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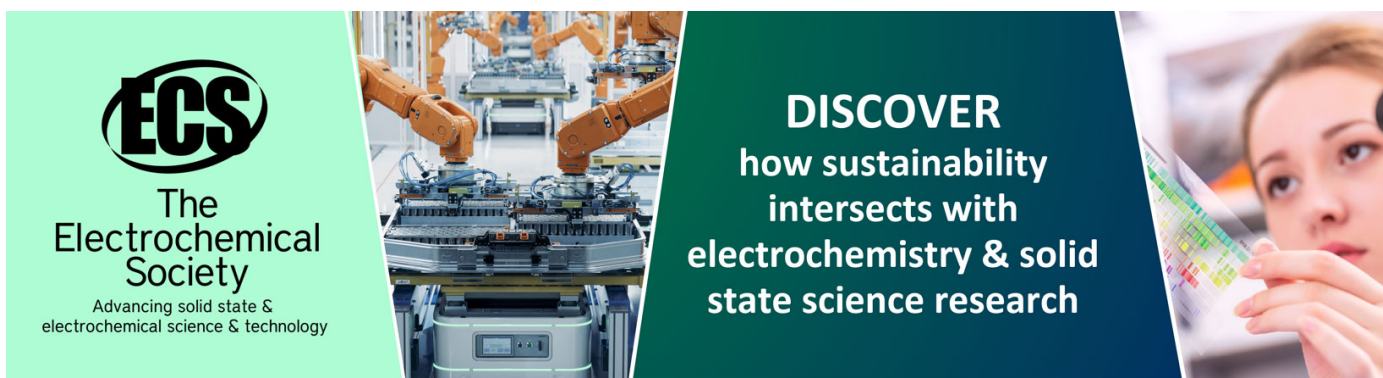
On the Elliptic Variational Inequality for A Simplified Signorini Problem.

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Abstract. The phrase "variational formulation" has been used recently in connection with generalized boundary formulation or initial – problems of value. However, in the classical sense of the phrase, minimizing the squared function as well, that involves all problem intrinsic feature, examples: border or / and starting conditions, the governing equations, constraint conditions, and even jump conditions. Variational formulations, in either sense of the phrase, new theories supposed, support methods to study the mathematical properties of solutions, also most importantly that approximation normal means. In three related topics, the variational formulations can be used. Firstly, in the terms of getting the extremum (example, maxima or minima) are posed for numerous mechanics problems, thus, according nature of that, in the variational statements terms can be formulated. Secondly, other means might formulate for other problems, as vector mechanics (example: Newton's laws), however, the means of variational principles can formulate for that. Thirdly, third, principle variation considers a strong basis for getting approximate solutions to problems in practice, a lot of it is unsolvable.variational inequality (V.I.) in mathematics, is an inequality involves a function, whose solutions to all probable values are for a variables given, mostly belong to a convex series. The variational inequalities of the theory of mathematic have developed in the beginning to deal with problems of equilibrium, Signorini problem specifically. The variational inequality of elliptic is the "A simplified Signorini problem" first type. This V. I of elliptic was associated with second-order partial differentiation.

Introduction

The Signorini problem considers elasto statistical problem in elasticity of linear, which composed in getting the formation of a non-homogeneous anisotropic elastic body elastic equilibrium, staying on a hard non-friction surface and just mass forces subjected. The ''Gaetano Fichera'' name was originated to his signori teacher honour: the original of the him name coined by a problem with the conditions of ambiguous boundary.

Be a highly useful and essential category of nonlinear problems emerging come from physics, mechanic, etc. compose of the called V. I.

variational inequality (V.I.) in mathematics, is an inequality involves a function, whose solutions to all probable values are for a variable given, usually belong to a convex series. The variational inequalities of the theory of mathematic have developed in the beginning to problems deal of equilibrium, Signorini problem specifically in that problem model, the functional concerned was achieved like the initially variation of potential energy implicated, so, origin variational it has, the general abstract problem name recalled . The theory applies since developed to involve economics problems, game theory, and optimization and finance [6]. The signorini problem is the first problem, including a variational inequality, at 1959 posed by Antonio Signorini, then Gaetano Fichera was solved it at 1963 [1], [4],[5] and [2].

Definition

Following [1], the variational inequality formal definition as following:

Given a Banach space E, a subset K of E, and a functional $F: K \to E^*$ From K to E^*

dual space of E space, the problem of V. I is the solving problem with variable respect of X related to K the inequality as following: $\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K$

Where is the duality pairing? Generally, the problem of V. I might formulate on any infinite –or finite- Banach space dimensional. The following, three problem steps of the study:

$\langle .,. \rangle: E^* \times E \to R$.

1- Prove solution existence: This step means the mathematical problem correctness, manifesting that there is a solution at least.

2- Prove the given solution is unique: step means the physical problem corrected, manifesting that can be used the solution to physical phenomenon represent. This especially step considers significant, as almost problems V. I modeled are it origin is physical.

3- Solution Finding.

 Examples

- *The finding problem of the value in a minimum of a real – valued function of a real variable.*

This problem considers a typical problem model, recorded according [1], the value in a minimum consider the finding problem with a f function differentiated, closed interval above a $= [a, b]$.

Let X^* be a point in I appears of minimum.

Can occur 3 cases:

- if $a < x^* b$ then $f'(x^*) = 0$,
- if $x^* = a$ then , $f'(x^*) \geq 0$
- if $x^* = b$ then . $f'(x^*) \leq 0$

Problem of finding $x^* \in I$ such that necessary conditions, can be summarized as the $f'(x^*)(y - x^*) \geq 0 \qquad \forall y \in I$.

Should be searched the absolute minimum amid solutions (if there are more than 1) of the previous inequality show to be a true number solution, so, this is an inequality of finite dimensional.

- *The general V. I. Finite dimensional*

A formulation of the general problem in R^n is the following: given a subset K of R^n and a mapping $: K \to R^N$, the finite- dimensional problem of V. I related to consist of V of getting a n-dimensional vector x belonging to K as:

$\langle F(x), y - x \rangle \geq 0 \qquad \qquad \forall y \in K$

Where $\langle .,. \rangle: R^n \times R^n \to R$ is the typical inner product on the vector space R^n.

Notations

V: space of scalar product with real Hilbert $(.,.)$ and related norm $\|.\|$

V^*: The V dual space of V.

$a(.,.):V \times V \to R$ be a continuous, bilinear and V −elliptic mapping on $V \times V$.

A form of bilinear $a(.,.)$ said that V −elliptic if a positive constant α present as

$(v,v) \geq \alpha\|v\|^2, \quad \forall v \in V$.

Generally, we don't suppose $a(.,.)$ it must be symmetrical, as in somewhat application bilinear non-symmetric forms naturally may occur [7].

$L:V \to R$ linear functional , continuous.

K: convex, closed, no empty V subset.

$j(.):V \to R^- = R \cup \{\infty\}$is a proper functional convex and low semi continuous (L.S.C).

$(j(.)$appropriate if $(j(v) > \infty \forall v \in V \ and \ j \neq \infty))$

Elliptic Variationl Inequality of the First Type (EVI)

To find $\in V$, as problem solution is u:

$$\left.\begin{matrix} a(u, v - u) \geq L(v - u), \forall v \in K \\ u \in k \end{matrix}\right\} \qquad (p_1)$$

First type of EVI Uniqueness and Existence Results

1.5.1   A Theorem of Uniqueness and Existence [7],[3]

The Problem (p_1) has one and only one solution

Proof:

1- Uniqueness

Let u_1 and u_2 be solutions of (p_1). We have then:

(1) $a(u_1, v - u_1) \geq L(v - u_1) \qquad \forall v \in k, u_1 \in k$

(2) $a(u_2, v - u_2) \geq L(v - u_2) \qquad \forall v \in k, u_2 \in k$

Placing u_2 for v in (1) and u_1 for v in (2) and add up to obtain, by V − ellipticity utilizing $a(.,.)$.

$\alpha\|u_2 - u_1\|^2 \leq a(u_2 - u_1, u_2 - u_1)0$

Which implies $u_1 = u_2$, since $\alpha > 0$.

2- Existence: we will problem reduce (p_1) to a point problem fixed. According to the Hilbert spaces theorem for Riesz representation there exist.

3- $A \in £(V,V), (A = A^t)$ if $a(.,.)$ is symmetric and $\ell \in V$ as:-

$(Au, v) = a(u, v) \qquad \forall u, v \in V$ and

(3) $L(v) = (\ell, v) \qquad \forall v \in V$

Latter, the (p_1)problem is equal to $u \in V$ finding such that:-

(4) $\left.\begin{matrix} (u - \rho(Au - \ell) - u, v - u) \leq 0 \qquad \forall v \in k \\ u \in k \qquad\qquad\qquad ,\rho > 0 \end{matrix}\right\}$

This is equal to finding u as that:

(5) $u = p_k(u - p(An - \ell))$, for some $p > 0$

p_k Indicate the operator projection from V to K in the norm $\|.\|$

the map $W_\rho:V \to V$ defined by:

(6) $W_\rho(V) = p_k(v - \rho(Av - \ell)$

let $v_1, v_2 \in V$, then as contraction we have a p_k

$\left\|w_\rho(v_1) - w(v_2)\right\|^2 \leq \|v_2 - v_1\|^2 + \rho^2\|A(v_2 - v_1)\|^2 - 2\rho a(v_2 - v_1, v_2 - v_1)$

Hence we have

(7) $\left\|w_\rho(v_1) - w(v_2)\right\|^2 \leq (1 - 2\rho\alpha + \rho^2\|A\|^2)\|(v_2 - v_1)\|^2$

Therefore w_ρ is a mapping rigorous retraction if $0 < \rho < \frac{2\alpha}{\|A\|^2}$. By ρ getting in this domain, unique solution we possess the fixed of our point problem which means the solution presence for (p_1).

An Example of EVI of the first type "A simplified signorini Problem"

Notations
1. Ω:a bounded domain in IR^2.
2. $\Gamma : \partial\Omega$
3. $X = \{x_1, x_2\}$a generic point of Ω
4. $\nabla = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$
5. $C^m(\overline{\Omega})$:space of m- times frequently real functions have differentiated real value in which all their derivatives are on frequently in $\overline{\Omega}$
6. $C_0^m(\Omega) = \{v \in C^m(\overline{\Omega}): sup\, p(v) is\ a\ compact subset\ of\ \Omega\}$
7. $\|v\|_{m,p,\Omega} = \sum_{|\alpha|\leq m}\|D^\alpha v\|_{L^p(\Omega)}$ for $v \in C^m(\overline{\Omega})$ where $\alpha = (\alpha_1, \alpha_2); \alpha_1, \alpha_2$ non negative integer , $|\alpha| = \alpha_1 + \alpha_2$ and $D^a = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$
8. $W^{m,p}(\Omega)$:completion of $C^m(\overline{\Omega})$ in the norm defined above.
9. $W_0^{m,p}(\Omega)$:completion of $C_o^m(\Omega)$ in the above norm.
10. $H^m(\Omega) = W^{m,2}(\Omega)$
$$H_0^m(\Omega) = W_0^{m,2}(\Omega)$$

1.1. The mathematical Problem Interpretation

Let Ω be a bounded domain of R^2 with a smooth boundary Γ, considered by us as the first type of EVI as the following:

(8) $\left.\begin{array}{l} a(u, v-u) \geq L(v-u), \forall v \in k \\ u \in k \end{array}\right\}$

(9)   Where $V = H^1(\Omega)$

(10) $0 a(u,v) = \int_\Omega \nabla u . \nabla_v dx + \int_\Omega uv dx$

(11) $L(v) = < f, x >, f \in V^*$

(12) $K = \{v \in H^1(\Omega): \gamma v \geq 0\ a.e\ on\ \Gamma\}$

Where γv denotes the trace of v on Γ.

" Signorini Problem" Uniqueness and Existence Results

Theorem
The $(V.I.)$

(13) $\left.\begin{array}{l} a(u, v-u) \geq L(v-u), \forall v \in k \\ u \in k \end{array}\right\}$

Has a unique solution

Proof

As the bilinear form $a(.,.)$ is scalar product usually in $H^1(\Omega)$ L is continuous, theorm accordance (1.5.1) , obtain that (13) unique solution possess supported that explain by us the K is a convex, closed , non- empty subset of V.

So, $0 \in K$ (actually (actually $H^1(\Omega) \subset K$) , K is non - empty.

The convexity of K is obvious.

If $(V_n)_n \subset K$ and $v_n \to v$ in $H^1(\Omega)$then

$\gamma v \to \gamma v$, since $\gamma: H^1(\Omega) \to L^2(\Gamma)$ is continuous since $v_n \in K, \gamma\alpha V_n \geq 0$ a.e on Γ .So that $\gamma v \geq 0$ a.e. on Γ .

So, $v \in K$ closed appears.

The theorem proved.

Remark [7]

Since symmetric $a(.,.)$, the solution u of (13) is characterized as a minimization problem the unique solution.

$$(14) \qquad \left.\begin{array}{c} J(u) \leq J(v) \qquad Vv \in k \\ u \in k \end{array}\right\}$$

$$(15) \qquad \text{The } J(v) = \frac{1}{2}a(v,v) - L(v)$$

Conclusion

The variational inequality (V.I.) represents an important class of non - linear problems and occurs in the mathematical description of a large variety of physical problems. The theory applies since developed to involve economics problems, game theory and optimization.

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