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# A model for incompressible fluids using finite element methods for the Brinkman problem

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# A model for incompressible fluids using finite element methods for the Brinkman problem

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**Abstract.** The partial differential equations for fluid flow dynamics based on the Brinkman equations, written in terms of velocity-vorticity and pressure in three dimensions, are essential for predicting climate, ocean currents, water flow in a pipe, the study of blood flow and any phenomenon involving incompressible fluids through porous media; having a significant impact in areas such as oceanographic engineering and biomedical sciences. This paper aims to study the Brinkman equations with homogeneous Dirichlet boundary are studied, the existence and uniqueness of solution at a continuous level through equivalence of problems is presented. It is discretized to approximate the solution using Nédélec finite elements and piecewise continuous polynomials to approximate vorticity and pressure. The velocity field is recovered, obtaining its a priori error estimation and order of convergence. As a result, ensuring a single prediction of the flow behavior of an incompressible fluid through porous media. Finally, a numerical example in 2D with the standard $L^2$ is presented, confirming the theoretical analysis.

## 1. Introduction

The Brinkman equations, which arise from a temporal discretization of the Navier-Stokes equations [1], provide an accurate way to calculate how fluids move, a key feature in countless scientific and technological problems, such as studying airflow around vehicles, aircraft, or projectiles without having to resort to costly wind tunnel testing. Furthermore, these equations govern the atmosphere of Earth, ocean currents, and any phenomenon involving Newtonian fluids [2, 3].

There are many numerical methods to solve the Brinkman equations; one technique is the finite element method [4–10]. The development of appropriate finite elements to simulate viscous and incompressible fluid flows through porous media can be seen in [11]. In [12,13], the Brinkman problem is in terms of velocity, vorticity and pressure. In the articles [14–18], the authors study the generalization of the Navier-Stokes equations, besides the Brinkman equations have been studied with homogeneous Dirichlet boundary with mixed variational formulation [19–21] and at discrete level [3].

This paper aims to complement the results of [3] by studying the Brinkman equations with homogeneous Dirichlet boundary, with a variational formulation in terms of vorticity and pressure at a continuous level. Therefore the novelty of the paper is the demonstration of the existence and uniqueness of the solution by the equivalence of problems; this allows ensuring a unique prediction of the flow behavior of an incompressible fluid through porous media. Finally, the purpose is to approximate the solution using Nédélec finite elements and continuous polynomials to approximate vorticity and pressure and then recover the velocity field.

## 2. Mathematical model

Let $\Omega \subseteq \mathbb{R}^d$ for $d = 2, 3$ be a bounded open set with Lipschitz boundary, $\boldsymbol{u}$ the velocity field, $p$ the pressure y $\boldsymbol{f}$ the external forces, $\nu > 0$ the kinematic viscosity of the fluid and $\kappa > 0$ the permeability of a porous medium. As the movement of a continuous medium is governed by the fundamental principles of classical mechanics and thermodynamics for the conservation of mass, we arrive at Equation (1), known as the Navier-Stokes equation, as shown in [1].

$$\rho \left( \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) + \nabla p - \mu \Delta \boldsymbol{u} = \boldsymbol{f}. \tag{1}$$

In the case of an incompressible fluid, this means $\operatorname{div} \boldsymbol{u} = 0$. From a temporal discretization of Equation (1), and disregarding the convective term, the Equation (2) known as the Brinkman equation is obtained, as shown in [1].

$$\kappa^{-1}\boldsymbol{u} + \nabla p - \mu \Delta \boldsymbol{u} = \boldsymbol{f}. \tag{2}$$

From Equation (1) and of Equation (2), the Brinkman model problem with homogeneous Dirichlet boundary in terms of velocity and pressure is presented as shown in Equation (3).

$$\begin{aligned} \kappa^{-1}\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{over} \quad \Gamma. \end{aligned} \tag{3}$$

To uniquely determine $p$, we can impose some additional condition, such $\int_\Omega p = 0$.

### 2.1. *Variational formulation of Brinkman problem in terms of velocity and pressure*

Multiplying by a suitable test function and integrating on $\Omega$ of the first equality of the Equation (3). In particular, for $\boldsymbol{v} \in [\mathrm{H}_0^1(\Omega)]^d$ it follows Equation (4).

$$\kappa^{-1} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} + \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in [\mathrm{H}_0^1(\Omega)]^3. \tag{4}$$

Now, multiplying the second equality of Equation (3) by a suitable test function $q \in \mathrm{L}_0^2(\Omega)$, it follows Equation (5).

$$\int_\Omega q \operatorname{div} \boldsymbol{u} = 0 \quad \forall q \in \mathrm{L}_0^2(\Omega). \tag{5}$$

Then, from Equation (4) and Equation (5) the following mixed variational formulation is obtained as observed in Equation (6). Find $(\boldsymbol{u}, p) \in [\mathrm{H}_0^1(\Omega)]^3 \times \mathrm{L}_0^2(\Omega)$, such that is expressed by Equation (6).

$$\begin{gathered} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in [\mathrm{H}_0^1(\Omega)]^3, \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in \mathrm{L}_0^2(\Omega), \end{gathered} \tag{6}$$

where the bilinear form $a(\boldsymbol{u}, \boldsymbol{v}) : [\mathrm{H}_0^1(\Omega)]^3 \times [\mathrm{H}_0^1(\Omega)]^3 \to \mathbb{R}$ is defined by $a(\boldsymbol{u}, \boldsymbol{v}) := \kappa^{-1} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} + \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}$, and the bilinear form $b(\boldsymbol{u}, q) : [\mathrm{H}_0^1(\Omega)]^3 \times \mathrm{L}_0^2(\Omega) \to \mathbb{R}$ is defined by $b(\boldsymbol{u}, q) := -\int_\Omega q \operatorname{div} \boldsymbol{u}$. The functional $F(\boldsymbol{v}) : [\mathrm{H}_0^1(\Omega)]^3 \to \mathbb{R}$ is defined by $F(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}$.

*2.2. Existence and uniqueness of solution*

For the existence and uniqueness of the solution, the results presented in [22] are used and for Stokes problem and classical theory of Babuška-Brezzi the results presented in [23] are used. The problem represented by Equation (3) at continuous level is said to have a single solution $(\boldsymbol{u}, p) \in [\mathrm{H}_0^1(\Omega)]^3 \times \mathrm{L}_0^2(\Omega)$. Furthermore, $\|\boldsymbol{u}\|_{1,\Omega} + \|p\|_{0,\Omega} \leq \|\boldsymbol{f}\|_{0,\Omega}$.

*2.3. Brinkman problem in terms of velocity-vorticity-pressure*

As obtained in Equation (3), a further variable called vorticity $\boldsymbol{\omega}$ is defined: the velocity rotation, *i.e.*, $\boldsymbol{\omega} = \sqrt{\nu}\mathbf{curl}\,\boldsymbol{u}$. The Brinkman problem is presented in terms of velocity, vorticity and pressure, as shown in Equation (7) as mentioned in [3].

$$\begin{aligned} \kappa^{-1}\boldsymbol{u} + \sqrt{\nu}\mathbf{curl}\,\boldsymbol{\omega} + \nabla p &= \boldsymbol{f} && \text{in} \quad \Omega, \\ \boldsymbol{\omega} - \sqrt{\nu}\mathbf{curl}\,\boldsymbol{u} &= \mathbf{0} && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in} \quad \Omega, \\ \boldsymbol{u} &= \mathbf{0} && \text{over} \quad \Gamma, \end{aligned} \tag{7}$$

where, $\nu > 0$ is the kinematic viscosity of the fluid and $\kappa > 0$ is the permeability of a porous medium.

*2.4. Variational formulation of the Brinkman problem in terms of vorticity-pressure*

The variational formulation of Equation (7) is introduced below. Multiplying by a suitable test function $\boldsymbol{\theta} \in \mathbf{Z} := \{\boldsymbol{\theta} \in [\mathrm{L}^2(\Omega)]^3 : \mathbf{curl}\,\boldsymbol{\theta} \in [\mathrm{L}^2(\Omega)]^3\}$ and integrating by parts in the second equality of Equation (7), using the boundary condition $\boldsymbol{u} = \mathbf{0}$ over $\Gamma$, the Equation (8) is obtained.

$$\kappa^{-1}\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} + \nu\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\mathbf{curl}\,\boldsymbol{\theta} + \sqrt{\nu}\int_\Omega \nabla p\cdot\mathbf{curl}\,\boldsymbol{\theta} = \sqrt{\nu}\int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\,\boldsymbol{\theta}. \tag{8}$$

Then, multiplying by $\nabla q$ the first the first equality of Equation (7), and integrating by parts by a suitable test function $q \in \mathrm{Q} := \mathrm{H}^1(\Omega) \cap \mathrm{L}_0^2(\Omega)$, considering the boundary conditions and making use of the third condition of the Equation (7), the following Equation (9) is obtained.

$$\sqrt{\nu}\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\nabla q + \int_\Omega \nabla p\cdot\nabla q = \int_\Omega \boldsymbol{f}\cdot\nabla q \quad \forall q \in \mathrm{Q}. \tag{9}$$

Adding Equation (8) and Equation (9), the following variational formulation, Equation (10) is obtained; find $(\boldsymbol{\omega}, p) \in \mathbf{Z} \times \mathrm{Q}$ such that.

$$\mathcal{A}((\boldsymbol{\omega}, p), (\boldsymbol{\theta}, q)) = \mathbf{F}(\boldsymbol{\theta}, q) \qquad \forall(\boldsymbol{\theta}, q) \in \mathbf{Z} \times \mathrm{Q}, \tag{10}$$

where the bilinear form $\mathcal{A} : (\mathbf{Z}\times\mathrm{Q})\times(\mathbf{Z}\times\mathrm{Q}) \to \mathbb{R}$ is defined by $\mathcal{A}((\boldsymbol{\omega}, p), (\boldsymbol{\theta}, q)) := \kappa^{-1}\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} + \nu\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\mathbf{curl}\,\boldsymbol{\theta} + \sqrt{\nu}\int_\Omega \nabla p\cdot\mathbf{curl}\,\boldsymbol{\theta} + \sqrt{\nu}\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\nabla q + \int_\Omega \nabla p\cdot\nabla q \quad \forall(\boldsymbol{\theta}, q) \in \mathbf{Z}\times\mathrm{Q}$ and the linear functional $F : \mathbf{Z}\times\mathrm{Q} \to \mathbb{R}$ is defined by $F(\boldsymbol{\theta}, q) := \sqrt{\nu}\int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\,\boldsymbol{\theta} + \int_\Omega \boldsymbol{f}\cdot\nabla q \quad \forall(\boldsymbol{\theta}, q) \in \mathbf{Z}\times\mathrm{Q}$.

*2.5. Existence and uniqueness of solution*

In this section, the existence and uniqueness of the solution of Equation (10) is established; to analyze the existence of a solution, the equivalence of problems is used and the uniqueness is the consequence of the following Lemma 1.

*Lemma 1.* Let $(\boldsymbol{\omega}, p) \in \mathbf{Z}\times\mathrm{Q}$ be such that, $\mathcal{A}((\boldsymbol{\omega}, p), (\boldsymbol{\theta}, q)) = 0 \quad \forall(\boldsymbol{\theta}, q) \in \mathbf{Z}\times\mathrm{Q}$. Then $(\boldsymbol{\omega}, p) = (\mathbf{0}, 0)$.

*Proof.* Taking $(\boldsymbol{\theta}, q) = (\boldsymbol{\omega}, p)$ as a test function the following Equation (11) is obtained.

$$\|\boldsymbol{\omega}\|_0^2 + \|\sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{\omega} + \nabla p\|_0^2 = 0. \tag{11}$$

The Equation (11) implies that $\boldsymbol{\omega} = \mathbf{0}$ in $\Omega$ and $p = 0$ in $\Omega$. As mentioned above, for the existence of the solution, the equivalence of problems from Equation (3) and Equation (10) will be used.

*Theorem 1.* The problems of Equation (3) and Equation (10) are equivalent.

*Proof.* To demonstrate that Equation (3) and Equation (10) are equivalent, the following implications are demonstrated: first Equation (3) $\Longrightarrow$ Equation (7) $\Longrightarrow$ Equation (10), this implication is trivial since it is only write the Equation (3) in terms of Equation (7) and testing with adequate functions the variational formulation of Equation (10) is obtained. Second, to demonstrate that Equation (10) implies Equation (3), the Equation (10) is tested with $(\boldsymbol{\theta}, 0)$ and with $(\mathbf{0}, q)$, obtaining the Equation (12) and Equation (13).

$$\kappa^{-1}\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} + \nu\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\mathbf{curl}\,\boldsymbol{\theta} + \sqrt{\nu}\int_\Omega \nabla p\cdot\mathbf{curl}\,\boldsymbol{\theta} = \sqrt{\nu}\int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\,\boldsymbol{\theta} \quad \forall\boldsymbol{\theta}\in\mathbf{Z}, \tag{12}$$

and

$$\sqrt{\nu}\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\nabla q + \int_\Omega \nabla p\cdot\nabla q = \int_\Omega \boldsymbol{f}\cdot\nabla q \quad \forall q\in \mathrm{Q}, \tag{13}$$

defining $\boldsymbol{u} := \kappa\boldsymbol{f} - \sqrt{\nu}\kappa\,\mathbf{curl}\,\boldsymbol{\omega} - \kappa\nabla p \in [\mathrm{L}^2(\Omega)]^3$. On the other hand, $X := \{\boldsymbol{\varphi}\in \mathrm{H}_0(\mathrm{div}\,;\Omega) : \mathrm{div}\,\boldsymbol{\varphi} = 0\} \subset \mathrm{H}_0(\mathrm{div}\,;\Omega)$ is a closed subspace of $[\mathrm{L}^2(\Omega)]^3$, then the following decomposition $[\mathrm{L}^2(\Omega)]^3 := X \oplus X^\perp$ where $X^\perp := \{\nabla q : q\in \mathrm{H}^1(\Omega)\}$ is obtained (see [22]; Theorem 2.7), then of Equation (13) and replacing $\boldsymbol{u}$, $\int_\Omega \boldsymbol{u}\cdot\nabla q = 0 \quad \forall q\in \mathrm{H}^1(\Omega)$ is obtained. Therefore, it follows that $\boldsymbol{u}\in X$, i.e. $\boldsymbol{u}\in \mathrm{H}_0(\mathrm{div}\,;\Omega)$, and $\mathrm{div}\,\boldsymbol{u} = 0$ in $\Omega$. on the other side, of the Equation (12) taking $\boldsymbol{\theta}\in[\mathcal{D}(\Omega)]^3$, the Equation (14) is obtained.

$$\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} - \sqrt{\nu}\int_\Omega (\kappa\boldsymbol{f} - \sqrt{\nu}\kappa\,\mathbf{curl}\,\boldsymbol{\omega} - \kappa\nabla p)\cdot\mathbf{curl}\,\boldsymbol{\theta} = 0, \tag{14}$$

then replacing $\boldsymbol{u}$ in Equation (14), the following Equation (15) is obtained.

$$\int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{\theta} - \sqrt{\nu}\int \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\theta} = 0 \quad \forall\boldsymbol{\theta}\in[\mathcal{D}(\Omega)]^3, \tag{15}$$

integrating by parts the Equation (15) obtain Equation (16).

$$\int_\Omega (\boldsymbol{\omega} - \sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{u})\cdot\boldsymbol{\theta} = 0 \quad \forall\boldsymbol{\theta}\in[\mathcal{D}(\Omega)]^3, \boldsymbol{\theta}|_{\partial\Omega} = 0. \tag{16}$$

The Equation (16) implies that $\boldsymbol{\omega} = \sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{u}$, in addition it is known that $\boldsymbol{\omega}\in \mathrm{H} \in (\mathbf{curl}\,,\Omega)$, *i.e.* $\boldsymbol{\omega}\in[\mathrm{L}^2(\Omega)]^3$. Then, $\mathbf{curl}\,\boldsymbol{u}\in[\mathrm{L}^2(\Omega)]^3$. Furthermore, integrating into Equation (15) by $\boldsymbol{\theta}\in[\mathrm{H}^1(\Omega)]^3$, the Equation (17) is obtained.

$$\int_\Omega (\boldsymbol{\omega} - \sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{u})\cdot\boldsymbol{\theta} + \langle \boldsymbol{u}\times\mathbf{n}, \boldsymbol{\theta}\rangle_{-\frac{1}{2},\frac{1}{2}} = 0 \quad \forall\boldsymbol{\theta}\in[\mathrm{H}^1(\Omega)]^3. \tag{17}$$

From Equation (17) it is obtained that $\langle \boldsymbol{u}\times\mathbf{n}, \boldsymbol{\theta}\rangle_{-\frac{1}{2},\frac{1}{2}} = 0 \in \boldsymbol{\theta}\in[\mathrm{H}^{1/2}(\Gamma)]^3$ given that $\boldsymbol{\omega} = \sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{u}$, thus $\boldsymbol{u}\times\mathbf{n} = \mathbf{0}$. Finally, it is found that $\boldsymbol{u}\in \mathrm{H}_0(\mathbf{curl}\,,\Omega)$. On the other hand, it is known that $[\mathrm{H}_0^1(\Omega)]^3 := \mathrm{H}_0(\mathrm{div}\,;\Omega) \cap \mathrm{H}_0(\mathbf{curl}\,;\Omega)$(see [22], Lemma 2.5). From the results above $\boldsymbol{u} \in [\mathrm{H}_0^1(\Omega)]^3$. So, by the definition of $\boldsymbol{u}$, $\kappa^{-1}\boldsymbol{u} + \sqrt{\nu}\mathbf{curl}\,\boldsymbol{\omega} + \nabla p = \boldsymbol{f}$, also $\boldsymbol{\omega} = \sqrt{\nu}\,\mathbf{curl}\,\boldsymbol{u}$, then $\boldsymbol{f}$ is determined by Equation (18).

$$\kappa^{-1}\boldsymbol{u} + \nu\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}) + \nabla p = \boldsymbol{f}. \tag{18}$$

Knowing that $-\Delta\boldsymbol{u} = \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}) - \nabla(\mathrm{div}\,\boldsymbol{u})$ and as $\mathrm{div}\,\boldsymbol{u} = 0$, replacing in Equation (18) it gets $\kappa^{-1}\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}$; therefore, the problem of the Equation (3) is obtained. Knowing that $-\Delta\boldsymbol{u} = \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}) - \nabla(\mathrm{div}\,\boldsymbol{u})$ and given that $\mathrm{div}\,\boldsymbol{u} = 0$; therefore, the problem of Equation (3) is obtained.

Therefore, by the Theorem 1 and the Lemma 1 the solution to the problem of Equation (10) exists and is unique on a continuous level.

## 3. Approximation of the solution

In this section we introduce the Galerkin schema for Equation (10), considering $\{\mathcal{T}_h(\Omega)\}_{h>0}$ is said to be regular if $c > 0$ exists, such that $\frac{h_{\mathrm{T}}}{\rho_{\mathrm{T}}} \leq c, \forall \mathrm{T} \in \mathcal{T}_h, \forall h > 0$ where $\rho_{\mathrm{T}} > 0$ is the diameter of the largest circumference or sphere contained in T and $h_{\mathrm{T}}$ is the longest side of the triangle or tetrahedron. The Brinkman problem with its variational formulation in the Equation (10), as already studied at the discrete level in [3]. The local space of Nédélec is introduced with order $k \geq 1$ for the vorticity and piecewise continuous polynomials for the pressure, as shown in the Equation (19), that was developed in [3].

$$\begin{aligned} \mathbf{Z}_h &:= \{\boldsymbol{\theta}_h \in \mathbf{Z} : \boldsymbol{\theta}_h|_{\mathrm{T}} \in \mathbb{N}_k(\mathrm{T})\forall \mathrm{T} \in \mathcal{T}_h(\Omega)\}, \\ \mathrm{Q}_h &:= \{q_h \in \mathrm{Q} : q_h|_T \in \mathcal{P}_k(\mathrm{T})\forall \in \mathrm{T} \in \mathcal{T}_h(\Omega)\}, \end{aligned} \tag{19}$$

are subspaces of $\mathbf{Z}$ y $\mathbf{Q}$, respectively. In addition, the following Theorem 2 is mentioned for the estimation of error in the vorticity and the pressure [3].

*Theorem 2.* Assuming $\boldsymbol{\omega} \in \mathrm{H}^s(\mathbf{curl}\,;\Omega)$ and $p \in \mathrm{H}^{1+s}(\Omega)$, for each $s \in (1/2, k]$; then, there is a constant $C > 0$, independent of $h$ and $\nu$, such that

$$\begin{aligned} \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\sqrt{\nu}\mathbf{curl}\,(\boldsymbol{\omega} - \boldsymbol{\omega}_h) + \nabla(p - p_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \\ \leq Ch^s(\|\boldsymbol{\omega}\|_{\mathrm{H}^s(\mathbf{curl}\,;\Omega)} + \|p\|_{\mathrm{H}^{1+s}(\Omega)}). \end{aligned} \tag{20}$$

*Proof.* The proof of Equation (20) is found in detail in [3], which determines the error estimation and the order of convergence in the $\mathrm{L}^2$; then, the veolocity is recovered by means of a post-processing technique that is found in detail in [3].

## 4. Numerical results and discussion

In this section, we present a numerical example in $\mathbb{R}^2$ of the a priori analysis to illustrate the proper functioning of the finite element scheme. The simulation is based on the FreeFem++ code [24]; individual errors and convergence rates are defined in [3].

*Example 1.* The domain $\Omega = (-1, 1)^2$, the coefficients $\kappa = 1$ and $\nu = 0.0001$ are considered, also $\boldsymbol{u} = \mathbf{0}$ at the boundary $\Gamma$. Taking adequate data for $\boldsymbol{f}$, so that the exact solution of the Equation (7) is given by the functions shown in Equation (21).

$$\boldsymbol{u} = \begin{pmatrix} y(x^2-1)^2(y^2-1) \\ -x(x^2-1)(y^2-1)^2 \end{pmatrix}, \quad p = x^5 - y^5, \tag{21}$$
$$\omega = \sqrt{\nu}((x^2-1)^2(y^2+1) + (y^2-1)^2(x^2+1)).$$

In Figure 1(a) and Figure 1(b), the approximate and exact solution for the vorticity is shown, in Figure 1(c) and Figure 1(d), the approximate and exact solution for the velocity and Figure 1(e) and Figure 1(f), shows the approximate and exact solution for the pressure. Observing that the approximate with respect to the exact have good similarity, this indicates that there is a good behavior of the method at the boundary.

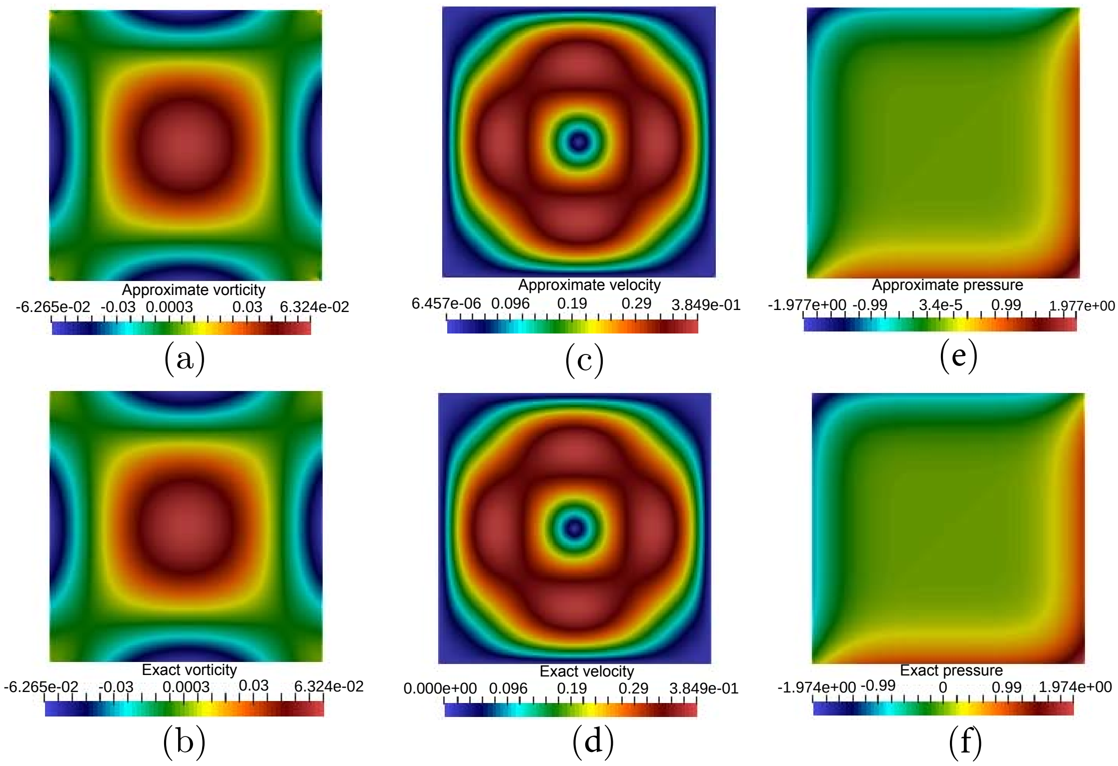


**Figure 1.** Represent (a) and (b) the approximate and exact solution for the vorticity, (c) and (d) the approximate and exact solution for the velocity, (e) and (f) the approximate and exact solution for the pressure.

In Table 1, shows the error estimates and the convergence rates in the $L^2$ norm and for a polynomial of degree $k = 1$, obtaining that the error tends to zero and an optimal convergence for the vorticity, velocity and pressure, $h$ is the refinement of the uniform mesh. In [21], they study the Brinkman problem written in terms of pressure velocity utilizing the discontinuous Galerkin finite element method, they obtain a superconvergence for the velocity and pressure, in the standard norm, while in this work we present a numerical example for the Brinkman Dirichlet problem homogeneous, with Nédélec finite elements and piecewise continuous polynomials, obtaining optimal convergences in its three variables as shown in Table 1, obtaining similar results as in article [3].

**Table 1.** Through Nédélec finite elements and continuous polynomials of order $k = 1$, error and convergence are observed for the vorticity, velocity and pressure.

| $h$ | $e(\boldsymbol{\omega})$ | $r(\boldsymbol{\omega})$ | $e(p)$ | $r(p)$ | $e(\boldsymbol{u})$ | $r(\boldsymbol{u})$ |
|---|---|---|---|---|---|---|
| 1.414214 | 0.208774 | 0.000000 | 3.704013 | 0.000000 | 1.760137 | 0.000000 |
| 0.707107 | 0.172852 | 0.272403 | 2.644768 | 0.485948 | 1.387364 | 0.343341 |
| 0.353553 | 0.114216 | 0.597778 | 1.477776 | 0.839714 | 0.817610 | 0.762861 |
| 0.176777 | 0.060729 | 0.911313 | 0.762346 | 0.954910 | 0.431725 | 0.921302 |
| 0.088388 | 0.028968 | 1.067922 | 0.384729 | 0.986603 | 0.220587 | 0.968767 |
| 0.044194 | 0.013222 | 1.131558 | 0.192917 | 0.995861 | 0.111336 | 0.986426 |

## 5. Conclusion

The methodology of this document, clearly shows its existence and uniqueness of solution at a continuous level for the homogeneous Brinkman Dirichlet problem by problems equivalence. Its solution was approximated by the method of finite elements Nédélec and piecewise continuous polynomials, obtaining its error estimate and its optimal convergence rate in its three variables, as shown in Table 1.

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