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# A Simple Mathematical Model of Pollutant Transfers on Ponds with Single Water Source

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**Abstract.** This paper discusses about a simple mathematical model of the pollutant transfer in n-ponds with three assumptions. First, the amount of pollutant come to the ponds always in constant rate, second, the volume of each pond is the same and third, the water discharge is constant. Using mass conservation rule, the models is constructed for the single water source for single ponds and multiple ponds. After getting the models, the solutions of the model can be achieved analytically. Afterward the solution of the models is investigated in several cases. Firstly, the pollutant that comes to the ponds is not zero, from the solutions the amount of pollutant in a pond is accumulated in the rate of pollutant additions. Secondly, there is no additional pollutant, in this case the amount of pollutant in the first ponds always decreases, but for the next ponds it slightly increases first until a time it start to decrease. The last case if the pond is in pure condition at first then the amount of pollutant in each pond has a linear correlation that increases from the first pond until the last pond.

## 1. Introduction

Quality water is the most important physical support in life. Peoples need water for feeding, metabolism processing, cleaning, sporting, etc [1]. To get the best support from water, peoples have to consider the water quality. Water quality depends on the chemicals substances and biological creatures those are existed in the water. Sometimes water contains some dangerous microbiological creatures or chemicals substances. Those not only affect in water use by humans, but also for some creatures those lives in the water [2]. Furthermore, if the creatures are food sources, the dangerous substances will go on the food chains.

Fish is one of creatures that live in the water which can be used as a cheap food sources and it is easy to cultivate [1]. Many people in the world cultivate fish in pond. But some chemical substance is contained in fish [5] or their foods, like chlorophyll a, total ammonia nitrogen, nitrite-nitrogen, carbon dioxide, and chemical oxygen demand (COD), etc, such that the chemical concentrations in the ponds slowly increases during the feeding time [2, 6, 7]. All of these chemical substances are not only affected the number of production of fish but also fish quality. So, we need to control amount of chemical in the ponds to get the best fish production [3].

Beside of some pollutants from the fish food, the water source of the pools is also considered. The water source not only best in quality but also free from some negative substance. Some ponds constructions just use single water sources and then the water flows through the pond to other ponds



[4]. This type of construction inflicts the pollutant from the previous pond move to the next pond and so on until it reaches the last pond (or waste shelter). This situation makes amount of pollutant in the next pond increased. Water source in often come from river that is not always pure [9, 10], but in this models we assume that the water source is the pure water (without negative chemical compound) the amount of pollutant in the pound will decrease.

This paper discusses the phenomenon how is the concentration of unneeded compound in pounds with single water source goes through the time. So, it leads to a simple mathematical model of the problem. By using this result, it is expected that the readers can control the water discharge and the amount of fish food to feed the fish. So the number of production and fish quality will increase.

## 2. Modelling process

Assume that there are  $n$ -ponds that are connected to a single water source, so the water flow through the first ponds, and then second ponds, and so on with a constant rate (constant water discharge). We also assume that the feeding rate of the fish is constant. Lastly, we assume that the water source is a pure water and no others pollutant that come into the ponds.

The modeling process will be done in some step. We assume that the volume of water in each pond is the same and then we can start the modeling process. Firstly, we create the model for a single pond and solve the resulting models. Secondly, we create the model for two ponds and then we generalize the results. Finally, we check our generalized result using mathematical induction and take the limit for infinite number of pounds.

### 2.1. Model for A Single Pond

Let  $u(t)$  is the concentration of pollutant in the water pond with volume  $V$ , and  $p$  is the rate of pollutant that come into pond. Since the rate of mass change of liquid in the pond in time  $t$  is the same with the rate of liquid that is flow out from the pond, then we have

$$Vu(t) + p\Delta t - Vu(t + \Delta t) = q\Delta tu(t) \quad (1)$$

where  $q$  is the water discharge.

By taken the limit at  $\Delta t \rightarrow 0$ , we get that

$$\frac{du(t)}{dt} = -\frac{q}{V}u(t) + \frac{p}{V} \quad (2)$$

With simple rule of solving ordinary differential equation (ODE) [8] we get

$$u(t) = \frac{p}{q} + k_1 e^{-\frac{q}{V}t} \quad (3)$$

where  $k_1$  is an arbitrary real number.

### 2.2. Model for Two Ponds

In the second pond, the water source has been contaminated in the first pond, so it contributes to pollute the second ponds. With the same idea with the first pond, we have

$$Vu_2(t) + q\Delta tu_1(t) + p\Delta t - Vu_2(t + \Delta t) = q\Delta tu_2(t) \quad (4)$$

By taken the limit at  $\Delta t \rightarrow 0$ , we get that

$$\frac{du_2(t)}{dt} = -\frac{q}{V}u_2(t) + \frac{q}{V}u_1(t) + \frac{p}{V} \quad (5)$$

By using the result of the first pond case we have that  $u_1(t)$  as equation (3), substituting to (5) then we get

$$\frac{du_2(t)}{dt} = -\frac{q}{V}u_2(t) + k_1 \frac{q}{V} e^{-\frac{q}{V}t} + \frac{2p}{V} \quad (6)$$

Again using the simple rule of solving ODE we get

$$u_2(t) = \frac{2p}{q} + \frac{k_1 q t}{V} e^{-\frac{q}{V}t} + k_2 e^{-\frac{q}{V}t}$$

$$\Leftrightarrow u_2(t) = \frac{2p}{q} + e^{-\frac{q}{V}t} \left( k_2 + k_1 \left( \frac{q}{V} t \right) \right) \quad (7)$$

where  $k_2$  is an arbitrary real number. By looking from (7) we can see that the number of pollutant in the second pond is greater than the first pond because we add some positive numbers on the right hand side of equation.

### 2.3. Model for $n$ -Ponds

For the first we make an ansatz for the solution of  $n$ -ponds, by looking from (7) we can guess the solution for  $n$ -ponds by

$$u_n(t) = \frac{np}{q} + e^{-\frac{q}{V}t} \left( k_n + \frac{k_{n-1} \left( \frac{q}{V} t \right)}{1!} + \dots + \frac{k_1 \left( \frac{q}{V} t \right)^{n-1}}{(n-1)!} \right) \quad (8)$$

We need to proof this ansatz using mathematical induction. By looking from (3) and (7) we can see that this ansatz is true for  $n = 1$  and  $n = 2$ . Let the solution is true for  $n = m$ , then we have

$$u_m(t) = \frac{mp}{q} + e^{-\frac{q}{V}t} \left( k_m + \frac{k_{m-1} \left( \frac{q}{V} t \right)}{1!} + \dots + \frac{k_1 \left( \frac{q}{V} t \right)^{m-1}}{(m-1)!} \right) \quad (9)$$

Now, we need to prove that it still true for  $n = m + 1$ . By using the same idea with the second pond, we know that amount of pollutant in the  $(m + 1)$ -th pond comes from the  $m$ -th, then we get

$$\frac{du_{m+1}(t)}{dt} = -\frac{q}{V}u_{m+1}(t) + \frac{q}{V}u_m(t) + \frac{(m+1)p}{V} \quad (10)$$

By using characteristics roots the homogeneous solution of (10) is given by

$$u_{m+1}^h(t) = k_{m+1} e^{-\frac{q}{V}t} \quad (11)$$

Because of the root is contains in  $u_m(t)$  then the particular solution of (10) is given by

$$u_{m+1}^p(t) = (a_1 + a_2 t + \dots + a_{m+1} t^m) e^{-\frac{q}{V}t} + A \quad (12)$$

where  $A, a_1, a_2, \dots, a_{m+1}$  some arbitrary real numbers. The solution of (10) is given by the sum of (11) and (12). By substituting this solution to (10), for  $m \neq 1$  we have

$$a_p = \frac{k_{m+2-p}}{p!} \left( \frac{q}{V} \right)^p \quad (13)$$

and

$$A = \frac{(m+1)p}{q} \quad (14)$$

Hence

$$u_{m+1}(t) = \frac{(m+1)p}{q} + e^{-\frac{q}{V}t} \left( k_{m+1} + \frac{k_m \left( \frac{q}{V} t \right)}{1!} + \dots + \frac{k_1 \left( \frac{q}{V} t \right)^m}{m!} \right) \quad (15)$$

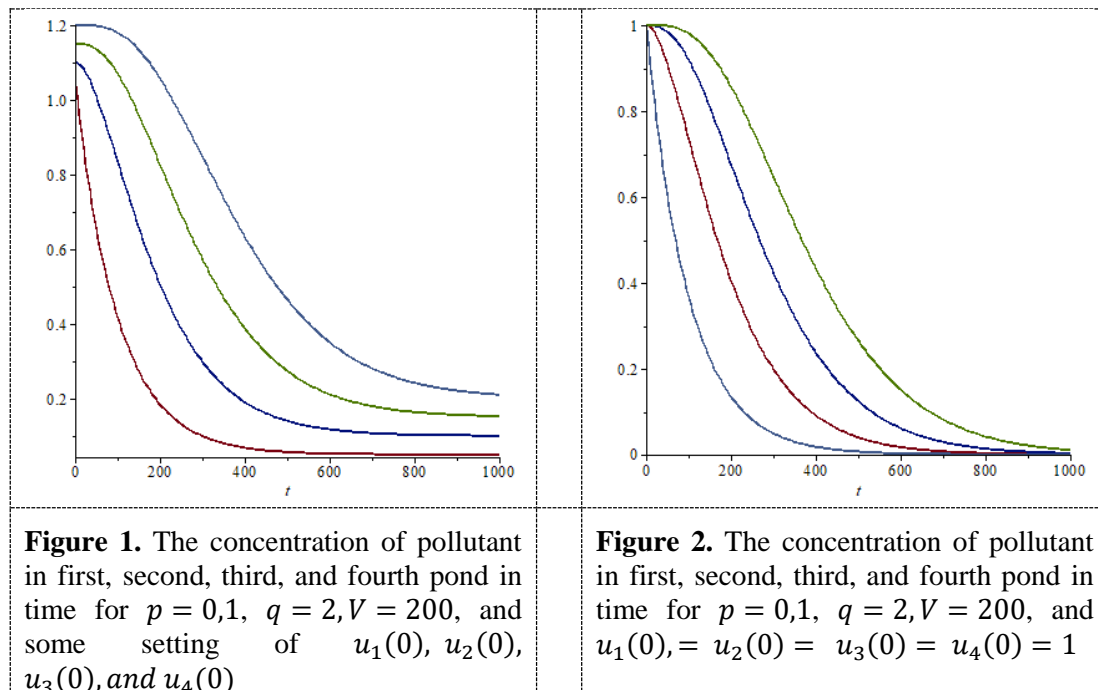
So, the proof process is completed.

### 3. Discussion

In the previous section we have created models and solution for the n-ponds with additional pollutant from fish and its food. In this section we discuss the interpretations of this model. Beside of discuss the interpretation, we also investigate some special case of this model.

Based on (3), (7), and (15), we see that if we add  $p$ (volume unit) pollutant to each pond each time, for amount of time the number of pollutants in the next ponds is accumulated with the previous. The last ponds will be full of pollutants. It is caused by the first assumption that in each pond the number of pollutant is increase such that the number of increasing in previous pond give effect, consequently it bring all pollutant proportionally with water discharge from the water source so it will accumulate until it reach the last ponds. By looking for constants part of each solution we see that if the water discharge is bigger then the amount of pollutant in each pond decrease. For the others term in the solution in the first pound its tell the decreasing of the amount of pollutant by time, differently for the next ponds this term firstly give an increasing to the amount of pollutant until one time then it always decreases. This phenomenon can be seen the Figure 1.

Now let see for the case  $p = 0$ , the amount of pollutant in each pond is only showed by exponential term such that the amount of pollutant in each pond and each time tend to zero but it never zero. Just like in the first case, the amount of pollutant in the second, third,..., until n-th ponds are rose until a time then it will decrease and tend to zero (see figure 2). From Figure 1 and Figure 2, we see that the effect of  $p$  is just like a shifting by  $\frac{p}{q}$  from one pond to the next pond.



Next, we discuss about the effect of the initial concentration of the pollutant in the pool. Firstly, if  $u(0) = 0$ , then the amount of pollutant in each pool are constant and only depends on  $p$  and  $q$ . and if we see the last condition the amount of pollutant linearly increase from the first pond to the next one. Secondly, if we created a condition such that  $\frac{k_i}{(n-i)!} = 1$  for each  $i \in \mathbb{N}$ , then the amount of pollutant in n-th pond for  $p = 0$  is given by

$$u_n(t) = e^{-\frac{q}{V}t} \left( 1 + \left( \frac{q}{V}t \right) + \cdots + \left( \frac{q}{V}t \right)^{n-1} \right)$$

$$\Leftrightarrow u_n(t) = e^{-\frac{q}{V}t} \left( \frac{1 - \left(\frac{q}{V}t\right)^n}{1 - \left(\frac{q}{V}t\right)} \right)$$

When  $n \rightarrow \infty$ , then

$$u_n(t) \rightarrow e^{-\frac{q}{V}t} \left( \frac{V}{V - qt} \right)$$

By this result we can see that the amount of pollutant will go to infinity, and the result will be negative after that. It just an impossible case.

Lastly, a unique case when we set  $u_1(0) = u_2(0) = \dots = u_n(0) = 1$  such that

$$k_n = \frac{1}{(n-1)!}$$

then we have

$$u_n(t) = e^{-\frac{q}{V}t} \left( 1 + \frac{1}{1!} \left(\frac{q}{V}t\right) + \dots + \frac{1}{(n-1)!} \left(\frac{q}{V}t\right)^{n-1} \right)$$

When  $n \rightarrow \infty$ , then

$$u_n(t) \rightarrow 1$$

So, for this single water source system the numbers of pollutant in each pool never goes to zero, and the amount of pollutant in each pond also depends on it position for  $p \neq 0$ .

#### 4. Conclusion

Good quality water is a necessary in the fish ponds. It depends on the water source and its velocity, pond's volume, and some pollutant from feeding process. By assuming that the liquid mass change in the ponds is proportional with mass of liquid that come out from the ponds each time, we can create a simple mathematical model for this case. In analytic ways, the solution of the models can be achieved. After that the solution of the models is investigated in several cases. The first case, there is a pollutant from fish and its food that come to the pond each time, from the solutions the amount of pollutant in a pond is accumulated in the rate of pollutant additions. The next case there is no additional pollutant, in this case we get that the amount of pollutant in the first ponds always decreases, but for the next ponds it slightly increases first until a time it starts to decrease. The last case if ponds are in pure condition at first then the amount of pollutant in each pond has a linear correlation that increases from the first ponds until the last pond.

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