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Globally stability analysis of the mathematical model in the IMTA system by using the energy-Casimir method

E Triyana[1], Widowati[2], S P Putro[3]

[1,2]Department of Mathematics, Faculty of Science and Mathematics, Diponegoro University, Jl. Prof Soedarto SH, Semarang 50275, Central Java, Indonesia
[3]Department of Biology, Faculty of Science and Mathematics, Diponegoro University, Jl. Prof Soedarto SH, Semarang 50275, Central Java, Indonesia.

Corresponding author: triyanaeka03@gmail.com

Abstract. Despite its benefits, fish farming has a potential impact on the water environment, such as algae bloom, fish death, and eutrophication. Integrated Multi-Trophic Aquaculture (IMTA) is designed to address environmental problems to reduce the excess of unfed pellets and fish feces under the cage. Mathematical modeling was built to describe a phenomenon in chemical, physical, and biological processes during the operation of IMTA to form a mathematical formula. The phenomenon was explained by the dynamic system, which is a method to describe, model, simulate and analyze dynamical systems. The model analyzed the interactions between nitrogen and phosphate concentrations and phytoplankton during the operation of IMTA. The model was a non-linear system of linear differential equations with three variables. Analysis of global stability is carried out at equilibrium points based on the Lyapunov stability theory using by Energy-Casimir method. Determine equilibrium point and Casimir functions of the dynamical systems, then assume that the Casimir functions are linearly independent. Find the value of the G matrix, then calculate the Lyapunov function with a positive definite value and test the validity of the Lyapunov function.

1. Introduction

Aquaculture activities are strongly influenced by water and sediment conditions can have an impact on the environment [1]. The aquaculture system produces a number of compounds such as suspended solids, total nitrogen, and total phosphorus [2]. The release of carbon (C), nitrogen (N), and phosphate (F) wastes are used to evaluate the environment for the influence of surrounding waters and the potential for IMTA cultivation [3]. Nitrogen and phosphorus are important factors in aquaculture systems, high concentrations of ammonia can be toxic to aquatic animals and can cause death. In addition, phosphorus also contributes to the eutrophication of water, which results from non-inedible feed residues, phytoplankton deaths, fish excretion and waste [4]. The IMTA system uses more than one species of biota that has a reciprocal relationship in the food chain. The application of IMTA allows farmers to obtain the same aquaculture products without increasing the area of cultivation. IMTA is different from traditional polyculture, where the cultivation is aimed at exclusive nutritional waste to supply fully or partially, nutritional inputs for other species. For example, fish farming will release soluble inorganic nutrients (eg ammonia, phosphate), resulting in suspended organic particles (feces and leftover feed) that settle on land [7][8].

Sediment is an important component in the aquatic environment, the exchange of substances between soil and water and pond bottom conditions greatly affect water quality. The surface of the sediment is flattering a number of pollutants such as organic matter, nitrogen and phosphorus concentration which can threaten ecosystem integration. The exchange of water is used to maintain ammonia concentration in aquaculture ponds. Phosphorus in marine sediments is important in evaluating responses to environmental changes. Regeneration of phosphorus dissolved through a long process, most of the phosphorus is buried in sediments [5][6].

The application of mathematical models to the IMTA system is very important for controlling and understanding interactions between species, maximizing productivity by utilizing the environment [9]. Aquatic biogeochemistry is a complex subject because of interactions between water and sediment compounds. The biogeochemical process becomes a model of calculating the concentration of compounds such as nitrogen, phosphate concentration, and oxygen which might have a negative impact on the environment [10]. Therefore, this paper discusses the dynamic model of nitrogen and phosphate concentrations on the growth rate of phytoplankton and sediments. This model is used to predict nitrogen and phosphate concentrations in IMTA.

2. Method

Natural phenomena can be modeled into mathematical modeling in the form of mathematical equations. Mathematical equations require some simplifications to form models in the form of ordinary non-linear differential equations. The equation includes the rate of change or growth of biota or chemical reactions. Limiting environmental factors to simplify models such as the level of concentration of substances and the growth of marine biota. In this paper, the formation of a model is limited to the concentration of nitrogen and phosphate against phytoplankton and sediments. Furthermore, we discussed the stability analysis of dynamic system models [16] concentrations of nitrogen and phosphate toward phytoplankton and sediments. Global stability analysis of the IMTA dynamic model [14] using the Lyapunov function [15] with the Casimir energy method

$$\begin{aligned}
\dot{P} &= \beta P - (\mu + \gamma) P, \\
\dot{N} &= q\Lambda - \gamma N - vN - \alpha\beta P \frac{N}{N+F}, \\
\dot{F} &= (1-q)\Lambda - \gamma F - \alpha\beta P \frac{F}{N+F},
\end{aligned} \tag{1}$$

Theorem 1. (Energy- Casimir Theorem)[13] The nonlinear dynamical system (1) where $f: \mathbb{D} \to \mathbb{R}^n$ is Lipschitz continuous on $\mathbb{D}$. Let $x_0 \in \mathbb{D}$ be an equilibrium point of (1) and let $C_k: \mathbb{D} \to \mathbb{R}^n, k = 1, \dots, r,$ be Casimir functions of (1). Assume that the vectors $C'_k(x_e), k = 1, \dots, r,$ are linearly independent, and suppose there exists $\mu = [\mu_1, \mu_2, \dots, \mu_r]^T \in \mathbb{R}^n$ such that $\mu_1 \neq 0, G'(x_e) = 0$ and $x^T G''(x_e) x > 0, x \in \mathcal{M}$ where $\mathcal{M} \triangleq \{x \in \mathbb{D} : C'_k(x_e) = 0, k = 2, \dots, r\}$. Then, there exists $\alpha \geq 0$ such that

$$G''(x_e) + \alpha \sum_{k=2}^{r} \left(\frac{\partial C_k}{\partial x}(x_e) \right)^T \left(\frac{\partial C_k}{\partial x}(x_e) \right) > 0 \tag{2}$$

Furthermore, the equilibrium solution $x(t) \equiv x_e$ of (1) is Lyapunov stable with Lyapunov function

$$V(x) = G(x) - G(x_e) + \frac{\alpha}{2} \sum_{k=2}^{r} \left[C_k(x) - C_k(x_e) \right]^2 \tag{3}$$

Proof. Note that

$$\dot{V}(x) = V'(x) f(x)$$

$$
\begin{aligned}
&= G'(x)\,f(x) + \alpha\sum_{k=2}^{r}\left[C_k(x) - C_k(x_e)\right]C'_k(x)\,f(x) \\
&= \sum_{k=2}^{r}\mu_k C'_k(x)\,f(x) + \alpha\sum_{k=2}^{r}\left[C_k(x) - C_k(x_e)\right]C'_k(x)\,f(x) \\
&= 0, x \in \mathrm{D}
\end{aligned}
$$

Now, it need only be shown that $V(x_e) = 0$ and $V(x) > 0, x \in \mathrm{D}, x \neq x_e$. Clearly $V(x_e) = 0$. Furthermore,

$$\dot{V}(x) = G'(x) + \alpha\sum_{k=2}^{r}\left[C_k(x) - C_k(x)\right]C'_k(x)$$

and hence, $V'(x) = 0$

next, note that

$$\dot{V}(x) = G''(x) + \alpha\sum_{k=2}^{r}\left\{\left[C'_k(x)\right]^T C'_k(x) + \left[C_k(x) - C_k(x_e)\right]C''_k(x)\right\} \tag{4}$$

And hence

$$\ddot{V}(x) = G''(x_e) + \alpha\sum_{k=2}^{r}\left(\frac{\partial C'_k}{\partial x}(x_e)\right)^T\left(\frac{\partial C_k}{\partial x}(x_e)\right) \tag{5}$$

$$G''(x_e) = S^T\begin{bmatrix} G_1 & G_{12} \\ G_{12}^T & G_2 \end{bmatrix}S \tag{6}$$

and

$$\sum_{k=2}^{r}\left(\frac{\partial C'_k}{\partial x}(x_e)\right)^T\left(\frac{\partial C_k}{\partial x}(x_e)\right) = S^T\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}S \tag{7}$$

Substituting (6) and (7) into (5) yields

$$\ddot{V}(x) = S^T\begin{bmatrix} G_1 & G_{12} \\ G_{12}^T & G_2 + \alpha N \end{bmatrix}S \,\square\, S^T Q S$$

Choosing $\alpha \geq 0$ such that $Q > 0$

The existence of energy-Casimir functions for (1) can be used to construct the Lyapunov function for (1). We can construct a function $H : \mathbb{D} \to \mathbb{R}$ such that $\dot{H}(x) = 0$ along the trajectories of the nonlinear dynamical system (1). If $C_1, \ldots, C_r$ are Casimir functions for (1), then $\frac{d}{dt}\left[H + G(C_1, \ldots, C_r)\right](x(t)) \equiv 0$, for every function $G: \mathbb{R}^n \to \mathbb{R}$. Hence, even if H is not positive definite at the equilibrium $x_e \in \mathbb{D}$, the function $V(x) = H(x) + G(C_1(t), \ldots, C_r(t))$ can be made positive definite at $x_e \in \mathbb{D}$ by properly choosing G so that $V(x)$ is a Lyapunov fuction for (1).

In the Energy-Casimir method, steps are used to construct the Lyapunov function on the system (1) are as follows: There is a Lipschitz constant $g(t)$ that satisfy $\left\| f(x_1(t), t) - f(x_2(t), t) \right\| \leq g(t)\left\| x_1 - x_2 \right\|$,so the system applies to every $t \in \mathbb{R}$. System (1) can be expressed in the form

$$
\begin{aligned}
\frac{dp}{dt} &= f(p(t), t) \\
\frac{dn}{dt} &= f(n(t), t) \\
\frac{df}{dt} &= f(f(t), t)
\end{aligned}
$$

Suppose there is a vector $p = (p_1, p_2), n = (n_1, n_2) and\ f = (f_1, f_2)$. will then be searched for who $g(t)$ is a Lipschitz constant

$$f(x_1(t),t) - f(x_2(t),t) \le g(t) x_1 - x_2$$

With

$$\left\| f(x_1(t),t) - f(x_2(t),t) \right\| = \left\| \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \right\|$$

Based on the system equation (1) can be formed as follows

- $a_{11} = (\beta - \mu - \gamma)(p_1 - p_2)$, suppose $(\beta - \mu - \gamma)(p_1 - p_2) = \delta_1(t)$ we obtained
  $$|a_{11}| = |\delta_1(t)| \le |\delta_1(t)|$$
- $a_{21} = q\Lambda - \gamma n_1 - v n_1 - \alpha\beta p_1 \frac{n_1}{n_1+f_1} - \left(q\Lambda - \gamma n_2 - v n_2 - \alpha\beta p_2 \frac{n_2}{n_2+f_2}\right)$
  $= (-\gamma - v)(n_1 - n_2) - \alpha\beta\left(\frac{p_1 n_1}{n_1+f_1} - \frac{p_3 n_2}{n_2+f_2}\right)$, suppose $-\alpha\beta\left(\frac{p_1 n_1}{n_1+f_1} - \frac{p_3 n_2}{n_2+f_2}\right) = \delta_2(t)(n_1 - n_2)$, then
  $a_{21} = \delta_2(t)(n_1 - n_2)$
- $a_{31} = (1-q)\Lambda - \gamma f_1 - \alpha\beta p_1 \frac{f_1}{n_1+f_1} - \left((1-q)\Lambda - \gamma f_2 - \alpha\beta p_2 \frac{f_2}{n_2+f_2}\right)$
  $= -\gamma(f_1 - f_2) - \alpha\beta\left(\frac{p_1 f_1}{n_1+f_1} - \frac{p_2 f_2}{n_2+f_2}\right)$, suppose $-\alpha\beta\left(\frac{p_1 f_1}{n_1+f_1} - \frac{p_2 f_2}{n_2+f_2}\right) = \delta_3(t)(f_1 - f_2)$

Can we write follows as

$$\left\| \begin{bmatrix} f(p_1(t),t) - f(p_2(t),t) \\ f(n_1(t),t) - f(n_2(t),t) \\ f(f_1(t),t) - f(f_2(t),t) \end{bmatrix} \right\| \le g(t) \left\| \begin{bmatrix} p_1 - p_2 \\ n_1 - n_2 \\ f_1 - f_2 \end{bmatrix} \right\|$$

Then the system satisfies the conditions of Lipschitz.

Suppose $C_1 = \frac{2\alpha\beta(n)^2}{n+f} - n$ and $C_2 = \frac{2\alpha\beta(f)^2}{n+f} - f$

We obtained

$C'_1 = \left(\frac{4\alpha\beta n}{n+f} - \frac{2\alpha\beta(n)^2}{(n+f)^2} - 1, 0, 0\right)$ and $C'_2 = \left(0, \frac{4\alpha\beta f}{n+f} - \frac{2\alpha\beta(f)^2}{(n+f)^2} - 1, 0\right)$ is linearly independent.

$C_1 = \frac{2\alpha\beta(n)^2}{n+f} - n$ and $C_2 = \frac{2\alpha\beta(f)^2}{n+f} - f$ are Casimir function for (1). Now, letting $G(x) = \mu_1 C_1(x) + \mu_2 C_2(x)$ i.e

$G(p,n,f) = \mu_1 C_1 + \mu_2 C_2 = \mu_1\left(\frac{2\alpha\beta(n)^2}{n+f} - n\right) + \mu_2\left(\frac{2\alpha\beta(f)^2}{n+f} - f\right)$ it follows that $G'(x_e) = 0$ and $x^T G''(x_e) x > 0, x \in \mathrm{M}$

$$G'(p,n,f) = \begin{bmatrix} \frac{\partial E}{\partial p} & \frac{\partial E}{\partial n} & \frac{\partial E}{\partial f} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mu_1\left(\frac{4\alpha\beta n}{n+f} - \frac{2\alpha\beta(n)^2}{(n+f)^2} - 1\right) - 2\left(\mu_2\left(\frac{2\alpha\beta(f)^2}{(n+f)^2}\right)\right) & \mu_2\left(\frac{4\alpha\beta f}{n+f} - \frac{2\alpha\beta(f)^2}{(n+f)^2} - 1\right) - 2\left(\mu_1\left(\frac{2\alpha\beta(n)^2}{(n+f)^2}\right)\right) \end{bmatrix}$$

$G'(T_1) = [0 \quad 0 \quad 0]$

so, we obtained

$$G''(p,n,f) = \begin{bmatrix} \frac{\partial \dot{G}}{\partial p} & \frac{\partial \dot{G}}{\partial n} & \frac{\partial \dot{G}}{\partial f} \end{bmatrix}^T$$

$$=\begin{bmatrix} 0 & 0 \\ 0 \;\; 4\left(\mu_1\left(\frac{\alpha\beta(n)^2+(-2f-2n)(\alpha\beta n)+\left(f^2+2fn+n^2\right)(\alpha\beta)}{(n+f)^3}\right)\right)+4\left(\mu_2\left(\frac{\alpha\beta(f)^2}{(n+f)^3}\right)\right)-4\left(\mu_2\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)\right)-4\left(\mu_2\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)\right) & 0 \\ 0 & 0 \\ & -4\left(\mu_1\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)\right)+4\left(\mu_1\left(\frac{\alpha\beta(n)^2}{(n+f)^3}\right)\right)-4\left(\mu_2\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)\right)-4\left(\mu_2\left(\frac{-\alpha\beta(f)^2+(2f+2n)\alpha\beta f+\left(f^2-2fn-n^2\right)\alpha\beta}{(n+f)^3}\right)\right) \end{bmatrix}$$

Next by selecting any μ_1 and μ_2 are the real number. For example $\mu_1 = 4$ and $\mu_2 = 4$ and μ_1 and $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x_1, x_2, x_3 \neq 0$. Because $C_1' = \left(\frac{4\alpha\beta n}{n+f}-\frac{2\alpha\beta(n)^2}{(n+f)^2}-1,0,0\right)$, then $C'(T_1)=(0,0,0)$, so obtained

$$C'(T_1)x=(0,0,0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

Because $C'_2 = \left(0,\frac{4\alpha\beta f}{n+f}-\frac{2\alpha\beta(f)^2}{(n+f)^2}-1,0\right)$, then $C'(T_2)=(0,0,0)$, so obtained

$$C'(T_2)x=(0,0,0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

So, $x^T G''(x_e)x>0$

$$=\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 \;\; 16\left(\left(\frac{\alpha\beta(n)^2+(-2f-2n)(\alpha\beta n)+\left(f^2+2fn+n^2\right)(\alpha\beta)}{(n+f)^3}\right)+\left(\frac{\alpha\beta(f)^2}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)\right) & 0 \\ 0 & 0 \\ & +16\left(-\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)+\left(\frac{\alpha\beta(n)^2}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+(2f+2n)\alpha\beta f+\left(f^2-2fn-n^2\right)\alpha\beta}{(n+f)^3}\right)\right) \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=16\left(\left(\frac{\alpha\beta(n)^2+(-2f-2n)(\alpha\beta n)+\left(f^2+2fn+n^2\right)(\alpha\beta)}{(n+f)^3}\right)+\left(\frac{\alpha\beta(f)^2}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)\right)x_2^2$$
$$+16\left(-\left(\frac{-\alpha\beta(n)^2+\alpha\beta n(n+f)}{(n+f)^3}\right)+\left(\frac{\alpha\beta(n)^2}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+\alpha\beta f(n+f)}{(n+f)^3}\right)-\left(\frac{-\alpha\beta(f)^2+(2f+2n)\alpha\beta f+\left(f^2-2fn-n^2\right)\alpha\beta}{(n+f)^3}\right)\right)x_2^2>0$$

fulfill assumptions

$$G''(x_e)+\alpha\sum_{k=2}^{r}\left(\frac{\partial C_k}{\partial x}(x_e)\right)^T\left(\frac{\partial C_k}{\partial x}(x_e)\right)>0$$

Lyapunov function at equilibrium point.is

$$V(x)=G(x)-G(x_e)+\frac{\alpha}{2}\sum_{k=2}^{r}\left(C_k(x)-C_k(x_e)\right)^2$$

$$=G(p,n,f)-G(p^*,n^*,f^*)+\frac{\alpha}{2}\sum_{k=2}^{r}\left(C_k(p,n,f)-C_k(p^*,n^*,f^*)\right)^2$$

$$=\mu_1\left(\frac{2\alpha\beta(n)^2}{n+f}-n\right)+\mu_2\left(\frac{2\alpha\beta(f)^2}{n+f}-f\right)-\mu_1\left(\frac{2\alpha\beta(n^*)^2}{n^*+f^*}-n^*\right)+\mu_2\left(\frac{2\alpha\beta(f^*)^2}{n^*+f^*}-f^*\right)$$

$$+\frac{\alpha}{2}\left(\left(\frac{2\alpha\beta(f)^2}{n+f}-f\right)-\left(\frac{2\alpha\beta(f^*)^2}{n^*+f^*}-f^*\right)\right)^2$$

$$=4\left(\frac{2\alpha\beta(n)^2}{n+f}-n\right)+4\left(\frac{2\alpha\beta(f)^2}{n+f}-f\right)-4\left(\frac{2\alpha\beta(n^*)^2}{n^*+f^*}-n^*\right)+4\left(\frac{2\alpha\beta(f^*)^2}{n^*+f^*}-f^*\right)$$

$$+\frac{1}{2}\left(\left(\frac{2\alpha\beta(f)^2}{n+f}-f\right)-\left(\frac{2\alpha\beta(f^*)^2}{n^*+f^*}-f^*\right)\right)^2$$

$$=2\alpha\beta\left(4\left(\frac{(n)^2}{n+f}-n\right)+4\left(\frac{(f)^2}{n+f}-f\right)-4\left(\frac{(n^*)^2}{n^*+f^*}-n^*\right)+4\left(\frac{(f^*)^2}{n^*+f^*}-f^*\right)+\frac{1}{2}\left(\left(\frac{(f)^2}{n+f}-f\right)-\left(\frac{(f^*)^2}{n^*+f^*}-f^*\right)\right)\right) \quad (8)$$

Lyapunov function (8) is definite positive because $V(x)>0$, for all p,n,f and $\dot{V}(x)<0$ for all p,n,f, so the dynamical system is globally asymptotically stable.

3. Simulation Numeric

In this simulation numeric, a review of equations is presented for nitrogen and phosphate concentrations and phytoplankton models as system dynamics. The equilibrium point on the data is $(P^*,N^*,F^*)=(0,30.3559,5.8529)$, analyze globally stability around the equilibrium point. Some data parameter i.e $\Lambda=3.98, v=0.05, q=0.9, \alpha=0.54, \beta=0.55, \mu=0.85, \gamma=0.068$

$$\frac{dP}{dt}=-0.41P$$
$$\frac{dN}{dt}=3.51-0.11N-0.25P\frac{N}{N+F} \quad (9)$$
$$\frac{dF}{dt}=0.39-0.06F-0.25P\frac{F}{N+F}$$

Suppose $C_1=\frac{0.5(n)^2}{n+f}-n$ and $C_2=\frac{0.5(f)^2}{n+f}-f$

We obtained

$C'_1=\left(\frac{n}{n+f}-\frac{0.5(n)^2}{(n+f)^2}-1,0,0\right)$ and $C'_2=\left(0,\frac{f}{n+f}-\frac{0.5(f)^2}{(n+f)^2}-1,0\right)$ is linearly independent.

$C_1=\frac{0.5(n)^2}{n+f}-n$ and $C_2=\frac{0.5(f)^2}{n+f}-f$ are Casimir's function for (9). Now, letting $G(x)=\mu_1C_1(x)+\mu_2C_2(x)$ i.e

$G(p,n,f)=\mu_1C_1+\mu_2C_2=\mu_1\left(\frac{(n)^2}{n+f}-n\right)+\mu_2\left(\frac{0.5(f)^2}{n+f}-f\right)$ it follows that $G'(x_e)=0$ and $x^TG''(x_e)x>0, x\in \mathrm{M}$

$$\dot{G}(p,n,f)=\begin{bmatrix}\frac{\partial E}{\partial p} & \frac{\partial E}{\partial n} & \frac{\partial E}{\partial f}\end{bmatrix}$$

$$=\left[0 \quad \mu_1\left(\frac{2n}{n+f}-\frac{(n)^2}{(n+f)^2}-1\right)-0.5\left(\mu_2\left(\frac{(f)^2}{(n+f)^2}\right)\right) \quad -\mu_1\left(\frac{(f)^2}{(n+f)^2}-1\right)+\mu_2\left(\frac{f}{n+f}-\frac{0.5(f)^2}{(n+f)^2}-1\right)\right]$$

$\dot{G}(T_1)=[0 \quad 0 \quad 0]$

so, we obtained

$$G''(p,n,f)=\left[\frac{\partial G'}{\partial p} \quad \frac{\partial G'}{\partial n} \quad \frac{\partial G'}{\partial f}\right]^T$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\left(\mu_1\left(\frac{(f^2)}{(n+f)^3}\right)\right)+\left(\mu_2\left(\frac{(f)^2}{(n+f)^3}\right)\right)-2\left(\mu_1\left(\frac{nf}{(n+f)^3}\right)\right)-\left(\mu_2\left(\frac{nf}{(n+f)^3}\right)\right) & 0 \\ 0 & 0 & -2\left(\mu_1\left(\frac{nf}{(n+f)^3}\right)\right)+2\left(\mu_1\left(\frac{(n)^2}{(n+f)^3}\right)\right)-\left(\mu_2\left(\frac{nf}{(n+f)^3}\right)\right)+\left(\mu_2\left(\frac{n^2}{(n+f)^3}\right)\right) \end{bmatrix}$$

Next by selecting any μ_1 and μ_2 are the real number. For example $\mu_1=4$ and $\mu_2=4$ and μ_1 and $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x_1, x_2, x_3 \neq 0$. Because $C'_1=\left(\frac{n}{n+f}-\frac{0.5(n)^2}{(n+f)^2}-1,0,0\right)$, then $C'(T_1)=(0,0,0)$, so obtained

$$C'(T_1)x=(0,0,0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

Because $C'_2=\left(0,\frac{f}{n+f}-\frac{0.5(f)^2}{(n+f)^2}-1,0\right)$, then $C'(T_2)=(0,0,0)$, so obtained

$$C'(T_2)x=(0,0,0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

So, $x^T G''(x_e)x>0$

$$=(x_1 \quad x_2 \quad x_3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(2\left(\frac{(f^2)}{(n+f)^3}\right)+\left(\frac{(f)^2}{(n+f)^3}\right)-2\left(\frac{nf}{(n+f)^3}\right)-\left(\frac{nf}{(n+f)^3}\right)\right) & 0 \\ 0 & 0 & 4\left(-2\left(\frac{nf}{(n+f)^3}\right)+2\left(\frac{(n)^2}{(n+f)^3}\right)-\left(\frac{nf}{(n+f)^3}\right)+\left(\frac{n^2}{(n+f)^3}\right)\right) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=4\left(2\left(\frac{(f^2)}{(n+f)^3}\right)+\left(\frac{(f)^2}{(n+f)^3}\right)-2\left(\frac{nf}{(n+f)^3}\right)-\left(\frac{nf}{(n+f)^3}\right)\right)x_2^2>0$$

$$+4\left(-2\left(\frac{nf}{(n+f)^3}\right)+2\left(\frac{(n)^2}{(n+f)^3}\right)-\left(\frac{nf}{(n+f)^3}\right)+\left(\frac{n^2}{(n+f)^3}\right)\right)x_3^2>0$$

fulfill assumptions

$$G''(x_e)+\alpha\sum_{k=2}^{r}\left(\frac{\partial C_k}{\partial x}(x_e)\right)^T\left(\frac{\partial C_k}{\partial x}(x_e)\right)>0$$

Lyapunov function at equilibrium point.is

$$\begin{aligned}
V(x) &= G(x)-G(x_e)+\frac{\alpha}{2}\sum_{k=2}^{r}\left(C_k(x)-C_k(x_e)\right)^2 \\
&= G(p,n,f)-G(p^*,n^*,f^*)+\frac{\alpha}{2}\sum_{k=2}^{r}\left(C_k(p,n,f)-C_k(p^*,n^*,f^*)\right)^2 \\
&= \mu_1\left(\frac{(n)^2}{n+f}-n\right)+\mu_2\left(\frac{0.5(f)^2}{n+f}-f\right)-\mu_1\left(\frac{(n^*)^2}{n^*+f^*}-n^*\right)-\mu_2\left(\frac{0.5(f^*)^2}{n^*+f^*}-f^*\right)+\frac{\alpha}{2}\left(\left(\frac{0.5(f)^2}{n+f}-f\right)-\left(\frac{0.5(f^*)^2}{n^*+f^*}-f^*\right)\right)^2 \\
&= 4\left(\frac{(n)^2}{n+f}-n\right)+4\left(\frac{0.5(f)^2}{n+f}-f\right)-4\left(\frac{(n^*)^2}{n^*+f^*}-n^*\right)+4\left(\frac{0.5(f^*)^2}{n^*+f^*}-f^*\right)+\frac{1}{2}\left(\left(\frac{0.5(f)^2}{n+f}-f\right)-\left(\frac{0.5(f^*)^2}{n^*+f^*}-f^*\right)\right)^2
\end{aligned} \tag{10}$$

Lyapunov function (3) is definite positive because $V(x)>0$, for all p,n,f and $\dot{V}(x)<0$ for all p,n,f , so the dynamical system is globally asymptotically stable.

4. Conclusion

Over the last few decades, aquaculture has become the fastest-growing food product and is expected to meet global fish production demand. However, increased cultivation can cause impacts on the environment. The environmental impact is very influential on the results of aquaculture production. Integrated multi-trophic cultivation is one way to reduce the ecological effects of fish farming. The IMTA system is complex and depends on between the species being cultivated or between the organism and its physical and chemical environment. In IMTA cultivation, waste from food scraps and fish excretion are used as feed for another marine biota. In this paper, mathematical models are used to analyze interactions between nitrogen and phosphate concentrations and phytoplankton for aquaculture. This model is a system of linear non-linear differential equations with three variables. Analysis of global stability is carried out at the equilibrium point based on Lyapunov's stability theory

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References

[1] Putro S P, Ferbria I J, and Muhammad F 2014 **22**. 3; 79-83
[2] Ariel E, 2014 *Turciosutta and Papenbrock Sustainability* **6**, 836-856; DOI:10.3390/su6020836
[3] Wang X, Olsen L M, Reitan K I, Olsen Y 2012 *Aquaculture Environment Interactions* 267–283
[4] Zhou L, Boyd C E. 2015 https://www.researchgate.net/publication/284447968
[5] Sajana T K, Ghangrekar M M, Mitra A 2013 *Aquacultural Engineering* **57** 101– 107
[6] Liu S M, Zhang J, Lib D J 2004 *Estuarine, Coastal and Shelf Science* **59** 209e218
[7] Putro S P, Widowati, Suhartana and Muhammad F 2015 *Internat. J. Sci. Eng* **85**-89
[8] Reid G K, Cranford P J, Robinson S M C, Filgueira R And Guyondet 2011 *Bulletin Of The Aquaculture Association Of Canada 109-2*
[9] Granada L, Lopes S, Novais S C,. Lemos M F L 2018 *Aquaculture* **495** 90–97
[10] Serpa D, Ferreira P P, Caetano M, Fonseca L C, Dinis M T, Duarte P 2012 *Ecological Modelling* **247** 286– 301
[11] Menéndez M, Herrera J And Comín F A 2002 *Sci. Mar* **66 (4):** 355-364
[12] Putro S P, Widowati, Rahmansyah, Suminto 2016 *AACL Bioflux* **9** Issue 4.

[13] Haddad W M and Chellaboina V 2007 Nonlinear Dynamical Systems and Control A Lyapunov-Based Approach. Princeton University Press Princeton and Oxford
[14] Widowati, Putro S P , and Silfiana 2018 *J. Phys.: Conf. Ser.* **1025** 012088
[15] Widowati, Putro S P, Maan N, Sulpiani R 2019 *J. Phys.: Conf. Ser.* **1217** 012059
[16] Silfiana, Widowati, Putro S P and Udjiani T 2018 *J. Phys.: Conf. Ser.* **983** 012122