{j\in I} E_j$. Then M = $\uplus_{j\in I} (M \cap E_j)$.

Proof. By Proposition. (3.4) M is Γ- invariant S_Γ-subact of E(M) and hence for each i ∈ J the projection S_Γ-endomorphism P_i : E(M) → E_i ⊆ E(M) is satisfy P_i(M) ⊆ M. That is mean M∩ E_i ⊆ M for all i ∈ I, and hence M = $\uplus_{j\in I}$ (M∩E_j)

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