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On differential invariants of an equivalence group and their geometric meaning

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On differential invariants of an equivalence group and their geometric meaning

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Abstract. Previously, the properties of the Lie group G, which is an equivalence group of the eikonal equation, wave equation, and other differential equations (DEs), have been studied by the author in the two-dimensional case; various applications to mathematical physics and differential geometry have been obtained. This paper presents a study of the three-dimensional analogue of the G group, the ten-parameter G_{10} group, which is a subgroup of the main equivalence group of the three-dimensional eikonal equation, acoustics equation, and other DEs. Its differential invariants (DIs) up to the third order and invariant differentiation operators (IDOs) were calculated. The geometric meaning of some DIs of the group G_{10} (the scalar curvature R of Riemannian space with the metric $dl^2 = n^2(x, y, z)(dx^2 + dy^2 + dz^2)$, its first and second Beltrami differential parameters $\Delta_1 u$ and $\Delta_2 u$, and other quantities) and IDOs was found. An expression for R was derived in terms of other DIs of the group G_{10}. To obtain this expression, and DIs and IDOs of the group G_{10}, we use the geometric analogy with the two-dimensional case and differential and Riemannian geometry.

1. Introduction

Below, group terms are understood in the sense of [1]. This paper is a continuation of the previous work of the author [2–15]. The line of research can be defined as the study of differential equations (DEs) of mathematical physics (the theory of propagation of waves of different nature in inhomogeneous media) based on group and geometric analysis. In [2, 5], a group approach to the study of DEs of the form $F[\boldsymbol{u}, \boldsymbol{a}] = 0$ (E_0) was proposed, in which the solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ and the parameters (arbitrary element) $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x})$ are considered as equivalent dependent variables $\boldsymbol{u}^1 = \boldsymbol{u}$, $\boldsymbol{u}^2 = \boldsymbol{a}$ (F is a given differential operator, and $\boldsymbol{x}$ are independent variables). This DE is considered as equation E of the form $F[\boldsymbol{u}^1, \boldsymbol{u}^2] = 0$ (with the same operator F); the Lie group G admitted by this equation is sought in the space $(\boldsymbol{x}, \boldsymbol{u}^1, \boldsymbol{u}^2)$ and is an equivalence group of the equation E_0, which is, generally speaking, extended compared with its equivalence group G_{eq} defined in [1]. In the two-dimensional case, the Lie group G of transformations of the five-dimensional space (x, y, t, u^1, u^2), which is an equivalence group of the eikonal equation $(u_x)^2 + (u_y)^2 = n^2(x, y)$, the wave equation $u_{xx} + u_{yy} = n^2(x, y)u_{tt}$ (where $u^1 = u$ and $u^2 = n^2$), and other DEs, was studied in [2, 5]. Its differential invariants and their basis were found and used to obtain the following various applications [2–15]. A group bundle of a wide class of DEs was constructed (its resolving system was found to admit the Lax representation) [4, 5, 15]; new differential identities were obtained [6–10]; a new description of the two-dimensional kinematic problem of seismics (geometric optics) was proposed [5,7,13,15]; exact solutions were found [3, 5, 15]; new transformations for a number of DEs and the relationships between different DEs were derived; using vector analysis and differential geometry, differential conservation laws were derived for the eikonal equation (for the first time, [7, 10, 15]) and other DEs [5, 9, 15] and for families of plane curves. These results show a number of new properties and capabilities of group analysis and are systematized in [5, 15]. Later, the equivalence group of the two-dimensional eikonal equation was used by Borovskikh [16] for group classification and search for particular solutions of this equation.

In this paper, we study the properties of the three-dimensional analogue of the group G — the ten-parameter Lie group G_{10} of the six-dimensional transformations of the space $(x, y, z, t, u^1 = u, u^2 = n^2)$. It is a subgroup of the main equivalence group of the three-dimensional eikonal equation

$$|\operatorname{grad} u|^2 \overset{\text{def}}{=} (u_x)^2 + (u_y)^2 + (u_z)^2 = n^2(x, y, z)$$

for the time field $u(x, y, z, t)$ in an inhomogeneous isotropic medium with the refractive index $n(x, y, z) = 1/v(x, y, z)$ (where v is the velocity of propagation of waves (signals) in the medium), the acoustics equation $\Delta u + (\operatorname{grad} u \cdot \operatorname{grad} \ln \rho)/2 = \rho u_{tt}$ (here $u^1 = u$ and $u^2 = \rho$), and other DEs. The eikonal equation is the main mathematical model of kinematic seismics and geometric optics. The variable t is not explicitly included in this DE; it is represented by the parameter (coordinate) of the point source. The group G_{10} was calculated using the above-mentioned approach based on general theory [1]. In [16], the equivalence group of the eikonal equation was obtained under the additional condition $n_u = 0$ and equivalence classification of this equation was performed. In this study, we have an additional variable t and, instead of $n_u = 0$, use the condition $n_t = 0$ (the parameter of the medium is independent of the position of the source), which, however, leads to the same group.

In the work, IDOs and DIs of the group G_{10} up to the third order are found. Some DIs and their geometric meaning are obtained using Riemannian geometry [17]: these DIs are the scalar curvature R of Riemannian space with the metric $dl^2 = n^2(x, y, z)(dx^2 + dy^2 + dz^2)$ and its first and second Beltrami differential parameters $\Delta_1 u$ and $\Delta_2 u$. The geometric meaning of the vector field $\boldsymbol{S}(\boldsymbol{\tau})$ included in one of the DIs of the second order is also given. An explicit expression for R is obtained in terms of other DIs of the group G_{10}. These quantities and formulas are three-dimensional analogues of the properties of the group G.

Systems of DEs for calculating DIs and IDOs of the group G based on general theory [1] are rather cumbersome. Their solutions can be found using analogies, including geometric ones (serving as heuristic arguments), with DIs and IDOs of the group G.

To make the text self-contained and to be able to compare corresponding formulas for the two-dimensional and three-dimensional cases, we briefly describe in Section 2 the properties of the group G as simpler and more compact than those of the group G_{10}. The symbols $(\boldsymbol{a} \cdot \boldsymbol{b})$ and $\boldsymbol{a} \times \boldsymbol{b}$ denote the scalar and vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, Δu is the Laplacian of the function u, and δ^i_j is the Kronecker symbol.

2. Two-dimensional case. The group G and its properties

These properties [2–6, 15] are described by the following theorem.

Theorem 1. *Let G be an infinite Lie group of point transformations of the space (x, y, t, u^1, u^2) for which the infinitesimal operator X of any of its one-parameter subgroup G_1 has the form $X = \Phi(x, y)\,\partial/\partial x + \Psi(x, y)\,\partial/\partial y - 2\Phi_x(x, y)u^2\,\partial/\partial u^2$, where Φ and Ψ are arbitrary conjugate harmonic functions. The second-order universal differential invariant $\underset{2}{J}$ of the group G is the set of invariants J^1–J^{15} of the form $J^1 = t$, $J^2 = u^1$, $J^3 = u^1_t = A_1 J^2$, $J^4 = \Delta_2 u^1 \overset{\text{def}}{=} \Delta u^1/u^2$, $J^5 = u^1_{tt} = A_1 J^3$, $J^6 = A_3 J^3$, $J^7 = \Delta_1 u^1 \overset{\text{def}}{=} ((u^1_x)^2 + (u^1_y)^2)/u^2 = A_2 J^2$,*

$J^8 = A_2 J^3$, $J^9 = A_2 J^7$, $J^{10} = A_3 J^7$, $J^{11} = K(x,y) = -\Delta \ln u^2/(2u^2)$, $J^{12} = u_t^2/u^2$, and $J^{12+i} = A_i J^{12}$, $i = 1,2,3$.

The basis of DIs of the group G is formed by the invariants $J^1 = t$ and $J^2 = u^1$. Here $A_1 = D_t$, $A_2 = \{u_x^1 D_x + u_y^1 D_y\}/u^2 = (J^7)^{1/2}(\boldsymbol{\tau}\cdot \mathrm{grad})/(u^2)^{1/2}$, $A_3 = \{u_y^1 D_x - u_x^1 D_y\}/u^2 = -(J^7)^{1/2}(\boldsymbol{\nu}\cdot \mathrm{grad})/(u^2)^{1/2}$ are the IDOs of the group G; D_t, D_x, and D_y are total differentiation operators, $\mathrm{grad}\, u^1 = (u_x^1, u_y^1)$, $\mathrm{grad} = (D_x, D_y)$; $\boldsymbol{\tau} = \mathrm{grad}\, u^1/|\mathrm{grad}\, u^1|$, $\boldsymbol{\nu} = -\mathrm{rot}\,(u^1\boldsymbol{k})/|\mathrm{rot}\, u^1\boldsymbol{k}| = -(\mathrm{grad}\, u^1 \times \boldsymbol{k})/|\mathrm{grad}\, u^1| = -\boldsymbol{\tau}\times\boldsymbol{\beta}$ is the Frenet basis ($\boldsymbol{\tau}$ is the unit tangent vector, and $\boldsymbol{\nu}$ is the normal unit vector) [18–20] of the plane curve L_τ which is the vector line of the vector field $\boldsymbol{v} = \mathrm{grad}\, u^1$; the unit vector $\boldsymbol{k}$ along the z axis plays the role of its binormal. The operators A_2 and A_3 are proportional to the differentiation operators of the scalar function $f(x,y)$ in the direction of the Frenet unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$. Furthermore, $J^{11} = K(x,y)$ is the Gaussian surface curvature in three-dimensional Euclidean space with the linear element (Riemannian metric) $dl^2 = n^2(x,y)(dx^2+dy^2)$; $J^7 = \Delta_1 u$ and $J^4 = \Delta_2 u$ are the first and second Beltrami differential parameters of the function $u(x,y)$. The following formula [3, 4, 13] is valid:

$$J^{11} = K(x,y) = \Delta \ln J^7/(2n^2) - A_2(J^4/J^7) - J^7(J^4/J^7)^2.$$

Any equation of the form $F(J^1, J^2, \ldots, J^{15}) = 0$, where F is some function, admits the group G.

3. Main (equivalence) group admitted by the three-dimensional eikonal equation in the space $(x, y, z, t, u^1 = u, u^2 = n^2)$. Group G_{10}

Theorem 2. *We denote $x = x^1$, $y = x^2$, $z = x^3$, $t = x^4$, $\boldsymbol{x} = (x^1, x^2, x^3, x^4)$, and $\partial u^k/\partial x^i = u_i^k$ ($k = 1,2$; $i = 1,2,3,4$). The main group of point transformations of the space (x, y, z, t, $u^1 = u$, $u^2 = n^2$) admitted by the eikonal equation $F[u^1,u^2] \equiv (u_1^1)^2 + (u_2^1)^2 + (u_3^1)^2 - u^2 = 0$ for $u_4^2 = 0$ has a Lie algebra of infinitesimal operators $X = \xi^i(\boldsymbol{x}, u^1, u^2)\,\partial/\partial x^i + \eta^1(\boldsymbol{x}, u^1, u^2)\,\partial/\partial u^1 + \eta^2(\boldsymbol{x}, u^1, u^2)\,\partial/\partial u^2$, where $\xi^i = A_j(2x^i x^j - |\boldsymbol{x}|^2\delta_j^i) + a_j^i x^j + bx^i + c^i$, $i, j = 1,2,3$, $a_j^i = -a_i^j$, A_j, a_j^i ($i < j$), c^i ($i, j = 1,2,3$), b are arbitrary constants (independent of x^1, x^2, x^3, t), $|\boldsymbol{x}|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, $\xi^4 = \xi(x^4)$, $\eta^1 = \sigma u^1 + \eta(x^4)$, $\eta^2 = 2u^2(\sigma - \mu/2)$, $\mu = 4\sum_{j=1}^3 A_j x^j + 2b$, σ is an arbitrary constant, $\xi(x^4)$, and $\eta(x^4)$ are arbitrary functions. It contains a ten-parameter subgroup G_{10} with the basic operators $X_i = \partial/\partial x^i$, $i = 1,2,3$ (shift operators), $X_{ij} = x^j\,\partial/\partial x^i - x^i\,\partial/\partial x^j$, $ij = 12, 13, 23$ (rotation operators), $Z = x^i\,\partial/\partial x^i - 2u^2\,\partial/\partial u^2$ (extension operators), $Y_i = (2x^i x^j - |\boldsymbol{x}|^2\delta_j^i)\,\partial/\partial x^j - 4x^i u^2\,\partial/\partial u^2$ ($i, j = 1,2,3$) (inversion operators).*

If we drop the terms with $\partial/\partial u^2$ in the operators Z and Y_i, we obtain the operators of the group of conformal transformations in the Euclidean space (x, y, z) [18, p. 371]; the group G_{10} is its extension to the space x, y, z, t, $u^1 = u$, $u^2 = n^2$.

Proof. According to the general theory [1], the invariance conditions of the system $F[u^1,u^2] = 0$, $u_4^2 = 0$ have the form $\underset{1}{X} F[u^1,u^2] \equiv 2u_i^1\zeta_i^1 - \eta^2(\boldsymbol{x}, u^1, u^2) = 0$, $\underset{1}{X} u_4^2 \equiv \zeta_4^2 = 0$, where $\underset{1}{X} = X + \zeta_i^k\,\partial/\partial u_i^k$ is the first extension operator with the coordinates $\zeta_i^k = \partial\eta^k/\partial x^i + u_i^l\,\partial\eta^k/\partial u^l - u_j^k\,\partial\xi^j/\partial x^i - u_i^l u_j^k\,\partial\xi^j/\partial u^l$. Moreover, both conditions must be satisfied identically for all variables x^i, u^k, u_i^1, and u_i^2 ($i = 1,2,3,4$; $k = 1,2$) on the manifold $\{F[u^1,u^2] = 0,\ u_4^2 = 0\}$. Splitting the invariance conditions into different variables, we obtain the following system of constitutive equations for ξ^i and η^k: $\partial\xi^i/\partial x^j + \partial\xi^j/\partial x^i = \mu\delta_j^i$ ($i, j = 1,2,3$); $\partial\xi^i/\partial x^4 = \partial\xi^i/\partial u^k = 0$, $\partial\xi^4/\partial x^i = \partial\xi^4/\partial u^k = 0$ ($i = 1,2,3$; $k = 1,2$), $\partial\eta^1/\partial x^i = \partial\eta^1/\partial u^2 = 0$ ($i = 1,2,3$); $\partial\eta^2/\partial x^4 = \partial\eta^2/\partial u^1 = 0$, $\eta^2 = 2u^2(\partial\eta^1/\partial u^1 - \mu/2)$. The first subsystem of DEs containing the function μ is equations for the group of conformal transformations of the space group (x, y, z) [1, 17]. The general solution ξ^i of these DEs is known and is given in Theorem 2, whence we have formulas for μ, η^1, and η^2. Sequentially setting one of the constants A_j, a^i_j $(i < j)$, b, and c^i equal to one and others equal to zero, we obtain ten basic operators of the group G_{10}. Calculating their commutators, we obtain Theorem 2.

4. DI sand IDOs of the group G_{10}

Lemma. *The scalar curvature R of Riemannian space with the metric $dl^2 = n^2(x, y, z)\{dx^2 + dy^2+dz^2\}$ (i.e., with the basic metric tensor $g_{ij} = \delta^i_j n^2(x, y, z)$) and the first and second Beltrami differential parameters $\Delta_1 u$ and $\Delta_2 u$ of the function $u(x, y, z)$ for this metric have the form*

$$R = 2\{\Delta \ln n^2 + |\operatorname{grad} \ln n^2|^2/4\}/n^2 = 2 \operatorname{div} \{n^{1/2} \operatorname{grad} \ln n^2\}/n^{5/2},$$

$$\Delta_1 u = |\operatorname{grad} u|^2/n^2 = \{(u_x)^2 + (u_y)^2 + (u_y)^2\}/n^2,$$
$$\Delta_2 u = \{\Delta u + (\operatorname{grad} u \cdot \operatorname{grad} \ln n^2)/2\}/n^2 = \operatorname{div} \{n \operatorname{grad} u\}/n^3.$$

The proof follows from the well-known formulas of Riemannian geometry [17, § 8, (8.14); § 11, ex. 14 or § 16, ex. 5; § 15, (15.8); § 14].

Theorem 3. *The first-order universal DI of the group G_{10} is the set of invariants $J^1 - J^3$, J^7, and J^{12} of the form ($u^1 = u$, $u^2 = n^2$) $J^1 = t$, $J^2 = u$, $J^3 = u_t = A_1J^2$, $J^7 = \Delta_1 u = |\operatorname{grad} u|^2/n^2 = A_2J^2$, and $J^{12} = (n^2_t)/n^2$. The expressions $J^4 = \Delta_2 u$, $J^5 = u_{tt} = A_1J^3$, $J^6 = A_3J^3$ or $(J^6)' = (\operatorname{grad} J^7 \cdot \operatorname{grad} u_t)/n^2 = A'_3 J^3$, $J^8 = (\operatorname{grad} u \cdot \operatorname{grad} u_t)/n^2 = A_2J^3$, $J^9 = (\operatorname{grad} u \cdot \operatorname{grad} J^7)/n^2 = A_2J^7 = A'_3J^2$, $(J^{10})' = |\operatorname{grad} J^7|^2/n^2 = A'_3J^7$ or $J^{10} = |\operatorname{grad} u \times \operatorname{grad} J^7|/n^2 = \{J^7(J^{10})' - (J^9)^2\}^{1/2}$, $J^{11} = R$, $J^{12+i} = A_iJ^{12}$, $i = 1, 2, 3, 4$, $J^{17} = A_4J^3$, $J^{18} = \operatorname{div} \{n[-\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau}(\boldsymbol{\tau} \cdot \operatorname{grad} \ln n^2)/2\}/n^3 + |\operatorname{grad} \ln n^2|^2/(8n^2)$, and $J^{19} = \operatorname{div} \{n[-\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau}(\boldsymbol{\tau} \cdot \operatorname{grad} \ln n^2)/2 + \operatorname{grad} \ln n^2]\}/n^3 = J^{18} + J^{11}/4$ and the expressions $J^{20} = \Delta_2 \ln J^7 = \operatorname{div}\{n \operatorname{grad} \ln J^7\}/n^3$ and $J^{21} = \operatorname{div} \{n(J^4/J^7) \operatorname{grad} u\}/n^3 = A_2(J^4/J^7) + J^7(J^4/J^7)^2 = J^{20}/2 + J^{19}$ are the second-order and third-order DIs, respectively, of the group G_{10}. Here the quantities $\Delta_1 u$, $\Delta_2 u$, and R are defined in the above lemma, $\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}$, $\boldsymbol{\tau} = \operatorname{grad} u/|\operatorname{grad} u|$ is the unit tangent vector of the vector line L_τ of the vector field $\operatorname{grad} u$. Moreover, $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{T} \stackrel{\text{def}}{=} \operatorname{grad} \ln |\operatorname{grad} u|^2/2 - \Delta u \operatorname{grad} u/|\operatorname{grad} u|^2$. The operators A_i $(i = 1, 2, 3, 4)$ and A'_3 are IDOs and are defined in Theorem 4.*

Theorem 4. *The IDOs of the group G have the form $A_1 = D_t$, $A_i = (\boldsymbol{\lambda}_i \cdot \boldsymbol{D}_3)$, where $i = 2, 3, 4$, $\boldsymbol{D}_3 = (D_x, D_y, D_z) = \operatorname{grad}$, $\boldsymbol{\lambda}_2 = \operatorname{grad} u/u^2$, $\boldsymbol{\lambda}_4 = (\operatorname{grad} u \times \operatorname{grad} J^7)/(u^2)^{3/2}$, $\boldsymbol{\lambda}_3 = -(u^2)^{1/2}(\boldsymbol{\lambda}_2 \times \boldsymbol{\lambda}_4)$ or in equivalent form $B_1 = D_t$, $B_2 = (J^7)^{-1/2}A_2|_{u^2\equiv 1} = (\boldsymbol{\tau} \cdot \operatorname{grad}) = \partial/\partial\tau$, $B_3 = (J^7)^{-1/2}(J^{10})^{-1}A_3|_{u^2\equiv 1} = (\boldsymbol{\nu} \cdot \operatorname{grad}) = \partial/\partial\nu$, $B_4 = (J^{10})^{-1}A_4|_{u^2\equiv 1} = (\boldsymbol{\beta} \cdot \operatorname{grad}) = \partial/\partial\beta$, so that the IDOs B_i $(i = 2, 3, 4)$ for $u^2 \equiv 1$ are differentiation operators of the scalar function in the direction of the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ of the vector line L_τ of the vector field $\operatorname{grad} u$ (tangent, principal normal, and binormal unit vectors). The operator $A'_3 = (\operatorname{grad} J^7 \cdot \operatorname{grad})/u^2 = (A_3 - J^9A_2)$ is also an IDO.*

Proof of Theorems 3 and 4 will be carried out jointly since they are interrelated. In order for the function J to be an invariant of the k-th order and in order for the operator $A = (\boldsymbol{\lambda} \cdot \boldsymbol{D})$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\boldsymbol{D} = (D_x, D_y, D_z, D_t)$ to be an IDO of the group G_{10}, it is necessary and sufficient [1] that the function J and the vector $\boldsymbol{\lambda}$ satisfy the system of first-order linear DEs

$$\underset{k}{X} J = 0 \qquad \text{(A)} \qquad\qquad \underset{k}{X} \boldsymbol{\lambda} = (\boldsymbol{\lambda} \cdot \boldsymbol{D})\boldsymbol{\xi} \qquad \text{(B)},$$

where $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3, \xi^4)$, ξ^i is the coordinate of the operator X in $\partial/\partial x^i$; the role of X is played by each of the ten operators X_i, X_{ij}, Z, and Y_i; $\underset{k}{X}$ is the k-th extension operator ($\underset{0}{X} = X$).

For $k=0$, system (A) is equivalent to the system $\partial J/\partial x^i = 0$ $(i=1,2,3)$, $\partial J/\partial u^2 = 0$; all its solutions are the invariants J^1 and J^2. In all ten operators of the group G_{10}, the coordinates ξ^1, ξ^2, and ξ^3 are independent of $x^4 = t$ and the coordinate $\xi^4 = 0$. Therefore, system (B) for any k is split into two subsystems $\underset{k}{X}\boldsymbol{\lambda}' = (\boldsymbol{\lambda}' \cdot D_3)\boldsymbol{\xi}'$ and $\underset{k}{X}\lambda_4 = 0$, where $\boldsymbol{\lambda}' = (\lambda_1, \lambda_2, \lambda_3)$ and $\boldsymbol{\xi}' = (\xi^1, \xi^2, \xi^3)$ for any of the ten operators X. For $k=0$, an obvious solution of this system is $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1$, which gives IDO A_1. Further we denote $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$ and $\boldsymbol{\xi}' = \boldsymbol{\xi}$.

For $k=1$, we have $\underset{1}{X}{}_i = \partial/\partial x^i \Rightarrow \partial J/\partial x^i = 0$, $\partial\lambda^j/\partial x^i = 0$, and $\underset{1}{X}{}_{12} = y\,\partial/\partial x - x\,\partial/\partial y + u^1_y\,\partial/\partial u^1_x - u^1_x\,\partial/\partial u^1_y + u^2_y\,\partial/\partial u^2_x - u^2_x\,\partial/\partial u^2_y$; the operators $\underset{1}{X}{}_{13}$ and $\underset{1}{X}{}_{23}$ are obtained from $\underset{1}{X}{}_{12}$ respectively, by replacing the symbols $y \to z$ and $y \to z$, $x \to y$; $\underset{1}{Z} = Z - u^1_x\,\partial/\partial u^1_x - u^1_y\,\partial/\partial u^1_y - u^1_z\,\partial/\partial u^1_z - 3u^2_x\,\partial/\partial u^2_x - 3u^2_y\,\partial/\partial u^2_y - 3u^2_z\,\partial/\partial u^2_z - 2u^2_t\,\partial/\partial u^2_t$; the operators $\underset{1}{Y}{}_i$ are linear combinations of $\underset{1}{X}{}_{ij}$, $\underset{1}{Z}$, and $\partial/\partial u^2_i$ when acting on J, λ^i: $\underset{1}{Z}{}_1 = 2(x\underset{1}{Z} - y\underset{1}{X}{}_{12} - z\underset{1}{X}{}_{13} - 2u^2\,\partial/\partial u^2_x)$, $\underset{1}{Y}{}_2 = 2(y\underset{1}{Z} + x\underset{1}{X}{}_{12} - z\underset{1}{X}{}_{23} - 2u^2\,\partial/\partial u^2_y)$, $\underset{1}{Y}{}_3 = 2(z\underset{1}{Z} + x\underset{1}{X}{}_{13} + y\underset{1}{X}{}_{23} - 2u^2\,\partial/\partial u^2_z)$; $\underset{1}{X}{}_{23}J$ is a linear combination of $\underset{1}{X}{}_{12}J$ and $\underset{1}{X}{}_{13}J$. Therefore, system (A) for $k=1$ is equivalent to the system of nine independent DEs $\partial J/\partial x^i = 0$, $\underset{1}{X}{}_{12}J = 0$, $\underset{1}{X}{}_{13}J = 0$, $\underset{1}{Z}\,J = 0$, and $\partial J/\partial u^2_i = 0$ $(i=1,2,3)$ with 14 independent variables, which has five functionally independent solutions J^1, J^2, J^3, J^7, and J^{12}. System (B) for $k=1$ is equivalent to the system $\partial\lambda_j/\partial x^i = 0$, $\partial\lambda_j/\partial u^2_i = 0$ $(i,j=1,2,3)$, $\underset{1}{Z}\boldsymbol{\lambda} = \boldsymbol{\lambda}$, $\underset{1}{X}{}_{12}\boldsymbol{\lambda} = (\lambda_2, -\lambda_1, 0)$, $\underset{1}{X}{}_{13}\boldsymbol{\lambda} = (\lambda_3, 0, -\lambda_1)$, $\underset{1}{X}{}_{23}\boldsymbol{\lambda} = (0, \lambda_3, -\lambda_2)$, which has the solution $\boldsymbol{\lambda} = \boldsymbol{\lambda}_2 = \operatorname{grad} u/u^2$, whence we have the IDO A_2. It can be shown that any solutions of this system has the form $\boldsymbol{\lambda} = \varphi(u_x, u_y, u_z, u^2)\boldsymbol{\lambda}_2$ and does not give other IDOs.

For $k=2$, we have $\underset{2}{X}{}_i = \partial/\partial x^i \Rightarrow \partial J/\partial x^i = 0$ and $\partial\lambda^j/\partial x^i = 0$, and the operators $\underset{2}{Y}{}_i$ are linear combinations of $\underset{2}{X}{}_{ij}$, $\underset{2}{Z}$ and the operators $\underset{2}{\tilde{Y}}{}_i$ (which are simpler than $\underset{2}{Y}{}_i$). The forms of $\underset{2}{X}{}_{ij}$, $\underset{2}{Z}$, $\underset{2}{Y}{}_i$, and $\underset{2}{\tilde{Y}}{}_i$ are not given here as they are cumbersome. Therefore, system (A) for $k=2$ is equivalent to the system $\partial J/\partial x^i = 0$, $\underset{2}{X}{}_{ij}J = 0$, $\underset{2}{Z}\,J = 0$, $\underset{2}{\tilde{Y}}{}_iJ = 0$. Substitution of the expressions for $J^4 = \Delta_2 u$, $J^{11} = R$, and J^{19} in terms of derivatives into this system shows that they satisfy it and hence are DIs (of the second order). Therefore, J^{21} and $J^{20} = 2(J^{21} - J^{19})$ are DIs (of the third order). System (B) for $k=2$ is equivalent to system (B*) of the form $\partial\lambda_i/\partial x^j = 0$ $(i,j=1,2,3)$, $\underset{2}{Z}\boldsymbol{\lambda} = \boldsymbol{\lambda}$, $\underset{2}{X}{}_{12}\boldsymbol{\lambda} = (\lambda_2, -\lambda_1, 0)$, $\underset{2}{X}{}_{13}\boldsymbol{\lambda} = (\lambda_3, 0, -\lambda_1)$, $\underset{2}{X}{}_{23}\boldsymbol{\lambda} = (0, \lambda_3, -\lambda_2)$, $\underset{2}{\tilde{Y}}{}_i\boldsymbol{\lambda} = 0$ $(i=1,2,3)$, which is very cumbersome. Its solution $\boldsymbol{\lambda}_3$, $\boldsymbol{\lambda}_4$ can be found using the analogy with the operators A_2 and A_3 of the group G from Theorem 1 in terms of the differential geometry of the vector lines L_τ of the field $\operatorname{grad} u$. In the three-dimensional case, the well-known formulas [18–20] for the binormal $\boldsymbol{\beta}$ and the principal normal $\boldsymbol{\nu}$ of the curve L_τ (k is its curvature) give $k\boldsymbol{\beta} = \operatorname{rot}\boldsymbol{\tau} = \operatorname{rot}\{\operatorname{grad} u/|\operatorname{grad} u|\} = |\operatorname{grad} u|^{-3}\boldsymbol{\Lambda}_\beta$ and $k\boldsymbol{\nu} = k\boldsymbol{\beta}\times\boldsymbol{\tau} = -\boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} = -|\operatorname{grad} u|^{-4}\boldsymbol{\Lambda}_\nu$, where $\boldsymbol{\Lambda}_\beta \overset{\text{def}}{=} \operatorname{grad} u \times \operatorname{grad}\,|\operatorname{grad} u|^2/2$, $\boldsymbol{\Lambda}_\nu \overset{\text{def}}{=} \operatorname{grad} u \times \boldsymbol{\Lambda}_\beta$. The literal analogy to the IDOs of the group G leads to the assumption that the vectors $\boldsymbol{\lambda}_3$ and $\boldsymbol{\lambda}_4$ in the IDOs A_3 and A_4 have the form $\boldsymbol{\lambda}_3 = a_3\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\lambda}_4 = a_4\boldsymbol{\Lambda}_\beta$, where a_3 and a_4 are scalar functions. However, such $\boldsymbol{\lambda}_3$ and $\boldsymbol{\lambda}_4$ do not satisfy system (B*). Its solutions $\boldsymbol{\lambda}_3$ and $\boldsymbol{\lambda}_4$ by the formulas of Theorem 4 contain expressions derived from the vectors $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\Lambda}_\beta$ by replacing the quantities $|\operatorname{grad} u|^2$ by $J^7 = |\operatorname{grad} u|^2/u^2$; such $\boldsymbol{\lambda}_3$ and $\boldsymbol{\lambda}_4$ satisfy (B*), and hence A_3 and A_4 are IDOs. The invariance of J^5–J^9, $(J^{10})'$, and J^{13}–J^{17} follows from the fact that A_i $(i=1,2,3,4)$ A'_3 are IDOs. The expression for A'_3 follows from the formula for A_3 and the identity [19, § 7] $(\boldsymbol{a}\cdot[\boldsymbol{b}\times\boldsymbol{c}]) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ for $\boldsymbol{a}=\boldsymbol{b}=\operatorname{grad}u$, and $\boldsymbol{c}=\operatorname{grad}J^7$. The invariance of J^{10} follows from the formula [20, Ch. 4, § 4] $(\boldsymbol{a}\times\boldsymbol{b})^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$ for $\boldsymbol{a}=\operatorname{grad}u/n$ and $\boldsymbol{b}=\operatorname{grad}J^7/n$.

Remark 1. Finding the DIs $R=J^{11}$, $\Delta_2 u = J^4$, J^{20}, and J^{21} and other DIs of Theorem 3 by solving the systems $\underset{2}{X}\,J=0$ and $\underset{3}{X}\,J=0$ is difficult due to their complexity. We find these expressions as three-dimensional analogues of the DIs $K=J^{11}$, $\Delta_2 u = J^4$, $\Delta\ln J^7/n^2$, and $\operatorname{div}\{(J^4/J^7)\operatorname{grad}u\}/n^2$, respectively, and other DIs of the group G. Verifying that they satisfy the system $\underset{2}{X}\,J=0$ and other systems is an easy (though laborious) task. The same is true for IDOs.

Remark 2. In [12], the geometric meaning of the field $\boldsymbol{S}(\boldsymbol{\tau})=\boldsymbol{T}$ was obtained as the sum of three curvature vectors of three curves associated with the surface orthogonal to the field $\boldsymbol{\tau}$; for its Gaussian curvature, we have [21]: $K=-\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})/2$.

Theorem 5. *The scalar curvature $R=J^{11}$ mentioned in the lemma and Theorem 3 is expressed in terms of other DIs of the group G_{10} as $R/4 = J^{21} - J^{20}/2 - J^{18}$.*

Proof is obtained either by direct calculation of the right side of this formula in terms of derivatives or (which is easier) from the identity $\boldsymbol{S}(\boldsymbol{\tau})=\boldsymbol{T}$ of Theorem 3 (obtained in [8]) by multiplying it by n and using the div operation, division by n^3, and the formula $\ln|\operatorname{grad}u|^2 = \ln J^7 + \ln n^2$. This formula for R is a three-dimensional analogue of the formula for $K(x,y)$ in Theorem 1.

5. Conclusions

This study shows that complex systems of DEs that arise when searching for DIs and IDOs of Lie groups admitted by DEs of mathematical physics can be solved using geometry (differential, Riemannian), vector analysis, and the method of analogies. In this study, they were used to find the DIs (and the relationships between them) and IDOs of the group G_{10} — an equivalence group of the three-dimensional eikonal equation and other DEs.

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