Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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Construction of lyapunov function using gradient method to stability analysis of the nitrogen-phosphate-phytoplankton-sediment interaction model

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Abstract. The significant growth in catch fisheries and aquaculture production has enhanced the world's capacity to consume diverse and nutritious food; however, this fast-growing industry has resulted in the environmental disturbance, especially caused by fish farming. The nutrient was generated from fish excretion and unfed pellet in the form of dissolved organic matter, especially as nitrogen and phosphate particulates. Therefore, applying mathematical models is crucial to understand and minimize the impact on the water ecosystem, thus maximizing productivity. This paper aims to research to analyze the global and stability of the equilibrium point the dynamical system that water and sediment in aquaculture Integrated Multi-Trophic Aquaculture (IMTA) system. Analysis stability global has been proved by constructing the Lyapunov function. The gradient method is used to construct the Lyapunov function in the most general form. The method is based on the assumption of a variable gradient function by determining V and ∇V. In the Lyapunov method construction, if the function $V(x)$ would be definite positive, whereas derivative $V(x)$ would be a definite negative. Models of nitrogen, phosphate, phytoplankton, and sediment system exhibited globally asymptotically stable. This implies that fish farming using the IMTA system on Karimunjawa island was still considerably under normal condition.

1. Introduction

Multi-species polyculture in central and eastern Europe, Asia and Latin America is used to increase fish production by utilizing leftover feed and waste recycling to improve water quality in production systems. Indonesia ranks second after China as a major producer of marine fisheries [1]. The development of the aquaculture sector is growing rapidly and attracting the attention of researchers, industry, and policymakers as a promising opportunity for the massive expansion of the aquaculture industry [5]. Integrated Multi-Trophic Aquaculture (IMTA) is polyculture cultivation with several marine biotas that are maintained in it to optimize the recycling of waste as a food source [9].

The aim of the Integrated Multi-Trophic Aquaculture (IMTA) is to become an ecologically balanced aquaculture practice by processing multi-species from various trophic levels to optimize the recycling of feed and stool [11] as a food source. This provides an opportunity to increase production and economic output if it is managed optimally for each species. The IMTA aquaculture system which combines dynamic energy budgets for several species from different trophic levels was developed to optimize the productivity of multi-species [7].

Water, sediment characteristics and agricultural practices are environmental variables that play an important role in aquaculture[2]. Seabed and sediment ecosystems can be affected by particulate nutrients that sink rapidly into the deep sea. Nutrients with organic molecules, dissolved organic N and P are absorbed and assimilated by phytoplankton [12]. Particulate nutrients (nitrogen and phosphorus) can be converted by microalgae from waste into biomass and bio-products, absorption of nutrients and assimilation by phytoplankton, thereby increasing the sustainability of wastewater treatment [13]. Nitrogen and phosphate concentrations in macroalgae differ morphologically which describes various processes in controlling nutrient availability and can be used as effective indicators of various sources of nutrients in the tropical system [6].

Nitrogen is a key element in the cultivation and management of important ponds. In aquaculture systems, nitrogen accumulation causes disturbance of ecosystem balance [14]. Also, carbon, nitrogen, and phosphate can estimate potential pollution from aquaculture facilities [15] [16]

The purpose of this study is to determine the stability analysis of dynamic models of nitrogen and phosphate concentrations in sediments and phytoplankton in the area of the Integrated Multi-Trophic Aquaculture (IMTA) Aquarium using the Lyapunov method [8].

2. Gradient Method to Construction Lyapunov Function

The Aquaculture growth is not a major cause of nutrient loading into aquatic ecosystems but is a result of fast-growing industries [19]. The impact on the environment cannot be ignored, to reduce environmental pollution caused by leftover food and metabolic products, which is done by implementing integrated multi-trophic cultivation (IMTA). IMTA can be in the form of land-based or open water systems such as marine or freshwater systems and several species are combined [10] to reduce environmental pollution. Phytoplankton in IMTA plays an important role because it has chlorophyll for photosynthesis [12]. The interaction of phytoplankton in aquaculture is related to its role in reducing nitrogen and phosphate concentrations. Dynamic models of nitrogen, phosphate, phytoplankton, and sediments in the IMTA will determine global analysis using the Variable Gradient Method is a formal approach to compiling Lyapunov functions [3]. The interaction model of the nitrogen, phosphate, phytoplankton, and sediments on the IMTA system is explained by the following ordinary differential equation

$$\begin{aligned}
\dot{P} &= \beta P-(\mu+\gamma)P, \\
\dot{N} &= q\Lambda+\omega\theta S-\gamma N-vN-\alpha\beta P\frac{N}{N+F}, \\
\dot{F} &= (1-q)\Lambda+(1-\omega)\theta S-\gamma F-\alpha\beta P\frac{F}{N+F}, \\
\dot{S} &= \mu P-\theta S,
\end{aligned} \tag{1}$$

where P, N, F, S are phytoplankton, nitrogen, fosfat and sediment, respectively at t is time. Linear equation depending on the time

$$\dot{x} = f(x,t) \text{ with } \dot{x} = \begin{bmatrix} \dot{P} \\ \dot{N} \\ \dot{F} \\ \dot{S} \end{bmatrix}$$

so obtained

$$f_1 = \beta P - (\mu + \gamma) P,$$
$$f_2 = q\Lambda + \omega\theta S - \gamma N - vN - \alpha\beta P \frac{N}{N+F},$$
$$f_3 = (1-q)\Lambda + (1-\omega)\theta S - \gamma F - \alpha\beta P \frac{F}{N+F},$$
$$f_4 = \mu P - \theta S,$$

Gradient equation of Lyapunov function

$$\nabla V = \begin{bmatrix} a_{11}P & a_{12}N & a_{13}F & a_{14}S \\ a_{21}P & a_{22}N & a_{23}F & a_{24}S \\ a_{31}P & a_{32}N & a_{33}F & a_{34}S \\ a_{41}P & a_{42}N & a_{43}F & a_{44}S \end{bmatrix}$$

$$\nabla V = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} P \\ N \\ F \\ S \end{bmatrix}$$

then from equation (1), the differential $\dot{V}$ can be determined from ∇V (the gradient of V) as follows [4]

$$\dot{V}(\boldsymbol{x}) = \nabla V^T f$$

$$\dot{V}(\boldsymbol{x}) = \begin{bmatrix} a_{11}P + a_{12}N + a_{13}F + a_{14}S \\ a_{21}P + a_{22}N + a_{23}F + a_{24}S \\ a_{31}P + a_{32}N + a_{33}F + a_{34}S \\ a_{41}P + a_{42}N + a_{43}F + a_{44}S \end{bmatrix}^T \begin{bmatrix} \beta P - (\mu+\gamma)P \\ q\Lambda + \omega\theta S - \gamma N - vN - \alpha\beta P \frac{N}{N+F} \\ (1-q)\Lambda + (1-\omega)\theta S - \gamma F - \alpha\beta P \frac{F}{N+F} \\ \mu P - \theta S \end{bmatrix}$$

$$\begin{aligned} &= (a_{11}P + a_{12}N + a_{13}F + a_{14}S)(\beta P - (\mu+\gamma)P) \\ &+ (a_{21}P + a_{22}N + a_{23}F + a_{24}S)\left(q\Lambda + \omega\theta S - \gamma N - vN - \alpha\beta P \frac{N}{N+F}\right) \\ &+ (a_{31}P + a_{32}N + a_{33}F + a_{34}S)\left((1-q)\Lambda + (1-\omega)\theta S - \gamma F - \alpha\beta P \frac{F}{N+F}\right) \\ &+ (a_{41}P + a_{42}N + a_{43}F + a_{44}S)(\mu P - \theta S) \end{aligned}$$

We assume that the gradient of the undetermined Lyapunov function has the following form

$$\nabla V_1 = a_{11}P + a_{12}N + a_{13}F + a_{14}S$$
$$\nabla V_2 = a_{21}P + a_{22}N + a_{23}F + a_{24}S$$
$$\nabla V_3 = a_{31}P + a_{32}N + a_{33}F + a_{34}S$$
$$\nabla V_4 = a_{41}P + a_{42}N + a_{43}F + a_{44}S$$

The curl equation is

$$\frac{\partial \nabla V_1}{\partial \mathrm{N}} = \frac{\partial \nabla V_2}{\partial \mathrm{P}} \Rightarrow a_{12} = a_{21}$$
$$\frac{\partial \nabla V_1}{\partial \mathrm{F}} = \frac{\partial \nabla V_3}{\partial \mathrm{P}} \Rightarrow a_{13} = a_{31}$$
$$\frac{\partial \nabla V_1}{\partial \mathrm{S}} = \frac{\partial \nabla V_4}{\partial \mathrm{P}} \Rightarrow a_{14} = a_{41}$$
$$\frac{\partial \nabla V_2}{\partial \mathrm{F}} = \frac{\partial \nabla V_3}{\partial \mathrm{N}} \Rightarrow a_{23} = a_{32}$$
$$\frac{\partial \nabla V_2}{\partial \mathrm{S}} = \frac{\partial \nabla V_4}{\partial \mathrm{N}} \Rightarrow a_{24} = a_{42}$$
$$\frac{\partial \nabla V_3}{\partial \mathrm{S}} = \frac{\partial \nabla V_4}{\partial \mathrm{F}} \Rightarrow a_{34} = a_{43}$$

Then $\dot{V}$can be computed as

$$\mathrm{V}(\boldsymbol{x}) = \int \nabla V dx$$

$$= \int_0^P \nabla V_1 dP + \int_0^N \nabla V_2 dN + \int_0^F \nabla V_3 dF + \int_0^S \nabla V_4 dS$$

$$= \int_0^P a_{11}P + a_{12}N + a_{13}F + a_{14}S\,dP + \int_0^N a_{21}P + a_{22}N + a_{23}F + a_{24}S\,dN$$

$$+ \int_0^F a_{31}P + a_{32}N + a_{33}F + a_{34}S\,dF + \int_0^S a_{41}P + a_{42}N + a_{43}F + a_{44}SdS$$

$$= \frac{1}{2}a_{11}P^2 + a_{12}NP + a_{13}FP + a_{14}SP + \frac{1}{2}a_{22}N^2 + a_{21}PN + a_{23}FN + a_{24}SN$$

$$+ \frac{1}{2}a_{33}F^2 + a_{31}PF + a_{32}NF + a_{34}SF + \frac{1}{2}a_{44}S^2 + a_{41}PS + a_{42}NS + a_{43}FS$$

$$V(x) = \frac{1}{2}a_{11}P^2 + \frac{1}{2}a_{22}N^2 + \frac{1}{2}a_{33}F^2 + \frac{1}{2}a_{44}S^2 + 2a_{12}NP + 2a_{13}FP + 2a_{14}SP + 2a_{23}FN + 2a_{24}SN + 2a_{34}SF \quad (2)$$

Next, make the positive definite matrix equation for the Lyapunov function as follows

$$V(x) = X^T PX$$

$$V(x) = \begin{bmatrix} P & N & F & S \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{23} & A_{33} & A_{34} \\ A_{41} & A_{24} & A_{34} & A_{44} \end{bmatrix} \begin{bmatrix} P \\ N \\ F \\ S \end{bmatrix}$$

$$V(x) = \begin{bmatrix} P & N & F & S \end{bmatrix} \begin{bmatrix} A_{11}P + A_{12}N + A_{13}F + A_{14}S \\ A_{21}P + A_{22}N + A_{23}F + A_{24}S \\ A_{31}P + A_{32}N + A_{33}F + A_{34}S \\ A_{41}P + A_{42}N + A_{43}F + A_{44}S \end{bmatrix}$$

$$V(x) = A_{11}P^2 + A_{12}PN + A_{13}PF + A_{14}PS + A_{21}NP + A_{22}N^2 + A_{23}NF + A_{24}NS$$
$$+ A_{31}FP + A_{32}FN + A_{33}F^2 + A_{34}FS + A_{41}SP + A_{42}SN + A_{43}SF + A_{44}S^2$$

$$V(x) = A_{11}P^2 + A_{22}N^2 + A_{33}F^2 + A_{44}S^2 + 2A_{12}PN + 2A_{13}PF + 2A_{14}PS + 2A_{23}NF + 2A_{24}NS + 2A_{34}FS \quad (3)$$

Equation (2) and (3) is same, than

$$V(x) = \frac{1}{2}a_{11}P^2 + \frac{1}{2}a_{22}N^2 + \frac{1}{2}a_{33}F^2 + \frac{1}{2}a_{44}S^2 + 2a_{12}NP + 2a_{13}FP + 2a_{14}SP + 2a_{23}FN + 2a_{24}SN + 2a_{34}SF$$
$$= A_{11}P^2 + A_{22}N^2 + A_{33}F^2 + A_{44}S^2 + 2A_{12}PN + 2A_{13}PF + 2A_{14}PS + 2A_{23}NF + 2A_{24}NS + 2A_{34}FS$$

We obtained

$$A_{11} = \frac{1}{2}a_{11}; A_{22} = \frac{1}{2}a_{22}; A_{33} = \frac{1}{2}a_{33};$$
$$2A_{12} = 2a_{12}; 2A_{13} = 2a_{13}; 2A_{14} = 2a_{14};$$
$$2A_{23} = 2a_{23}; 2A_{24} = 2a_{24}; 2A_{34} = 2a_{34};$$

$$A = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & \frac{1}{2}a_{44} \end{bmatrix}$$

Using Sylvester's criterion to determine the first, second, third and fourth-order respectively from the *A* matrix.

$$\Delta_1 = \frac{1}{2}a_{11} < 0$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{2}a_{11} & a_{12} \\ a_{12} & \frac{1}{2}a_{22} \end{vmatrix}$$
$$= \left(\frac{1}{2}a_{11}.\frac{1}{2}a_{22}\right) - \left(a_{12}.a_{12}\right)$$
$$= \frac{1}{4}a_{11}.a_{22} - a_{12}.a_{12} > 0$$
$$\Delta_3 = \begin{vmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} \end{vmatrix}$$
$$= \left(\frac{1}{2}a_{11}.\frac{1}{2}a_{22}.\frac{1}{2}a_{33}\right) + \left(a_{12}.a_{23}.a_{13}\right) + \left(a_{13}.a_{12}.a_{23}\right) - \left(a_{13}.\frac{1}{2}a_{22}.a_{13}\right) - \left(a_{23}.\frac{1}{2}a_{11}.a_{23}\right) - \left(a_{12}.\frac{1}{2}a_{33}.a_{12}\right) < 0$$
$$\Delta_4 = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & \frac{1}{2}a_{44} \end{bmatrix}$$
$$\begin{aligned}\Delta_4 = {} & \frac{1}{16}\left(-4a_{33}a_{44} + 16a_{34}^2\right)a_{12}^2 + \frac{1}{16}\left(\left(16a_{23}a_{44} - 32a_{24}a_{34}\right)a_{13} + \left(-32a_{23}a_{34} + 16a_{23}a_{33}\right)a_{14}\right)a_{12} \\ & + \frac{1}{16}\left(-4a_{22}a_{44} + 16a_{24}^2\right)a_{13}^2 + a_{14}\left(a_{22}a_{34} - 2a_{23}a_{24}\right)a_{13} + \frac{1}{16}\left(-4a_{22}a_{33} + 16a_{23}^2\right)a_{14}^2 \\ & + \frac{1}{16}\left(-4a_{23}^2a_{44} + 16a_{23}a_{24}a_{34} - 4a_{24}^2a_{33} + a_{22}\left(a_{33}a_{44} - 4a_{34}^2\right)\right)a_{11} > 0\end{aligned}$$

if. $\beta, \alpha, \gamma, \mu, \Lambda, q, v > 0; a_{11}a_{12}, a_{13}, a_{14} < 0; a_{21}, a_{22}, a_{23}, a_{24} < 0; a_{31}, a_{32}, a_{33}, a_{34} < 0; a_{41}, a_{42}, a_{43}, a_{44} < 0$
Than $V(x)$ positive definite. Thus the system of equation (1) is globally asymptotically stable.

3. Numerical simulation

The simulation stability of the nitrogen, phosphate, phytoplankton, and sediment of the system (1) has been demonstrated in this section. Numerical simulation will be used to analyze the dynamics of the system (1).

Table 1. Model Parameters

Parameters	Definition	Value	Dimension	Source
Λ	Total waste input	4.0	mg g^{-1} day^{-1}	[9]
β	phytoplankton growth rate	0.3	day^{-1}	[17]
α	the growth of phytoplankton which is affected by nitrogen and phosphate levels	0.5	day^{-1}	[17]
q	the proportion of waste entering the water	0.4	Dimensionless	[18]
μ	sedimentation rate of phytoplankton	0.8	day^{-1}	[9]
γ	water exchange rate	0.05	day^{-1}	[9]
θ	total remineralization sediment	1	Dimensionless	[18]
ω	the proportion of remineralization rate of N in the sludge	0.02	Dimensionless	[18]
v	Volatilization rate	0.05	day^{-1}	[9]

Table 1 provides the value of the other fixed parameter, their units, and the source. The mathematical model is a non-linear system as follows,

$$\begin{aligned}
\dot{P} &= -0.55P,\\
\dot{N} &= 1.6+0.02S-0.1N-0.15P\frac{N}{N+F},\\
\dot{F} &= 2.4+0.98S-0.05F-0.15P\frac{F}{N+F},\\
\dot{S} &= 0.8P-S,
\end{aligned} \tag{4}$$

equation depending on the time

$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t)$, with $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{P} \\ \dot{N} \\ \dot{F} \\ \dot{S} \end{bmatrix}$

so obtained

$$\begin{aligned}
f_1 &= -0.55P,\\
f_2 &= 1.6+0.02S-0.1N-0.15P\frac{N}{N+F},\\
f_3 &= 2.4+0.98S-0.05F-0.15P\frac{F}{N+F},\\
f_4 &= 0.8P-S,
\end{aligned} \tag{5}$$

Gradient equation of Lyapunov function

$$\nabla V = \begin{bmatrix} a_{11}P & a_{12}N & a_{13}F & a_{14}S \\ a_{21}P & a_{22}N & a_{23}F & a_{24}S \\ a_{31}P & a_{32}N & a_{33}F & a_{34}S \\ a_{41}P & a_{42}N & a_{43}F & a_{44}S \end{bmatrix}$$

$$\nabla V = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} P \\ N \\ F \\ S \end{bmatrix}$$

then from equation (4), the differential $\dot{V}$ can be determined from ∇V (the gradient of V) as follows [4]

$$\dot{V}(\boldsymbol{x}) = \nabla V^T f$$

$$\begin{aligned}
\dot{V}(\boldsymbol{x}) &= \begin{bmatrix} a_{11}P + a_{12}N + a_{13}F + a_{14}S \\ a_{21}P + a_{22}N + a_{23}F + a_{24}S \\ a_{31}P + a_{32}N + a_{33}F + a_{34}S \\ a_{41}P + a_{42}N + a_{43}F + a_{44}S \end{bmatrix}^T \begin{bmatrix} -0.55P \\ 1.6+0.02S-0.1N-0.15P\frac{N}{N+F} \\ 2.4+0.98S-0.05F-0.15P\frac{F}{N+F} \\ 0.8P-S \end{bmatrix}\\
&= (a_{11}P + a_{12}N + a_{13}F + a_{14}S)(-0.55P)\\
&\quad + (a_{21}P + a_{22}N + a_{23}F + a_{24}S)\left(1.6+0.02S-0.1N-0.15P\frac{N}{N+F}\right)\\
&\quad + (a_{31}P + a_{32}N + a_{33}F + a_{34}S)\left(2.4+0.98S-0.05F-0.15P\frac{F}{N+F}\right)\\
&\quad + (a_{41}P + a_{42}N + a_{43}F + a_{44}S)(0.8P-S)
\end{aligned}$$

That obtained $\dot{V}(\boldsymbol{x})$ is a negative definite function if $\beta, \alpha, \gamma, \theta, \omega, \mu, \Lambda, q, v > 0, a_{11}a_{12}, a_{13}, a_{14} < 0; a_{21}, a_{22}, a_{23}, a_{24} < 0; a_{31}, a_{32}, a_{33}, a_{34} < 0; a_{41}, a_{42}, a_{43}, a_{44} < 0$

Obtained

$$\dot{V}(\boldsymbol{x}) = -(a_{11}P + a_{12}N + a_{13}F + a_{14}S)(-0.55P)$$

$$-\left(a_{21}P+a_{22}N+a_{23}F+a_{24}S\right)\left(1.6+0.02S-0.1N-0.15P\frac{N}{N+F}\right)-\left(a_{31}P+a_{32}N+a_{33}F+a_{34}S\right)\left(2.4+0.98S-0.05F-0.15P\frac{F}{N+F}\right)$$
$$-\left(a_{41}P+a_{42}N+a_{43}F+a_{44}S\right)\left(0.8P-S\right)$$

Then $\dot{V}$ can be computed as

$$\begin{aligned}
\mathrm{V}(\boldsymbol{x}) &= \int \nabla V dx \\
&= \int_0^P \nabla V_1 dP + \int_0^N \nabla V_2 dN + \int_0^F \nabla V_3 dF + \int_0^S \nabla V_4 dS \\
&= \int_0^P a_{11}P + a_{12}N + a_{13}F + a_{14}S\, dP + \int_0^N a_{21}P + a_{22}N + a_{23}F + a_{24}S\ dN \\
&+ \int_0^F a_{31}P + a_{32}N + a_{33}F + a_{34}S\, dF + \int_0^S a_{41}P + a_{42}N + a_{43}F + a_{44}SdS \\
&= \frac{1}{2}a_{11}P^2 + a_{12}NP + a_{13}FP + a_{14}SP + \frac{1}{2}a_{22}N^2 + a_{21}PN + a_{23}FN + a_{24}SN \\
&+ \frac{1}{2}a_{33}F^2 + a_{31}PF + a_{32}NF + a_{34}SF + \frac{1}{2}a_{44}S^2 + a_{41}PS + a_{42}NS + a_{43}FS
\end{aligned}$$

$$V(x) = \frac{1}{2}a_{11}P^2 + \frac{1}{2}a_{22}N^2 + \frac{1}{2}a_{33}F^2 + \frac{1}{2}a_{44}S^2 + 2a_{12}NP + 2a_{13}FP + 2a_{14}SP + 2a_{23}FN + 2a_{24}SN + 2a_{34}SF \qquad (6)$$

Next, make the positive definite matrix equation for the Lyapunov function as follows

$$V(x) = X^T PX$$

$$V(x) = \begin{bmatrix} P & N & F & S \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{23} & A_{33} & A_{34} \\ A_{41} & A_{24} & A_{34} & A_{44} \end{bmatrix} \begin{bmatrix} P \\ N \\ F \\ S \end{bmatrix}$$

$$V(x) = \begin{bmatrix} P & N & F & S \end{bmatrix} \begin{bmatrix} A_{11}P + A_{12}N + A_{13}F + A_{14}S \\ A_{21}P + A_{22}N + A_{23}F + A_{24}S \\ A_{31}P + A_{32}N + A_{33}F + A_{34}S \\ A_{41}P + A_{42}N + A_{43}F + A_{44}S \end{bmatrix}$$

$$\begin{aligned}
V(x) &= A_{11}P^2 + A_{12}PN + A_{13}PF + A_{14}PS + A_{21}NP + A_{22}N^2 + A_{23}NF + A_{24}NS \\
&+ A_{31}FP + A_{32}FN + A_{33}F^2 + A_{34}FS + A_{41}SP + A_{42}SN + A_{43}SF + A_{44}S^2
\end{aligned}$$

$$V(x) = A_{11}P^2 + A_{22}N^2 + A_{33}F^2 + A_{44}S^2 + 2A_{12}PN + 2A_{13}PF + 2A_{14}PS + 2A_{23}NF + 2A_{24}NS + 2A_{34}FS \qquad (7)$$

Equation (6) and (7) is the same, so:

$$\begin{aligned}
V(x) &= \frac{1}{2}a_{11}P^2 + \frac{1}{2}a_{22}N^2 + \frac{1}{2}a_{33}F^2 + \frac{1}{2}a_{44}S^2 + 2a_{12}NP + 2a_{13}FP + 2a_{14}SP + 2a_{23}FN + 2a_{24}SN + 2a_{34}SF \\
&= A_{11}P^2 + A_{22}N^2 + A_{33}F^2 + A_{44}S^2 + 2A_{12}PN + 2A_{13}PF + 2A_{14}PS + 2A_{23}NF + 2A_{24}NS + 2A_{34}FS
\end{aligned}$$

We obtained

$$\begin{gathered}
A_{11} = \frac{1}{2}a_{11}; A_{22} = \frac{1}{2}a_{22}; A_{33} = \frac{1}{2}a_{33}; \\
2A_{12} = 2a_{12}; 2A_{13} = 2a_{13}; 2A_{14} = 2a_{14}; \\
2A_{23} = 2a_{23}; 2A_{24} = 2a_{24}; 2A_{34} = 2a_{34};
\end{gathered}$$

Diperoleh matrix

$$A = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & \frac{1}{2}a_{44} \end{bmatrix}$$

Using Sylvester's criterion to determine the first, second, third and fourth-order respectively from the A matrix.

$$\Delta_1 = \frac{1}{2}a_{11} < 0$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{2}a_{11} & a_{12} \\ a_{12} & \frac{1}{2}a_{22} \end{vmatrix}$$

$$= \left(\frac{1}{2}a_{11}.\frac{1}{2}a_{22}\right) - \left(a_{12}.a_{12}\right)$$

$$= \frac{1}{4}a_{11}.a_{22} - a_{12}.a_{12} > 0$$

$$\Delta_3 = \begin{vmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} \end{vmatrix}$$

$$= \left(\frac{1}{2}a_{11}.\frac{1}{2}a_{22}.\frac{1}{2}a_{33}\right) + \left(a_{12}.a_{23}.a_{13}\right) + \left(a_{13}.a_{12}.a_{23}\right) - \left(a_{13}.\frac{1}{2}a_{22}.a_{13}\right) - \left(a_{23}.\frac{1}{2}a_{11}.a_{23}\right) - \left(a_{12}.\frac{1}{2}a_{33}.a_{12}\right) < 0$$

$$\Delta_4 = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & \frac{1}{2}a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & \frac{1}{2}a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & \frac{1}{2}a_{44} \end{bmatrix}$$

$$\Delta_4 = \frac{1}{16}\left(-4a_{33}a_{44} + 16a_{34}^2\right)a_{12}^2 + \frac{1}{16}\left(\left(16a_{23}a_{44} - 32a_{24}a_{34}\right)a_{13} + \left(-32a_{23}a_{34} + 16a_{23}a_{33}\right)a_{14}\right)a_{12}$$
$$+ \frac{1}{16}\left(-4a_{22}a_{44} + 16a_{24}^2\right)a_{13}^2 + a_{14}\left(a_{22}a_{34} - 2a_{23}a_{24}\right)a_{13} + \frac{1}{16}\left(-4a_{22}a_{33} + 16a_{23}^2\right)a_{14}^2$$
$$+ \frac{1}{16}\left(-4a_{23}^2a_{44} + 16a_{23}a_{24}a_{34} - 4a_{24}^2a_{33} + a_{22}\left(a_{33}a_{44} - 4a_{34}^2\right)\right)a_{11} > 0$$

if. $\beta, \alpha, \gamma, \mu, \Lambda, q, v > 0; a_{11}a_{12}, a_{13}, a_{14} < 0; a_{21}, a_{22}, a_{23}, a_{24} < 0; a_{31}, a_{32}, a_{33}, a_{34} < 0; a_{41}, a_{42}, a_{43}, a_{44} < 0$

Than $V(x)$ positive definite. Thus the system of equation (4) is globally asymptotically stable.

4. Conclusion

This paper has discussed the models of nitrogen, phosphate, phytoplankton, and sediment in IMTA and the model is a system of non-linear differential equations. Nitrogen and phosphate are important factors in phytoplankton growth and sediment deposition globally asymptotically stable. Analysis stability globally has been proved by constructing the Lyapunov function. From analysis and numerical result, we obtained that models of nitrogen, phosphate, phytoplankton, and sediment system is globally asymptotically stable. This shows that the cultivation of the IMTA system on Karimunjawa island is secure as fish farming.

Acknowledgment

The authors would like to thank the Directorate General of Higher Education (Simlitabmas-DIKTI) for financially supporting this research project under the scheme of Excellent Research For National Strategic Grant year 2018 project. The study was supported by the Laboratory of Computer Modelling, Mathematics Departement, Center of Marine Ecology and Biomonitoring for Sustainable Aquaculture (Ce-MEBSA), and Laboratory of Ecology and Biosystematics, Biology Departement, Faculty of Science and Mathematics, Diponegoro University.

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