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A class of numerical methods for determining the inclusion relation of positive invariant sets in linear dynamic systems

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A class of numerical methods for determining the inclusion relation of positive invariant sets in linear dynamic systems

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Abstract. Positive invariant sets play an important role in the system theory and a:lication of dynamic systems. Firstly, in order to judge the inclusion relation between positive invariant sets of systems, the equivalent conditions for judging the inclusion relation of special positive invariant sets are derived for two special positive invariant sets. Three specific positive invariant subsets are discussed, and the verification of the inclusion relation of positive invariant sets can be transformed into a class of numerical problem in the form of optimization, which simplify the conditions of readily implemented. Finally, numerical examples are given to illustrate our conclusion.

1. Introduction

Positive invariant sets play an important role in the theory and a:lication of dynamic systems. With the help of a positive invariant set, geometrically, if the initial state belongs to a positive invariant set, then the trajectory cannot leave the positive invariant set. The stability, control and retention of constraints of a dynamic system can be expressed geometrically. Because there is a wide range of linear constraints on state or control vectors in practical control problems, and the polyhedron invariant set can more conveniently a:roximate the reachable set and the attraction domain of the system than the ellipse invariant set, the study of the positive invariant set has important theoretical and practical value.

The necessary and sufficient conditions for the existence of polyhedral sets are mainly divided into two categories: one is the condition in the form of matrix equation, the other is the spectral property derived from the quadratic Lyapunov function of the system is invariant set, and the result of judging the existence of the system is invariant set [1,3-5]. These two kinds of results are expressed by algebraic conditions.Study of dynamic system are invariant set can be traced back to the (1966) Lyapunov, [8] for controlled performance research of dynamic system stability, usually by constructing a maximum (or minimum) is invariant set, the literature [11] by means of solving the improved functional differential inequality is the smallest volume of robust control invariant set looming.To solve these problems, we usually need to determine the inclusion relationship between positive invariant sets. In [2,12], we give some new results. In [6,7,9,10], we study the relationship between feedback control and controlled invariant sets.

2. Theorems and definitions

Consider a linear discrete system S_1

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$$x(k+1) = Ax(k) \tag{1}$$

and linear continuous systems S_2

$$\dot{x} = Ax \tag{2}$$

where $A \in R^{n\times n}$, $k \in N$.

Theorem 1[1](Ellipsoidal invariant set) For a symmetric and positive-definite matrix $Q \in R^{n\times n}$ the corresponding hyperellipsoid $\varphi(P, c) = \{x \in R^n : x^T Px \le c\}$ defined in (1) is a positively invariant set of system S_1, if and only if there exists a positive semi-definite matrix $P \in R^{n\times n}$ such that

$$A^T PA - P = -Q \tag{3}$$

Definition 1[4] (Polyhedral set)A convex polyhedral set is a set of the form

$$P(F,g) = \{x : Fx \le g\} = \{x : f_i \le g_i, i = 1,2,\cdots,s\} \tag{4}$$

where f_i denotes the i-th row of the $s\times n$ matrix F and g_i the i-th component of the $s\times 1$ vector g.

3. Positively invariant polyhedral sets

Consider a polyhedron $P = P(F) = V(X)$,For simplicity, assume that the origin is an internal point.

3.1 Determination of the inclusion relation of positive invariant sets

Theorem 2[4] The inclusion $P\left[F^{(1)}, g^{(1)}\right] \subset P\left[F^{(2)}, g^{(2)}\right]$ holds if and only if there exists a non-negative matrix H such that $HF^{(1)} = F^{(2)}$, $Hg^{(1)} \le g^{(2)}$.

Theorem 3 A polyhedron form such as $P_2 = P\left(F^{(2)}, g^{(2)}\right) = \left\{x \in R^n : F^{(2)}x \le g^{(2)}\right\}$ is the positive invariant set of the system S_1, if and only if there exist a non-negative matrix $H \in R^{m\times m}$, such that $\begin{pmatrix} F^T & 0 \\ g^T & 1 \end{pmatrix}\begin{pmatrix} h_i \\ w \end{pmatrix} = \begin{pmatrix} A^T f_i \\ g_i \end{pmatrix}, i = 1,2,\cdots,m$.

Proof. For fixed i , $i = 1,2,\cdots,m$, $h_i^T F = f_i^T A^T$,take the transpose of both sides,available $F^T h_i = A^T f_i$, $g^T h_i \le g_i$,then $g^T h_i + w_i = g_i, w \ge 0, i = 1,2,\cdots,m$ .Rewrite these equations as matrix products, and we get the result.

Theorem 4 $P\left[F^{(1)}, g^{(1)}\right] \subset P\left[F^{(2)}, g^{(2)}\right]$ if and only if the current optimization problem has a solution for all $i = 1,2,\cdots,m$.

$$\min g(y) = \left(u^T, \theta^T\right)\begin{pmatrix} t \\ x \end{pmatrix} = \sum_{j=1}^{m+1} t_j \ ,$$

$$s.t. \ \ B\begin{pmatrix} t \\ x \end{pmatrix} \ge \begin{pmatrix} \begin{pmatrix} F_i^{(2)^T} \\ g_i^{(2)} \end{pmatrix} \\ -\begin{pmatrix} F_i^{(2)^T} \\ g_i^{(2)} \end{pmatrix} \end{pmatrix}, \ \begin{pmatrix} t \\ x \end{pmatrix} \ge 0 \,. \tag{5}$$

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where $B = \begin{pmatrix} I_{m+1} & \begin{pmatrix} F^{(1)^T} & 0 \\ g^{(1)^T} & 1 \end{pmatrix} \\ I_{m+1} & -\begin{pmatrix} F^{(1)^T} & 0 \\ g^{(1)^T} & 1 \end{pmatrix} \end{pmatrix}$, $t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{m+1} \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} t \\ x \end{pmatrix}$, $\theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0_{m+1} \end{pmatrix}$.

3.2 Determination of the inclusion relation between ellipsoid and polyhedron

Consider the ellipsoid set

$$\varepsilon(P,1) = \{x \in R^n : x^T P x \le 1\} \tag{6}$$

and the set of symmetric polyhedral set

$$S(|Q|, \rho) = \{x \in R^n : |Qx| \le \rho\} \tag{7}$$

$$S(|Q|, \rho) = \{x \in R^n : |Q_{(i)}x| \le \rho_{(i)}, \forall i\} \tag{8}$$

If $x^T \dfrac{Q_{(i)}{}^T Q_{(i)}}{\rho_{(i)}{}^2} x \le x^T P x \le 1, \forall i$, then $\varepsilon(P,1) \subset S(|Q|, \rho)$,

3.3 Determination of the inclusion relation of positive invariant set of ellipsoid

Consider the ellipsoid set

$$\varepsilon(P_1,1) = \{x \in R^n : x^T P_1 x \le 1\} \tag{9}$$

$$\varepsilon(P_2,1) = \{x \in R^n : x^T P_2 x \le 1\} \tag{10}$$

If $\varepsilon(P_2,1) \subset \varepsilon(P_1,1)$,then $P_2 > P_1$, $P_2 - P_1 > 0$,so $P_2 - P_1 = (P_2 - P_1)^T$,and $\dfrac{tr(P_2 - P_1)}{n} > R_{\min}$.

4. Numerical examples

Example1. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3.2 \\ -0.1 & -0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can calculate existence $P_1 = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$,so get the positive invariant set $\varepsilon(P_1,1)$ of the system.

Then $P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix}$,make the inclusion relationship $\varepsilon(P_2,1) \subset \varepsilon(P_1,1)$ is true.

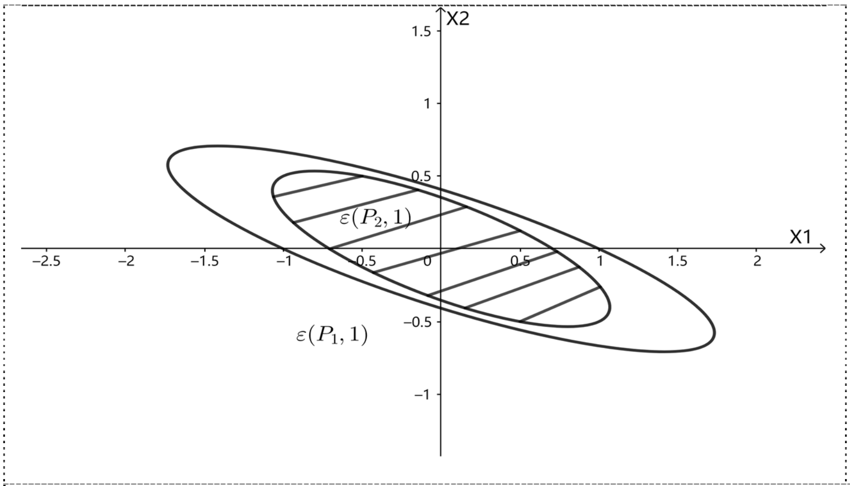

Figure 1. A subset of an ellipsoid set

Example2. Consider the system

$$x(k+1)=\begin{bmatrix}0.8 & 0.5\\ -0.4 & 1.2\end{bmatrix}x(k)+\begin{bmatrix}0\\1\end{bmatrix}u(k)$$

It can be seen from [6] that the polyhedron set $P(F,w)$ is the positive invariant set of the system

$$F=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},w=\begin{bmatrix}1\\1\end{bmatrix},$$

According to theorem 4, the $P=\begin{pmatrix}1.246 & -0.125\\ -0.125 & 6.5\end{pmatrix}$ existence is calculated, such that

$$\varepsilon(P,1)\subset P(F,w).$$

It's verified by theorem 1, $Q=\begin{bmatrix}0.0008 & -3.0686\\ 4.2064 & -44.1312\end{bmatrix}$, $\varepsilon(P,1)$ is the positive invariant set of the system.As shown in figure 2.

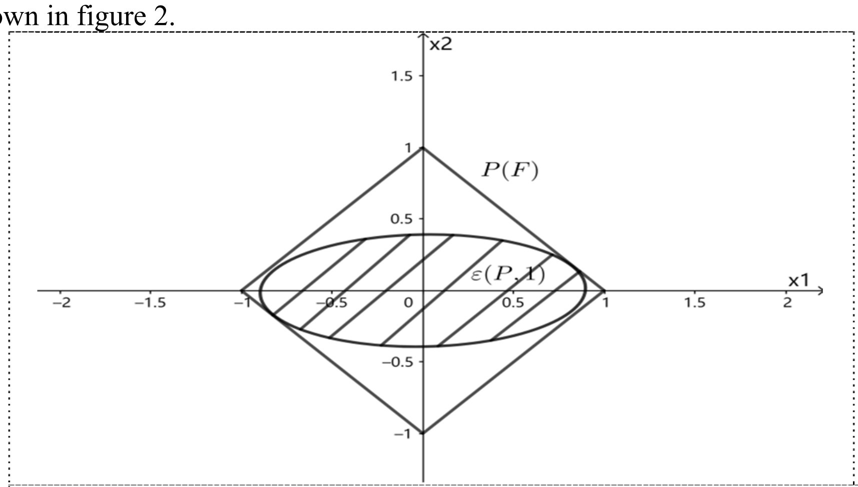

Figure 2. Subsets of ellipsoid sets of polyhedron sets

5. Conclusions

In this paper, the boundedness and inclusion relation of the polyhedral positive invariant sets of linear dynamic systems are studied. A necessary and sufficient condition is obtained to determine the inclusion relation of two sets of special invariant sets, and the subset determination problem of the set is transformed into an optimization model to solve the problem.

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