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Using of fast expansions in the construction of two-dimensional exact solutions of the Poisson equation

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Abstract. Using the fast expansion method, we obtained several exact solutions of the boundary value problem for the Poisson equation in a rectangular domain. We have given graphs of exact solutions corresponding to different boundary conditions and different types of the Poisson equation free term. We showed the influence of the type of boundary conditions and the right-hand side on the type of exact solution. We obtained a solution to the problem of membrane deflections under the action of variable load. We have given graphs of the stress components, from the analysis of which it follows that the greatest stress is in the middle of the rectangular membrane long sides.

1. Introduction

Various methods are used to solve mechanics boundary value problems with the Poisson equation (see [1–5]). So, in [1] a perturbation method was developed. In [2], the method of special orthonormal polynomials and the regular asymptotic method of “large λ” are used to solve the problem. In [3], relations of the generalized theory of elasticity are used that contain a structural parameter and allow one to obtain a regular solution, in [4] Fourier series are used, and in [5] the integral Mellin transform is used to solve contact problems of elasticity. Works [6–12] are devoted to solving the Poisson equation using numerical methods. The collocation method was used in [6], the quadrature element method was used in [7], the modified cubic B-spline differential-quadrature method was used in [8, 10], and the Haar wavelet method was used in [9, 11]. In [12] an analysis of multigrid correction of defects of compact discretization schemes of solving the Poisson equation is presented. In this paper some Poisson equation exact solutions will be obtained by the method of fast expansions [13].

2. Materials and methods

We write the Poisson equation for the rectangular domain $\Omega_{\square}$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + F(x,y) = 0,\ \ (x,y)\in \Omega_{\square},\ 0\le x\le a,\ 0\le y\le b, \tag{1}$$

where $F(x,y)$ is the internal source.

We set the boundary conditions in the form

$$U\big|_{x=0} = f_1(y),\ U\big|_{y=0} = f_2(x),\ U\big|_{x=a} = f_3(y),\ U\big|_{y=b} = f_4(x). \tag{2}$$

The solution of the boundary value problem (1), (2) must satisfy the consistency conditions in the corners of the rectangle:

$$\begin{gathered} f_1(0) = f_2(0),\ \ f_2(a) = f_3(0),\ \ f_3(b) = f_4(a),\ \ f_1(b) = f_4(0), \\ U_{xx}(0,0) + U_{yy}(0,0) + F(0,0) = 0,\ \ U_{xx}(a,0) + U_{yy}(a,0) + F(a,0) = 0, \\ U_{xx}(0,b) + U_{yy}(0,b) + F(0,b) = 0,\ \ U_{xx}(a,b) + U_{yy}(a,b) + F(a,b) = 0. \end{gathered} \tag{3}$$

The equations (3) follow from the independence of the function value $U(x, y)$ from the direction of the approach to these corners.

We represent the function $U(x, y)$ as the sum of the boundary function and the sine Fourier series, in which two Fourier coefficients are taken into account

$$U(x, y) = M_2(x, y) + A_5(y)\sin \pi\frac{x}{a} + A_6(y)\sin 2\pi\frac{x}{a},\ 0 \le x \le a. \tag{4}$$

Here $M_2(x, y)$ is the second-order boundary function

$$\begin{gathered} M_2(x, y) = A_1(y)P_1(x) + A_2(y)P_2(x) + A_3(y)P_3(x) + A_4(y)P_4(x), \\ P_1(x) = 1 - \frac{x}{a},\ \ P_2(x) = \frac{x}{a},\ \ P_3(x) = \frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3},\ \ P_4(x) = \frac{x^3}{6a} - \frac{ax}{6}. \end{gathered}$$

Unknowns are functions $A_i(y)$ $i = 1 \div 6$ that depend on only one variable y. The functions $A_i(y)$ $i = 1 \div 6$ are also represented by fast sine expansions:

$$A_i(y) = M_2^i(y) + A_{i,5}\sin \pi\frac{y}{b} + A_{i,6}\sin 2\pi\frac{y}{b},\ i = 1 \div 6,\ \ 0 \le y \le b, \tag{5}$$

where $M_2^i(y)$ $i = 1 \div 6$ is the second-order boundary function

$$\begin{gathered} M_2^i(y) = A_{i,1}P_1(y) + A_{i,2}P_2(y) + A_{i,3}P_3(y) + A_{i,4}P_4(y), \\ P_1(y) = 1 - \frac{y}{b},\ \ P_2(y) = \frac{y}{b},\ \ P_3(y) = \frac{y^2}{2} - \frac{y^3}{6b} - \frac{by}{3},\ \ P_4(y) = \frac{y^3}{6b} - \frac{by}{6}. \end{gathered}$$

Thus, the function $U(x, y)$ is represented in the form of a double fast expansion containing 36 unknown coefficients

$$A_{i,j},\ i = 1 \div 6,\ \ j = 1 \div 6. \tag{6}$$

We define the functions $f_1(y),\ f_2(x),\ f_3(y),\ f_4(x)$ included in the boundary conditions (2) as follows:

$$\begin{aligned} f_1(y) &= f_{1,1}P_1(y) + f_{1,2}P_2(y) + f_{1,3}P_3(y) + f_{1,4}P_4(y) + f_{1,5}\sin \pi\frac{y}{b} + f_{1,6}\sin 2\pi\frac{y}{b}, \\ f_2(x) &= f_{2,1}P_1(x) + f_{2,2}P_2(x) + f_{2,3}P_3(x) + f_{2,4}P_4(x) + f_{2,5}\sin \pi\frac{x}{a} + f_{2,6}\sin 2\pi\frac{x}{a}, \\ f_3(y) &= f_{3,1}P_1(y) + f_{3,2}P_2(y) + f_{3,3}P_3(y) + f_{3,4}P_4(y) + f_{3,5}\sin \pi\frac{y}{b} + f_{3,6}\sin 2\pi\frac{y}{b}, \end{aligned} \tag{7}$$

$$f_4(x)=f_{4,1}P_1(x)+f_{4,2}P_2(x)+f_{4,3}P_3(x)+f_{4,4}P_4(x)+f_{4,5}\sin\pi\frac{x}{a}+f_{4,6}\sin 2\pi\frac{x}{a},$$

where the constants $f_{i,j}$, $i=1\div 4,\ j=1\div 6$ are known.

We write the internal source $F(x,y)$ in the form a finite sum by analogy with dependencies (4), (5):

$$F(x,y)=F_1(y)P_1(x)+F_2(y)P_2(x)+F_3(y)P_3(x)+F_4(y)P_4(x)+F_5(y)\sin\pi\frac{x}{a}+F_6(y)\sin 2\pi\frac{x}{a},$$
$$F_i(y)=F_{i,1}P_1(y)+F_{i,2}P_2(y)+F_{i,3}P_3(y)+F_{i,4}P_4(y)+F_{i,5}\sin\pi\frac{y}{b}+F_{i,6}\sin 2\pi\frac{y}{b},\ i=1\div 6. \tag{8}$$

We consider all coefficients F_{ij}, $i=1\div 6,\ j=1\div 6$ in expression (8) for the source to be known, since the source $F(x,y)$ is a given function.

Thus, it is required to find a solution to equation (1) with a given internal source in the form (8) that exactly satisfies the boundary conditions (2) and the consistency conditions (3).

To find the unknown coefficients $A_{i,j}$ from (6), we apply the fast expansion method, according to which we substitute the double of the fast expansion of the function into boundary conditions (2), consistency conditions (3), and differential equation (1).

From the boundary conditions (2) we obtain

$$U\big|_{x=0}=f_1(y)\Rightarrow\sum_{j=1}^{4}A_{1,j}P_j(y)+A_{1,5}\sin\pi\frac{y}{b}+A_{1,6}\sin 2\pi\frac{y}{b}=\sum_{j=1}^{4}f_{1,j}P_j(y)+f_{1,5}\sin\pi\frac{y}{b}+f_{1,6}\sin 2\pi\frac{y}{b},$$
$$U\big|_{y=0}=f_2(x)\Rightarrow\sum_{i=1}^{4}A_{i,1}P_i(x)+A_{5,1}\sin\pi\frac{x}{a}+A_{6,1}\sin 2\pi\frac{x}{a}=\sum_{j=1}^{4}f_{2,j}P_j(x)+f_{2,5}\sin\pi\frac{x}{a}+f_{2,6}\sin 2\pi\frac{x}{a},$$
$$U\big|_{x=a}=f_3(y)\Rightarrow\sum_{j=1}^{4}A_{2,j}P_j(y)+A_{2,5}\sin\pi\frac{y}{b}+A_{2,6}\sin 2\pi\frac{y}{b}=\sum_{j=1}^{4}f_{3,j}P_j(y)+f_{3,5}\sin\pi\frac{y}{b}+f_{3,6}\sin 2\pi\frac{y}{b}, \tag{9}$$
$$U\big|_{y=b}=f_4(x)\Rightarrow\sum_{i=1}^{4}A_{i,2}P_i(x)+A_{5,2}\sin\pi\frac{x}{a}+A_{6,2}\sin 2\pi\frac{x}{a}=\sum_{j=1}^{4}f_{4,j}P_j(x)+f_{4,5}\sin\pi\frac{x}{a}+f_{4,6}\sin 2\pi\frac{x}{a}.$$

The consistency conditions (3) give the following relations

$$f_{1,1}=f_{2,1},\ f_{2,2}=f_{3,1},\ f_{3,2}=f_{4,2},\ f_{1,2}=f_{4,1},$$
$$A_{3,1}+A_{1,3}+F_{1,1}=0,\ A_{4,1}+A_{2,3}+F_{2,1}=0,\ A_{3,2}+A_{1,4}+F_{1,2}=0,\ A_{4,2}+A_{2,4}+F_{2,2}=0. \tag{10}$$

Now we substitute $U(x,y)$ from (4) into differential equation

$$\sum_{i=1}^{4}\left(\sum_{j=1}^{4}A_{i,j}P_j(y)+A_{i,5}\sin\pi\frac{y}{b}+A_{i,6}\sin 2\pi\frac{y}{b}\right)P_i''(x)$$
$$-\frac{\pi^2}{a^2}\cdot\left(\sum_{j=1}^{4}A_{5,j}P_j(y)+A_{5,5}\sin\pi\frac{y}{b}+A_{5,6}\sin 2\pi\frac{y}{b}\right)\sin\pi\frac{x}{a}$$
$$-\frac{4\pi^2}{a^2}\left(\sum_{j=1}^{4}A_{6,j}P_j(y)+A_{6,5}\sin\pi\frac{y}{b}+A_{6,6}\sin 2\pi\frac{y}{b}\right)\sin 2\pi\frac{x}{a}$$
$$+\sum_{i=1}^{4}\left(\sum_{j=1}^{4}A_{i,j}P_j''(y)-A_{i,5}\frac{\pi^2}{b^2}\sin\pi\frac{y}{b}-A_{i,6}\frac{4\pi^2}{b^2}\sin 2\pi\frac{y}{b}\right)P_i(x)$$

$$+\left(\sum_{j=1}^{4}A_{5,j}P_j''(y)-A_{5,5}\frac{\pi^2}{b^2}\sin\pi\frac{y}{b}-A_{5,6}\frac{4\pi^2}{b^2}\sin2\pi\frac{y}{b}\right)\sin\pi\frac{x}{a}$$
$$+\left(\sum_{j=1}^{4}A_{6,j}P_j''(y)-A_{6,5}\frac{\pi^2}{b^2}\sin\pi\frac{y}{b}-A_{6,6}\frac{4\pi^2}{b^2}\sin2\pi\frac{y}{b}\right)\sin2\pi\frac{x}{a}$$
$$+\sum_{i=1}^{4}\left(\sum_{j=1}^{4}F_{i,j}P_j(y)+F_{i,5}\sin\pi\frac{y}{b}+F_{i,6}\sin2\pi\frac{y}{b}\right)P_i(x) \tag{11}$$
$$+\left(\sum_{j=1}^{4}F_{5,j}P_j(y)+F_{5,5}\sin\pi\frac{y}{b}+F_{5,6}\sin2\pi\frac{y}{b}\right)\sin\pi\frac{x}{a}$$
$$+\left(\sum_{j=1}^{4}F_{6,j}P_j(y)+F_{6,5}\sin\pi\frac{y}{b}+F_{6,6}\sin2\pi\frac{y}{b}\right)\sin2\pi\frac{x}{a}=0,\ 0\le x\le a,\ 0\le y\le b.$$

Equation (11) must be satisfied for any $0\le x\le a$, $0\le y\le b$. In equality (11), the following functions are linearly independent:

$$P_1(x),\, P_2(x),\, P_3(x)\,,\, P_4(x),\, P_1(y)\,,\, P_2(y),\, P_3(y),\, P_4(y),$$
$$\sin\pi\frac{x}{a}\,,\ \sin\pi\frac{y}{b},\ \sin2\pi\frac{x}{a}\,,\ \sin2\pi\frac{y}{b}. \tag{12}$$

We equate the coefficients in (11) left and right of the linearly independent functions (12), taking into account $P_1''=P_2''=0,\ \ P_3''=P_1,\ \ P_4''=P_2$. As a result, we have the following equations from (11)

- when equating the coefficients of the $P_1(x)$:

$$A_{3,1}+A_{1,3}+F_{1,1}=0,\ A_{3,2}+A_{1,4}+F_{1,2}=0,\ A_{3,3}+F_{1,3}=0,\ A_{3,4}+F_{1,4}=0,$$
$$A_{3,5}-\frac{\pi^2}{b^2}\mathrm{A}_{1,5}+F_{1,5}=0,\ A_{3,6}-\frac{4\pi^2}{b^2}\mathrm{A}_{1,6}+F_{1,6}=0. \tag{13}$$

- when equating the coefficients of the $P_2(x)$

$$A_{4,1}+A_{2,3}+F_{2,1}=0,\ A_{4,2}+A_{2,4}+F_{2,2}=0,\ A_{4,3}+F_{2,3}=0,\ A_{4,4}+F_{2,4}=0,$$
$$A_{4,5}-\frac{\pi^2}{b^2}A_{2,5}+F_{2,5}=0,\ A_{4,6}-\frac{4\pi^2}{b^2}A_{2,6}+F_{2,6}=0. \tag{14}$$

- when equating the coefficients of the $P_3(x)$

$$A_{3,3}+F_{3,1}=0,\ A_{3,4}+F_{3,2}=0,\ F_{3,3}=0,\ F_{3,4}=0,\ -\frac{\pi^2}{b^2}A_{3,5}+F_{3,5}=0,-\frac{4\pi^2}{b^2}A_{3,6}+F_{3,6}=0. \tag{15}$$

- when equating the coefficients of the $P_4(x)$

$$A_{4,3}+F_{4,1}=0,\ A_{4,4}+F_{4,2}=0,\ F_{4,3}=0,\ F_{4,4}=0,\ -\frac{\pi^2}{b^2}A_{4,5}+F_{4,5}=0,\ -\frac{4\pi^2}{b^2}A_{4,6}+F_{4,6}=0. \tag{16}$$

- when equating the coefficients of the $\sin\pi\frac{x}{a}$

$$
\begin{gathered}
-\frac{\pi^2}{a^2}A_{5,1}+A_{5,3}+F_{5,1}=0,\quad -\frac{\pi^2}{a^2}A_{5,2}+A_{5,4}+F_{5,2}=0,\quad -\frac{\pi^2}{a^2}A_{5,5}-\frac{\pi^2}{b^2}A_{5,5}+F_{5,5}=0,\\
-\frac{\pi^2}{a^2}A_{5,6}-\frac{4\pi^2}{b^2}A_{5,6}+F_{5,6}=0,\quad -\frac{\pi^2}{a^2}A_{5,3}+F_{5,3}=0,\quad -\frac{\pi^2}{a^2}A_{5,4}+F_{5,4}=0.
\end{gathered}
\tag{17}
$$

● when equating the coefficients of the $\sin 2\pi\frac{x}{a}$

$$
\begin{gathered}
-\frac{4\pi^2}{a^2}A_{6,1}+A_{6,3}+F_{6,1}=0,\quad -\frac{4\pi^2}{a^2}A_{6,2}+A_{6,4}+F_{6,2}=0,\quad -\frac{4\pi^2}{a^2}A_{6,5}-\frac{\pi^2}{b^2}A_{6,5}+F_{6,5}=0,\\
-\frac{4\pi^2}{a^2}A_{6,6}-\frac{4\pi^2}{b^2}A_{6,6}+F_{6,6}=0,\quad -\frac{4\pi^2}{a^2}A_{6,3}+F_{6,3}=0,\quad -\frac{4\pi^2}{a^2}A_{6,4}+F_{6,4}=0.
\end{gathered}
\tag{18}
$$

Similarly, from equalities (9) we obtain

$$
\begin{gathered}
f_{1,1}=A_{1,1},\ f_{1,2}=A_{1,2},\ f_{1,3}=A_{1,3},\ f_{1,4}=A_{1,4},\ f_{1,5}=A_{1,5},\ f_{1,6}=A_{1,6},\\
f_{2,1}=A_{1,1},\ f_{2,2}=A_{2,1},\ f_{2,3}=A_{3,1},\ f_{2,4}=A_{4,1},\ f_{2,5}=A_{5,1},\ f_{2,6}=A_{6,1},\\
f_{3,1}=A_{2,1},\ f_{3,2}=A_{2,2},\ f_{3,3}=A_{2,3},\ f_{3,4}=A_{2,4},\ f_{3,5}=A_{2,5},\ f_{3,6}=A_{2,6},\\
f_{4,1}=A_{1,2},\ f_{4,2}=A_{2,2},\ f_{4,3}=A_{3,2},\ f_{4,4}=A_{4,2},\ f_{4,5}=A_{5,2},\ f_{4,6}=A_{6,2}.
\end{gathered}
\tag{19}
$$

Therefore, the functional system (9), (11) reduces to the overdetermined system linear algebraic equations (13) - (19). Due to the fulfillment of the coordination conditions (3), this redefined system has a solution. It can be seen from system (13) - (19) that relations (10) obtained from the matching conditions (3) are satisfied automatically, since all equalities (10) are included in system (13) - (19). 36 equations are needed from system (13) - (19) to find unknowns (6), and the rest of the equations are applicable for compiling relations between the coefficients $f_{i,j}$, $i=1\div 4$, $j=1\div 6$ of functions (7) and the coefficients F_{ij}, $i=1\div 6$, $j=1\div 6$ of the internal source $F(x,y)$.

Thus, from the system of equations (13) - (19) we find the following values of the coefficients $A_{i,j}$:

$$
\begin{gathered}
A_{1,j}=f_{1,j},\ A_{2,j}=f_{3,j},\ j=1\div 6,\\
A_{3,1}=f_{2,3},\ A_{3,2}=f_{4,3},\ A_{3,3}=-F_{1,3},\ A_{3,4}=-F_{1,4},\ A_{3,5}=\frac{\pi^2}{b^2}f_{1,5}-F_{1,5},\ A_{3,6}=\frac{4\pi^2}{b^2}f_{1,6}-F_{1,6},\\
A_{4,1}=f_{2,4},\ A_{4,2}=f_{4,4},\ A_{4,3}=-F_{4,1},\ A_{4,4}=-F_{2,4},\ A_{4,5}=\frac{\pi^2}{b^2}f_{3,5}-F_{2,5},\ A_{4,6}=\frac{4\pi^2}{b^2}f_{3,6}-F_{2,6},\\
A_{5,1}=f_{2,5},\ A_{5,2}=f_{4,5},\ A_{5,3}=\frac{\pi^2}{a^2}f_{2,5}-F_{5,1},\ A_{5,4}=\frac{\pi^2}{a^2}f_{4,5}-F_{5,2},\\
A_{5,5}=F_{5,5}\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^{-1},\ A_{5,6}=F_{5,6}\left(\frac{\pi^2}{a^2}+\frac{4\pi^2}{b^2}\right)^{-1},\\
A_{6,1}=f_{2,6},\ A_{6,2}=f_{4,6},\ A_{6,3}=\frac{4\pi^2}{a^2}f_{2,6}-F_{6,1},\ A_{6,4}=\frac{4\pi^2}{a^2}f_{4,6}-F_{6,2},\\
A_{6,5}=F_{6,5}\left(\frac{4\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^{-1},\ A_{6,6}=F_{6,6}\left(\frac{4\pi^2}{a^2}+\frac{4\pi^2}{b^2}\right)^{-1}.
\end{gathered}
\tag{20}
$$

Substituting the coefficients from (20) into expression (4), we obtain the problem exact solution.

From (10), (15), (16) it follows that when the boundary conditions (2) and the internal source (8) are specified, the following conditions must be satisfied:

$$f_{1,1} = f_{2,1}, \quad f_{2,2} = f_{3,1}, \quad f_{3,2} = f_{4,2}, \quad f_{1,2} = f_{4,1}, \tag{21}$$

$$F_{3,3} = F_{3,4} = F_{4,3} = F_{4,4} = 0. \tag{22}$$

From system (13) - (19), the following equations remained unused, which must be satisfied exactly:

$$\begin{gathered} A_{3,1} + A_{1,3} + F_{1,1} = 0, \quad A_{4,1} + A_{2,3} + F_{2,1} = 0, \quad A_{3,2} + A_{1,4} + F_{1,2} = 0, \\ A_{4,2} + A_{2,4} + F_{2,2} = 0, \quad A_{3,3} + F_{3,1} = 0, \quad A_{3,4} + F_{3,2} = 0, \quad A_{4,3} + F_{2,3} = 0, \\ A_{4,4} + F_{4,2} = 0, \quad -\frac{\pi^2}{b^2} A_{3,5} + F_{3,5} = 0, \quad -\frac{4\pi^2}{b^2} A_{3,6} + F_{3,6} = 0, \\ -\frac{\pi^2}{b^2} A_{4,5} + F_{4,5} = 0, \quad -\frac{4\pi^2}{b^2} A_{4,6} + F_{4,6} = 0, \quad -\frac{\pi^2}{a^2} A_{5,3} + F_{5,3} = 0, \\ -\frac{\pi^2}{a^2} A_{5,4} + F_{5,4} = 0, \quad -\frac{4\pi^2}{a^2} A_{6,3} + F_{6,3} = 0, \quad -\frac{4\pi^2}{a^2} A_{6,4} + F_{6,4} = 0. \end{gathered} \tag{23}$$

Substituting the found coefficients (20) into equations (23), we obtain additional equalities:

$$f_{1,3} = -f_{2,3} - F_{1,1}, \quad f_{1,4} = -f_{4,3} - F_{1,2}, \quad f_{3,3} = -f_{2,4} - F_{2,1}, \quad f_{3,4} = -f_{4,4} - F_{2,2}, \tag{24}$$

$$F_{1,3} = F_{3,1}, \quad F_{1,4} = F_{3,2}, \quad F_{4,1} = F_{2,3}, \quad F_{2,4} = F_{4,2}, \tag{25}$$

$$\begin{gathered} f_{1,5} = \frac{b^2}{\pi^2}\left(\frac{b^2}{\pi^2} F_{3,5} + F_{1,5}\right), \quad f_{1,6} = \frac{b^2}{4\pi^2}\left(\frac{b^2}{4\pi^2} F_{3,6} + F_{1,6}\right), \quad f_{2,5} = \frac{a^2}{\pi^2}\left(\frac{a^2}{\pi^2} F_{5,3} + F_{5,1}\right), \\ f_{2,6} = \frac{a^2}{4\pi^2}\left(\frac{a^2}{4\pi^2} F_{6,3} + F_{6,1}\right), \quad f_{3,5} = \frac{b^2}{\pi^2}\left(\frac{b^2}{\pi^2} F_{4,5} + F_{2,5}\right), \quad f_{3,6} = \frac{b^2}{4\pi^2}\left(\frac{b^2}{4\pi^2} F_{4,6} + F_{2,6}\right), \\ f_{4,5} = \frac{a^2}{\pi^2}\left(\frac{a^2}{\pi^2} F_{5,4} + F_{5,2}\right), \quad f_{4,6} = \frac{a^2}{4\pi^2}\left(\frac{a^2}{4\pi^2} F_{6,4} + F_{6,2}\right). \end{gathered} \tag{26}$$

Thus, the exact solution (20) exist when conditions (21), (22), (24) - (26) are satisfied.

3. Results and discussion

Let us consider several options for setting problem (1) - (3), depending on the presence or absence of a source $F(x, y)$ and various combinations of setting functions (7) in boundary conditions (2).

Let $F(x, y) = 0$. Then from formulas (20) and additional equalities (26) follows that the problem exact solution will be determined only by the values of the coefficients $f_{i,j}$, $i = 1 \div 4$, $j = 1 \div 4$ included in the boundary conditions (2), and the exact solution form will contain only polynomials from the boundary function $M_2(x, y)$.

If boundary conditions are specified by a linear law (only polynomials $P_i(x)$ and $P_i(y)$, $i = 1 \div 2$ will be used in functions (7)), the coefficients $f_{i,3} = f_{i,4} = 0$, $i = 1 \div 4$, and the choice of values $f_{i,j}$, $i = 1 \div 2$, $j = 1 \div 2$ is determined from conditions (21). We denote

$$f_{1,1} = T_1, \; f_{1,2} = T_2, \; f_{2,1} = T_1, \; f_{2,2} = T_3, \; f_{3,1} = T_3, \; f_{3,2} = T_4, \; f_{4,1} = T_2, \; f_{4,2} = T_4. \tag{27}$$

Boundary conditions given by the linear law can be written as

$$\begin{gathered} U\big|_{x=0} = f_1(y) = T_1 P_1(y) + T_2 P_2(y), \quad U\big|_{y=0} = f_2(x) = T_1 P_1(x) + T_3 P_2(x), \\ U\big|_{x=a} = f_3(y) = T_3 P_1(y) + T_4 P_2(y), \quad U\big|_{y=b} = f_4(x) = T_2 P_1(x) + T_4 P_2(x). \end{gathered} \tag{28}$$

Taking into account equalities (27), using formulas (20) we find

$$A_{1,1} = T_1,\ A_{1,2} = T_2,\ A_{2,1} = T_3,\ A_{2,2} = T_4,$$
$$A_{1,j} = A_{2,j} = 0,\ j = 3 \div 6,\ A_{3,k} = A_{4,k} = A_{5,k} = A_{6,k} = 0,\ \ k = 1 \div 6.$$

We write the problem solution in the form (4) as follows

$$U(x,y) = \left[T_1 P_1(y) + T_2 P_2(y)\right] P_1(x) + \left[T_3 P_1(y) + T_4 P_2(y)\right] P_2(x). \tag{29}$$

The distribution of the function $U(x,y)$ from equality (29), constructed for $T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 4$ in domain $\Omega_{\square}$ $(0 \le x \le 1,\ 0 \le y \le 2)$, is shown in figure 1a.

If we use all the polynomials $P_i(x)$ and $P_i(y)$, $i = 1 \div 4$ in functions (7), then we obtain the specification of the boundary conditions by a nonlinear law, with respect to the variables x and y. To draw up such boundary conditions, we will take into account equalities (27) and conditions (24). Based on (24), we introduce the notation

$$f_{1,3} = T_5,\ f_{1,4} = T_6,\ f_{2,3} = -T_5,\ f_{2,4} = -T_7,\ f_{3,3} = T_7,\ f_{3,4} = T_8,\ f_{4,3} = -T_6,\ f_{4,4} = -T_8. \tag{30}$$

Considering (27) and (30), we write the boundary conditions in the form

$$\begin{aligned} U\big|_{x=0} &= f_1(y) = T_1 P_1(y) + T_2 P_2(y) + T_5 P_3(y) + T_6 P_4(y), \\ U\big|_{y=0} &= f_2(x) = T_1 P_1(x) + T_3 P_2(x) - T_5 P_3(x) - T_7 P_4(x), \\ U\big|_{x=a} &= f_3(y) = T_3 P_1(y) + T_4 P_2(y) + T_7 P_3(y) + T_8 P_4(y), \\ U\big|_{y=b} &= f_4(x) = T_2 P_1(x) + T_4 P_2(x) - T_6 P_3(x) - T_8 P_4(x). \end{aligned} \tag{31}$$

Using equation (27) and (30) according to the formulas (20) we find

$$A_{1,1} = T_1,\ A_{1,2} = T_2,\ A_{1,3} = T_5,\ A_{1,4} = T_6,\ A_{2,1} = T_3,\ A_{2,2} = T_4,\ \ A_{2,3} = T_7,\ A_{2,4} = T_8,$$
$$A_{3,1} = -T_5,\ A_{3,2} = -T_6,\ A_{4,1} = -T_7,\ A_{4,2} = -T_8,$$
$$A_{1,j} = A_{2,j} = 0,\ j = 5 \div 6,\ \ A_{3,i} = A_{4,i} = 0,\ i = 3 \div 6,\ \ A_{5,k} = A_{6,k} = 0,\ \ k = 1 \div 6.$$

The problem exact solution in the form of (4) will have the form

$$\begin{aligned} U(x,y) = &\left[T_1 P_1(y) + T_2 P_2(y) + T_5 P_3(y) + T_6 P_4(y)\right] P_1(x) \\ &+ \left[T_3 P_1(y) + T_4 P_2(y) + T_7 P_3(y) + T_8 P_4(y)\right] P_2(x) \\ &- \left[T_5 P_1(y) + T_6 P_2(y)\right] P_3(x) - \left[T_7 P_1(y) + T_8 P_2(y)\right] P_4(x). \end{aligned} \tag{32}$$

The distribution of the function $U(x,y)$ from equality (32), constructed for $T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 4,\ T_5 = 1,\ T_6 = 2,\ T_7 = 3,\ T_8 = 4$ in domain $\Omega_{\square}$ $(0 \le x \le 1,\ 0 \le y \le 2)$, is shown in figure 1b.

If $F(x,y) \ne 0$, then from formulas (20) and equalities (26) follows that the form of the exact solution will depend not only on the boundary conditions form, but also on the form of the function describing the source $F(x,y)$.

Let the boundary conditions be given by equations (28). Consequently, equalities (27) hold. We choose $F(x,y)$ in such a way that only the coefficients from equalities (25) are not equal to zero. We denote

$$F_{3,1} = F_{1,3} = Q_1,\ F_{1,4} = F_{3,2} = Q_2,\ F_{4,1} = F_{2,3} = Q_3,\ F_{2,4} = F_{4,2} = Q_4. \tag{33}$$

Then, considering (22) and (33), we write the source $F(x,y)$ in the form

$$\begin{aligned}F(x,y)=&\left(Q_1P_3(y)+Q_2P_4(y)\right)P_1(x)+\left(Q_3P_3(y)+Q_4P_4(y)\right)P_2(x)\\&+\left(Q_1P_1(y)+Q_2P_2(y)\right)P_3(x)+\left(Q_3P_1(y)+Q_4P_2(y)\right)P_4(x).\end{aligned}\tag{34}$$

Substituting the coefficients from (27) and (33) into formulas (20) we find

$$\begin{gathered}A_{1,1}=T_1,\ A_{1,2}=T_2,\ A_{2,1}=T_3,\ A_{2,2}=T_4,\ A_{3,3}=-Q_1,\ A_{3,4}=-Q_2,\ A_{4,3}=-Q_3,\ A_{4,4}=-Q_4,\\A_{1,j}=A_{2,j}=0,\ j=3\div 6,\ A_{3,i}=A_{4,i}=0,\ i=1\div 2,\ i=5\div 6,\ A_{5,k}=A_{6,k}=0,\ k=1\div 6.\end{gathered}$$

Therefore, the solution of the problem in the form (4)

$$\begin{aligned}U(x,y)=&\left[T_1P_1(y)+T_2P_2(y)\right]P_1(x)+\left[T_3P_1(y)+T_4P_2(y)\right]P_2(x)\\&-\left[Q_1P_3(y)+Q_2P_4(y)\right]P_3(x)-\left[Q_3P_3(y)+Q_4P_4(y)\right]P_4(x).\end{aligned}\tag{35}$$

The distribution of the function $U(x,y)$ from equality (35), constructed for $T_1=1,\ T_2=2,\ T_3=3,\ T_4=4,\ Q_1=20,\ Q_2=20,\ Q_3=20,\ Q_4=20$, in domain $\Omega_\square$ $(0\le x\le 1,$ $0\le y\le 2)$ is shown in figure 1c.

We choose equality (31) as the boundary conditions, and set the source $F(x,y)$ by formula (34). Then the coefficients found according to the formulas (20) will take values

$$\begin{gathered}A_{1,1}=T_1,\ A_{1,2}=T_2,\ A_{1,3}=T_5,\ A_{1,4}=T_6,\ A_{2,1}=T_3,\ A_{2,2}=T_4,\ A_{2,3}=T_7,\ A_{2,4}=T_8,\\A_{3,1}=-T_5,\ A_{3,2}=-T_6,\ A_{4,1}=-T_7,\ A_{4,2}=-T_8,\ A_{3,3}=-Q_1,\ A_{3,4}=-Q_2,\ A_{4,3}=-Q_3,\ A_{4,4}=-Q_4,\\A_{1,j}=A_{2,j}=A_{3,j}=A_{4,j}=0,\ j=5\div 6,\ A_{5,k}=A_{6,k}=0,\ k=1\div 6.\end{gathered}$$

The problem exact solution will take the form

$$\begin{aligned}U(x,y)=&\left[T_1P_1(y)+T_2P_2(y)+T_5P_3(y)+T_6P_4(y)\right]P_1(x)\\&+\left[T_3P_1(y)+T_4P_2(y)+T_7P_3(y)+T_8P_4(y)\right]P_2(x)\\&-\left[T_5P_1(y)+T_6P_2(y)+Q_1P_3(y)+Q_2P_4(y)\right]P_3(x)\\&-\left[T_7P_1(y)+T_8P_2(y)+Q_3P_3(y)+Q_4P_4(y)\right]P_4(x).\end{aligned}\tag{36}$$

The distribution of the function $U(x,y)$ from equality (36), constructed for $T_1=1,\ T_2=2,\ T_3=3,\ T_4=4,\ T_5=1,\ T_6=2,\ T_7=3,\ T_8=4,\ Q_1=20,\ Q_2=20,\ Q_3=20,\ Q_4=20$ in domain $\Omega_\square$ $(0\le x\le 1,\ 0\le y\le 2)$, is shown in figure 1d.

As figure 1 demonstrates, the behavior of the function $U(x,y)$ in the corners of the domain $\Omega_\square$ is completely described by constants $T_1,\ T_2,\ T_3,\ T_4$, regardless of the boundary conditions (28) or (31) and the internal source presence $F(x,y)$.

Expressions (37), (38), (39) and (40), allowing to compute the values of the function $U(x,y)$ in the center of the domain $\Omega_\square$, are obtained respectively from the formulas of the exact solution (29), (35), (32) and (36)

$$U\left(\frac{a}{2};\frac{b}{2}\right)=\frac{1}{4}\left(T_1+T_2+T_3+T_4\right),\tag{37}$$

$$U\left(\frac{a}{2};\frac{b}{2}\right)=\frac{1}{4}\left(T_1+T_2+T_3+T_4\right)-\frac{a^2b^2}{256}\left(Q_1+Q_2+Q_3+Q_4\right), \tag{38}$$

$$U\left(\frac{a}{2};\frac{b}{2}\right)=\frac{1}{4}\left(T_1+T_2+T_3+T_4\right)+\frac{a^2}{32}\left(T_5+T_6+T_7+T_8\right)-\frac{b^2}{32}\left(T_5+T_6+T_7+T_8\right), \tag{39}$$

$$\begin{aligned}U\left(\frac{a}{2};\frac{b}{2}\right)=&\frac{1}{4}\left(T_1+T_2+T_3+T_4\right)+\frac{a^2}{32}\left(T_5+T_6+T_7+T_8\right)-\frac{b^2}{32}\left(T_5+T_6+T_7+T_8\right)\\&-\frac{a^2b^2}{256}\left(Q_1+Q_2+Q_3+Q_4\right).\end{aligned} \tag{40}$$

Formulas (37) - (40) show that the presence of an internal source $F(x,y)$ decreases the value $U(x,y)$ in the center of the domain $\Omega_{\square}$ in comparison with similar values when $F(x,y)=0$. This result is shown in figure 1.

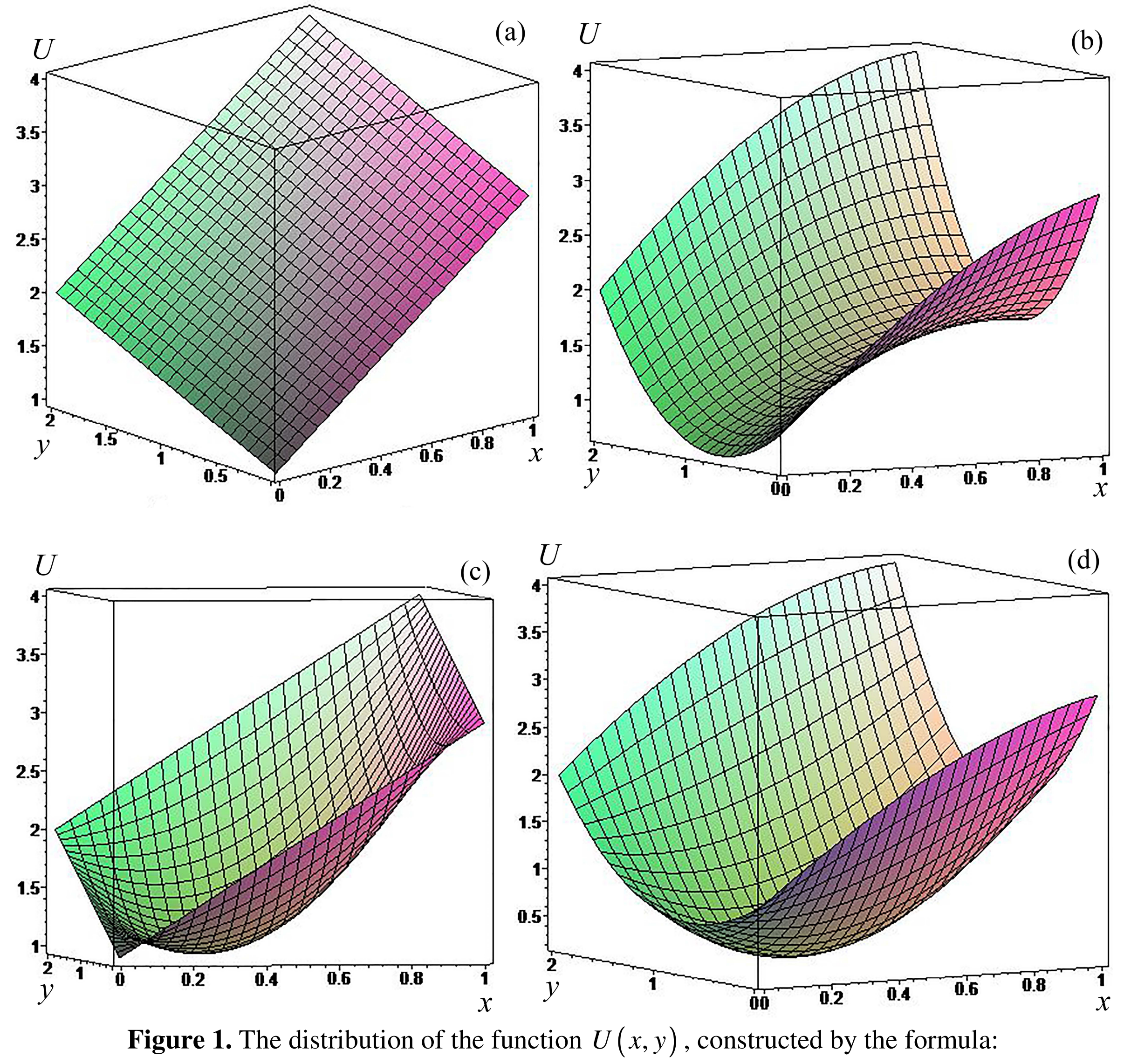

Figure 1. The distribution of the function $U(x,y)$, constructed by the formula:
(a) (29); (b) (32); (c) (35); (d) (36).

Formula (37) show that the value $U(x, y)$ in the center of the rectangular domain is equal to the arithmetical mean value $U(x, y)$ in the corners when boundary conditions are specified by formulas (28). From formulas (39) and (40) we can conclude that in the case $a = b$ the value of the function $U(x, y)$ in the center of the domain $\Omega_{\square}$ will be equal to the value calculated according to the formulas (37) and (38), respectively.

Now we show the application of the obtained two-dimensional exact solutions the Poisson equation for solving mechanics problems on the example of the membrane deflections problem. In [4], solution is given to the deflection of a membrane, which contour lies in the plane xoy, and the membrane load is constant. In this paper, we consider the case when the membrane contour will lie in a plane other than xoy, and the load acting on the membrane is variable. The boundary conditions for such a problem are written in the form (28). The membrane load is given by formula (34). The exact solution of this problem has the form (35).

The deflection values $U(x, y)$ in the center of the membrane are described by the formula (38). We see that at constant values T_1, T_2, T_3, T_4, Q_1, Q_2, Q_3, Q_4 on the deflection of the membrane is influenced by its sizes. The deflection of the membrane increases with their increasing.

Structural carbon steel of ordinary quality of VSt3ps brand was chosen as the membrane material. It has the following characteristics [14]

$$R_y = 2.35 \cdot 10^8 \text{ Pa}, \ \nu = 0{,}25, \ E = 2{,}13 \cdot 10^{11} \text{ Pa},$$

where R_y is calculated resistance of the membrane material.

Different values of the parameters T_1, T_2, T_3, T_4, Q_1, Q_2, Q_3, Q_4, a, b were selected so that the stresses did not exceed the calculated resistance of the membrane material under a biaxial stress state [15]

$$\sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2} = \tilde{\sigma} \le R_y, \tag{41}$$

where

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right), \ \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right), \ \varepsilon_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \ \varepsilon_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2. \tag{42}$$

Type load (34) and the deflection of the membrane, calculated by the formula (35) shown in figure 2 and figure 3 respectively. Calculations were performed with

$$T_1 = T_2 = 0.011, \ T_3 = T_4 = 0.0113, \ Q_1 = Q_2 = Q_3 = Q_4 = 2 \cdot 10^{-2}, \ a = 1 \text{ m}, \ b = 2 \text{ m}. \tag{43}$$

Computational experiments showed that the maximum deflection $U_{\max}$ of the membrane for the variable load $F(x, y)$ specified by equality (34) is not in the center of the membrane, but in its vicinity.

The stress components in the membrane calculated according to formulas (42) according to data (43) are shown in figure 4, and the distribution $\tilde{\sigma}$ calculated by formula (41) is shown in figure 5. Figure 4 and figure 5 show that stresses increase with movement from the center of the membrane to its boundaries and reach their maximum in the middle of the membrane sides. The greatest tension is in the middle of the long sides. This result coincides with the results described in [4] for a constant load on the membrane, which lies in the plane xoy.

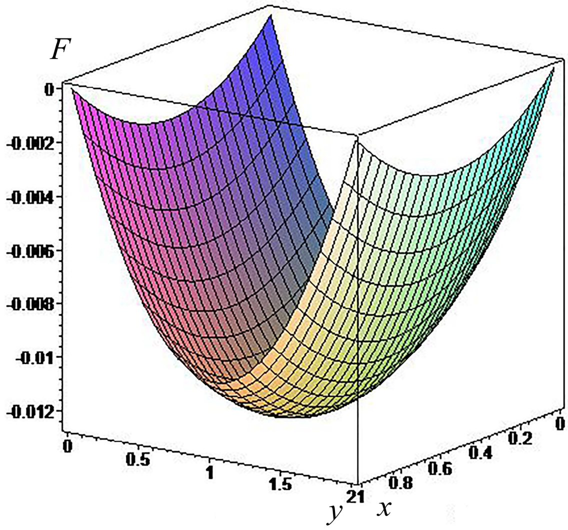

Figure 2. Type of variable load.

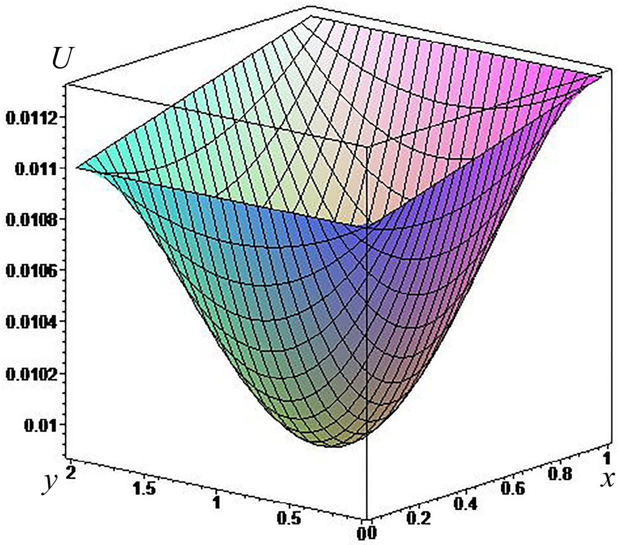

Figure 3. The deflection of rectangular membranes under the action of variable loads.

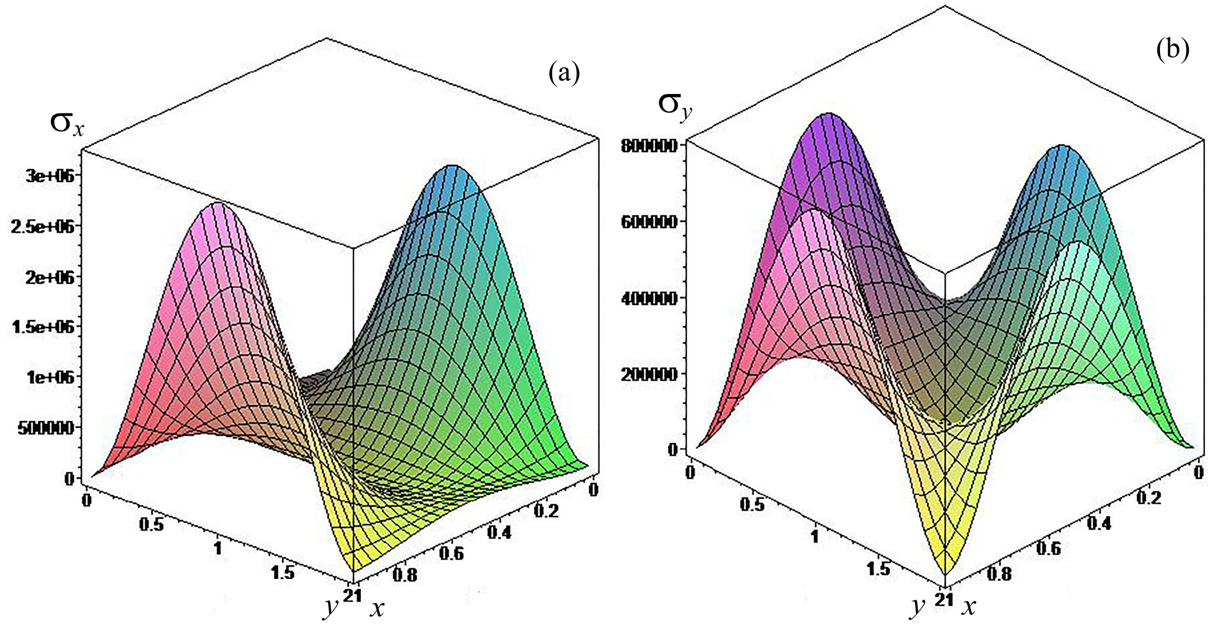

Figure 4. Stress components in the membrane: (a) σ_x ; (b) σ_y .

4. Conclusion

The method of fast expansions allows to obtain not only approximate solutions [16-18], but also exact ones. The obtained solutions in this article are convenient to use both for theoretical research and for the numerical experiments formulation. The coefficients numerical values selection of functions included in the boundary conditions and the source $F(x, y)$ should be performed taking into account the equalities (21), (22), (24) – (26). Calculations with a variable load on the membrane showed that the stresses begin to increase from the center to the boundaries of the membrane, reaching their maximum in the middle of the membrane sides. The greatest stress is in the middle of the long sides.

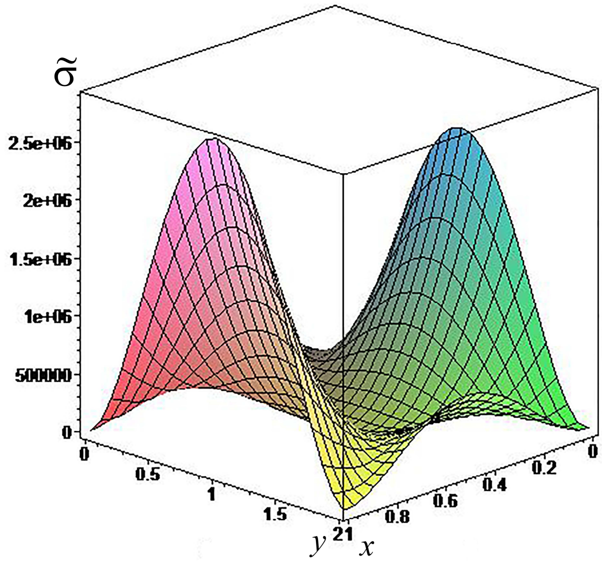

Figure 5. Distribution $\tilde{\sigma}$.

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