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Development of solution for control problem based on cascade decomposition method

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Abstract. The problem of simplifying solution of linear dynamic systems with preset conditions at the edges and internal points of the segment is considered. In this regard a modification technique of the existing cascade decomposition method based on obtaining a general formula for the system state and control functions is described. Coefficients of this formula are identified in a recursive way. The formula is presented to obtain the cascade decomposition method and kernel functions of the intermediate step matrices being important for the last step function. Illustratively, a practice-oriented example proves the method advantages in convenience and speed comparing with the conventional method of cascade decomposition.

1. Introduction

The fully controlled dynamic system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{1}$$

(see [1–7]) is considered. Here, vector function $x(t)$ is called a system state and $u(t)$ is system control. The present paper examines a solution for system (1) when the state and control function of the system meets the following boundary and intermediate conditions:

$$x(t_i) = x_i^{0\,0}. \tag{2}$$

$$\frac{d^q u(t)}{dt^q}|_{t_i} = u_i^{q\,0},\ q = \overline{0,p2},\ i = \overline{0,l}, \tag{3}$$

when $t_i :\ i = \overline{0,l},\ T_0 = t_0 < t_1 < ... < t_l = T_1$.

A system with conditions (2) is used, for example, in solving the task of movement of a material point moving in reactive force over a challenging terrain from position $x(T_0) = x_0$ to point $x(T_1) = x_1$ and getting through checkpoints $x(t_i) = x_i : i = \overline{0,l}$. The task having (3)-type conditions is set up and solved in case of seamless matching of various technological modes. Analytical solutions given in [1] are known, which have some complexity due to matrix exponentials.

[8–10] provide a technique for solving the stated tasks, i.e. the method of system cascade decomposition basing on sequential transfers from (1)–(3) to systems having matrices of smaller size and higher number of imposed conditions of (2), (3) type. Transfer performed untill controllability matrix B_p becomes surjective on the p-th step. A vector-function of state

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$x^p(t)$ is then being sought in a selected class of basis functions by the method of indefined coefficients. After that a state and control function of the task stated is determined by recurrent formulae. Here one can apply different classes as classes of linearly independent functions, such as polynomial, exponential, linear fractional and others. One of the drawbacks of the described above method is that when changing the applied functions class, as well as the imposed system conditions of types (2), (3), all steps of the cascade decomposition method should be performed again, thus providing computational time delays if the task is solved with systems of high dimension, or tasks where multiple solution is required using the cascade decomposition method of the same system (1) with different conditions (2), (3) [11, 12]. The present paper proposes a method for developing a universal formula for state and control functions, thus accelerating and facilitating solution of such tasks as constructing a constant-bounded continuous control and finding a feedback matrix.

2. Forward Path of Cascade Decomposition Method

The study is based on the concepts of a kernel, a co-kernel, kernel projector $P(B)$, co-kernel projector $Q(B)$, half-inverse operator B^- of [9]. Below is a brief description of cascade decomposition method [9] for further study. Conditions imposed on higher derivatives of function $x(t)$ are pre-calculated

$$x_i^{j+1\,0} = \frac{d^{j+1}x(t)}{(dt)^{j+1}}|_{t_i} = A\frac{d^j x(t)}{(dt)^j}|_{t_i} + B \cdot \frac{d^j u(t)}{(dt)^j}|_{t_i} = Ax_i^{j\,0} + Bu_i^{j\,0}.\ j = \overline{0, p2},\ i = \overline{0, l}. \tag{4}$$

If matrix B of (1) is non- surjective, variables are replaced

$$\begin{gathered} x^1(t) = Q(B) \cdot x(t),\ y^1(t) = (E - Q(B)) \cdot x(t), \\ A_1 = Q(B) \cdot A \cdot Q(B),\ B_1 = Q(B) \cdot A \cdot (E - Q(B)). \end{gathered} \tag{5}$$

From this point on E denotes a single matrix with the size equals the size of matrix A of (1). Conditions for new state and control functions $x^1(t)$ and $y^1(t)$ are re-calculated by

$$\begin{gathered} x_i^{j\,1} = Q(B) \cdot x_i^{j\,0},\ y_i^{j\,1} = (E - Q(B)) \cdot x_i^{j\,0}, \\ x_i^{p2+2\,1} = A_1 x_i^{p2+1\,1} + B_1 y_i^{p2+1\,1}\ j = \overline{0, p2+1},\ i = \overline{0, l}. \end{gathered} \tag{6}$$

A new system of equations of lesser dimensions is obtained as, in the classical solution of the problem by the cascade decomposition method, only linear independent components are kept in $x^1(t)$ number of which is equal to matrix $Q(B)$ rank.

$$\frac{dx^1(t)}{dt} = A_1 \cdot x^1(t) + B_1 \cdot y^1(t).$$

The procedure of variables replacement goes on and the following equation is obtained on the p-th step

$$\frac{dx^p(t)}{dt} = A_p \cdot x^p(t) + B_p \cdot y^p(t)$$

Having surjective matrix B_p and conditions

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$$\frac{d^j x^p(t)}{(dt)^j}|_{t_i} = x_i^{j\ p},\ \frac{d^{p2+p+1} x^p(t)}{(dt)^{p2+p+1}}|_{t_i} = x_i^{p2+p+1\ p},\ \frac{d^j y^p(t)}{(dt)^j}|_{t_i} = y_i^{j\ p}\ j=\overline{0,p2+p},\ i=\overline{0,l}, \tag{7}$$

Then, $x^p(t)$ is found by uncertainty coefficient method and $y^p(t)$ is determined by $y^p(t) = B_p^-(\dot{x^p}(t) - A_p x^p(t)) + z^p(t)$. Functions of the previous steps are distorted by

$$x^{m-1}(t) = x^m(t) + y^m(t),\ y^{m-1}(t) = B_{m-1}^-(\dot{x}^{m-1}(t) - A_{m-1}x^{m-1}(t)) + z^{m-1}(t),\ \ m=\overline{1,p}. \tag{8}$$

Here $z^m(t) : m = \overline{0,p}$ functions of matrix B_m kernels meeting conditions $B_m z^m(t) = 0$ and $z_i^{j\ m} = (I - B_m^- B_m) y_i^{j\ m}$ with $j = \overline{0,p2+m},\ i=\overline{0,l}$. Therefore, functions $x(t)$ and $u(t)$ are determined by formula (8). Now, to accomplish the task, we change the order of the actions performed comparing with the conventional cascade decomposition method algorithm. That is, first we get a general formula to express functions $x(t)$ and $u(t)$ by $x^p(t)$ and kernel elements $z^m(t) : m = \overline{0,p}$. Then, we obtain a formula to express conditions imposed on $x^p(t)$ and $z^m(t) : m = \overline{0,p}$ through (2), (3). To obtain a general formula, matrix dimensions are not changed in the course of forward path of the cascade decomposition method. Let $q = \overline{0,p}$ be any number. Hence,

$$x^{p-q}(t) = W_q^q(x^p(t))^{(q)} + W_{q-1}^q(x^p(t))^{(q-1)} + ... + W_0^q(x^p(t)) +$$
$$+Q_{p\ q-1}^q(z^p(t))^{(q-1)} + ... + Q_{p\ 0}^q(z^p(t)) + Q_{p-1\ q-2}^q(z^{p-1}(t))^{(q-2)} + ... + Q_{p-q+1\ 0}^q z^{p-q+1}(t), \tag{9}$$
$$y^{p-q}(t) = V_{q+1}^q(x^p(t))^{(q+1)} + V_q^q(x^p(t))^{(q)} + V_{q-1}^q(x^p(t))^{(q-1)} + ... + V_0^q(x^p(t)) +$$
$$+R_{p\ q}^q(z^p(t))^{(q)} + ... + R_{p\ 0}^q(z^p(t)) + R_{p-1\ q-1}^q(z^{p-1}(t))^{(q-1)} + ... + R_{p-q\ 0}^q z^{p-q}(t).$$

Here $W_i^q : i = \overline{0,q},\ V_i^q : i = \overline{0,q+1},\ Q_{i\ j}^q : i = \overline{p-q+1,p},\ j = \overline{0,q-1},\ R_{i\ j}^q : i = \overline{p-q,p},\ \ j = \overline{0,q}$ are any constant matrices independent on the selection of edge conditions and basic function and dependent on $\varphi(t)$ and dependent on system matrices A and B. The matrices are calculated by the recurrent formula. Hence, we get the following formula from this equation and (8)

$$W_0^0 = E,\ V_1^0 = B_p^-,\ V_0^0 = -B_p^- A_p^-,\ R_{p\ 0}^0 = E,\ Q_{p\ 0}^1 = E,\ W_{q+1}^{q+1} = V_{q+1}^q,$$
$$V_{q+2}^{q+1} = B_{p-(q+1)}^- V_{q+1}^q, V_{q+1}^{q+1} = B_{p-(q+1)}^-(W_q^q + V_q^q) - B_{p-(q+1)}^- A_{p-(q+1)} V_{q+1}^q,$$
$$V_0^{q+1} = -B_{p-(q+1)}^- A_{p-(q+1)}(W_0^q + V_0^q),\ \ W_i^{q+1} = V_i^q + W_i^q, i = \overline{0,q}$$
$$V_i^{q+1} = B_{p-(q+1)}^-(W_{i-1}^q + V_{i-1}^q) - B_{p-(q+1)}^- A_{p-(q+1)}(W_i^q + V_i^q), i = \overline{1,q},$$

$$R_{i\ 0}^{q+1} = -B_{p-(q+1)}^- A_{p-(q+1)}^-,\ R_{p-(q+1)\ 0}^{q+1} = I. : i = \overline{p-q-1,p},\ \ j = \overline{0,q+1}.$$

$$Q_{i\ i-p+q+1}^{q+1} = R_{i\ i-p+q}^q,$$

otherwise with

$$j \neq i-p+q,\ Q_{i\ j}^{q+1} = Q_{i\ j}^q + R_{i\ j}^q :,\ i = \overline{p-q,p},\ j = \overline{0,q},$$

$$R_{i\ i-p+(q+1)}^{q+1} = B_{p-(q+1)}^- R_{i\ i-p+q}^q,$$

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$$R^{q+1}_{i\, i-p+q} = (B^{-}_{p-(q+1)}(Q^{q}_{i\, i-p+q-1} + R^{q}_{i\, i-p+q-1}) - B^{-}_{p-(q+1)}A^{-}_{p-(q+1)}R^{q}_{i\, i-p+q}),$$

otherwise with $j \neq i-p+(q+1),\ j \neq i-p+q$

$$R^{q+1}_{i\, j} = (B^{-}_{p-(q+1)}(Q^{q}_{i\, j-1} + R^{q}_{i\, j-1}) - B^{-}_{p-(q+1)}A^{-}_{p-(q+1)}(Q^{q}_{i\, j} + R^{q}_{i\, j})).$$

This formula can be proved by induction.

The required functions $x(t)$ and $u(t)$ are obviously obtained by formula of type (9) when $p = q$.

Now a recurrent formula is obtained to express conditions imposed on $x^p(t)$ through (2), (3).

$$\begin{aligned} x_i^{j\ s} &= (\widetilde{X}_{j,s})x_i^{0\ 0} + \sum_{k=0}^{p2}(\widetilde{U}^k_{j,s})u_i^{k\ 0},\ s=\overline{0,p},\ j=\overline{0,s+p2+1},\ i=\overline{0,l}. \\ u_i^{j\ s} &= (\widehat{X}_{j,s})x_i^{0\ 0} + \sum_{k=0}^{p2}(\widehat{U}^k_{j,s})u_i^{k\ 0}, s=\overline{0,p},\ j=\overline{0,s+p2},\ i=\overline{0,l}. \\ z_i^{j\ s} &= (\overline{X}_{j,s})x_i^{0\ 0} + \sum_{k=0}^{p2}(\overline{U}^k_{j,s})u_i^{k\ 0},\ s=\overline{0,p},\ j=\overline{0,s+p2},\ i=\overline{0,l}. \end{aligned} \tag{10}$$

Here $\widetilde{X}_{j,s}, \widehat{X}_{j,s}, \overline{X}_{j,s}, \widetilde{U}^k_{j,s}, \widehat{U}^k_{j,s}, \overline{U}^k_{j,s}$ are some constant matrices depending on system matrices A and B in the same way (9). The following recurrent formula is obtained from (5)

$$\widetilde{X}_{0,0} = E,\ \widetilde{U}^k_{0,0} = 0;\ \widehat{X}_{j,0} = 0\ ,\widehat{U}^j_{j,0} = E,\ \widehat{U}^k_{j,0} = 0, if\ j \neq k,\ k=\overline{0,p2},\ j=\overline{0,p2},$$

$$\widetilde{X}_{j,0} = \widetilde{X}_{j-1,0} + \widehat{X}_{j-1,0},\ \widetilde{U}^k_{j,0} = \widetilde{U}^k_{j-1,0} + \widehat{U}^k_{j-1,0},\ k=\overline{0,p2},\ j=\overline{1,p2},$$

$$\widetilde{X}_{j,s+1} = Q(B_s)\widetilde{X}_{j,s},\ \widetilde{U}^k_{j,s+1} = Q(B_s)\widetilde{U}^k_{j,s},\ \widehat{X}_{j,s+1} = (E-Q(B_s))\widetilde{X}_{j,s},\ \widehat{U}^k_{j,s+1} = (E-Q(B_s))\widetilde{U}^k_{j,s},$$

$$\widetilde{X}_{s+p2+2,s+1} = A_s\widetilde{X}_{s+p2+1,s} + B_s\widehat{X}_{s+p2+1,s},\ \widetilde{U}_{s+p2+2,s+1} = A_s\widetilde{U}_{s+p2+1,s} + B_s\widehat{U}_{s+p2+1,s},$$

$$s=\overline{0,p-1},\ j=\overline{0,s+p2+1}$$

$$\overline{X}_{j,s} = (I - B_s^{-}B_s)\widehat{X}_{j,s},\ \overline{U}^k_{j,s} = (I - B_s^{-}B_s)\widehat{U}^k_{j,s},\ s=\overline{0,p},\ j=\overline{0,s+p2}.$$

It is evident that conditions imposed on $x^p(t)$ can be obtained by (10) with $s = p$.

3. Backward Path of Cascade Decompositioon Method

Let $\varphi_1(t),\ \varphi_2(t), ...,\ \varphi_r(t)$ be linearly independent functions applied to determine function $x^p(t)$ by the uncertainty coefficients method. A number of conditions imposed on $x^p(t)$ is denoted by r . Matrix F is a matrix of the following form

$$F = \begin{pmatrix} \varphi_1(t_0) & \varphi_2(t_0) & ... & \varphi_r(t_0) \\ \varphi_1(t_1) & \varphi_2(t_1) & ... & \varphi_r(t_1) \\ & & ... & \\ \varphi_1^{(p2+p+2)}(t_1) & \varphi_2^{(p2+p+2)}(t_1) & ... & \varphi_r^{(p2+p+2)}(t_1) \\ & & ... & \\ \varphi_1^{(p2+p+2)}(t_s) & \varphi_2^{(p2+p+2)}(t_s) & ... & \varphi_r^{(p2+p+2)}(t_s) \end{pmatrix}. \tag{11}$$

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The limitation is a need of matrix F nonsingularity. Let matrix val be

$$val = \begin{pmatrix} (x_0^{0\,p})_1 & (x_0^{0\,p})_2 & \dots & (x_0^{0\,p})_n \\ & & \dots & \\ (x_l^{0\,p})_1 & (x_l^{0\,p})_2 & \dots & (x_l^{0\,p})_n \\ & & \dots & \\ (x_0^{p2+p+2\,p})_1 & (x_0^{p2+p+2\,p})_2 & \dots & (x_0^{p2+p+2\,p})_n \\ & & \dots & \\ (x_l^{p2+p+2\,p})_1 & (x_l^{p2+p+2\,p})_2 & \dots & (x_l^{p2+p+2\,p})_n \end{pmatrix} \tag{12}$$

a matrix of imposed conditions of (7). Here, n is matrix dimension A of (1). Function $x^p(t)$ is determined by the uncertainty coefficients method in the following form.

$$x^p(t) = (\varphi(t) \cdot a)^T, \tag{13}$$

with the use of its conditions (7). Here $\varphi_i(t) : i = \overline{1, r}$ means some linear independent pre-set functions, a is basis decomposition coefficient matrix $x^p(t)$ from $\varphi_i(t) : i = \overline{1, r}$. Matrix a can be determined by $a = F^{-1} \cdot val$. Hence,

$$x^p(t) = (\varphi(t) \cdot F^{-1} \cdot val)^T. \tag{14}$$

Arguments for finding kernel functions $z^m(t) : m = \overline{0, p}$ with the uncertainty coefficient method are the same. F_{z_m}, $valz_m$, $\varphi_{z_m}(t) : m = \overline{0, p}$ denote their respect matrices and functions. Hence, the formula below is obtained for the kernel function.

$$z^m(t) = (\varphi_{z_m}(t) \cdot F_{z_m}^{-1} \cdot valz_m)^T. \tag{15}$$

Subsequently, determined $x^p(t)$ and $z^m(t) : m = \overline{0, p}$ are placed into (9).

4. Example of System Solution For Boeing-747 movement

Bellow is an illustration of application of the desclosed method using an example of [13] which shows the transition process for Boeing 747 heavy aircraft lateral motion model in a landing configuration. Its shown that, having performed the cascade decomposition method once and obtained necessary formulas, in the future when the edge conditions are changed, the new solution will be completed much faster than when the cascade decomposition method is re-solved. The linearized model of aircraft movement in the configuration used during landing without an automatic stability improvement system has the following form [13]:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{pmatrix} = \begin{pmatrix} -0.089 & -2.19 & 0.328 & 0.319 & 0 & 0 \\ 0.076 & -0.217 & -0.166 & 0 & 0 & 0 \\ -0.602 & 0.327 & -0.975 & 0 & 0 & 0 \\ 0 & 0.15 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} + \begin{pmatrix} 0 & -0.0327 & 0 & 0 & 0 & 0 \\ 0.0264 & 0.151 & 0 & 0 & 0 & 0 \\ 0.227 & -0.0636 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}. \tag{16}$$

Here, the following symbols for phase variables are applied:

x_1 — rate of sideslip;
x_2 — yawing rate;
x_3 — rolling rate;
x_4 — lateral attitude;
x_5 — yawing angle;
x_6 — lateral error.
The system controls are:
u_1 — angle of aileron deflection;
u_2 — angle of rudder deflection.
Here u_3, u_4, u_5, u_6 are imaginery control components to make the control matrix square.

The equations are written with the use of dimensionless values. 0.01 rad unit is accepted as a unit for angles and 0.305 mps for rates.

To maintain the experimental integrity, conditions of types (2), (3) are not pre-imposed. The general solution is obtained

$$x(t) = W_2^3(x^3(t))'' + W_1^3(x^3(t))' + W_0^3(x^3(t))$$
$$+Q_{3\,2}^3(z^3(t))'' + Q_{3\,1}^3(z^3(t))' + Q_{3\,0}^3(z^3(t)) + Q_{2\,1}^3(z^2(t))' + Q_{2\,0}^3(z^2(t)) + Q_{1\,0}^3(z^1(t)),$$

$$y(t) = V_3^3(x^3(t))^{(3)} + V_2^3(x^3(t))'' + V_1^3(x^3(t))' + V_0^3(x^3(t))$$
$$+R_{3\,3}^3(z^3(t))^{(3)}+R_{3\,2}^3(z^3(t))''+R_{3\,1}^3(z^3(t))'+R_{3\,0}^3(z^3(t))+R_{2\,2}^3(z^2(t))''+R_{2\,1}^3(z^2(t))'+R_{2\,0}^3(z^2(t))+$$
$$R_{1\,1}^3(z^1(t))' + R_{1\,0}^3(z^1(t)) + R_{0\,0}^3(z^0(t)).$$

Matrices of (9) have the following form. Below is a part of these matrices.

$$V_0^3 = \begin{pmatrix} 2.45 & -1.48 & 4.45 & -0.3 & 0.0 & 0.0 \\ -0.97 & 1.8 & 0.31 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix},$$

$$W_0^3 = \begin{pmatrix} 1.01 & -0.21 & 0.02 & -0.03 & 0.0 & 0.0 \\ -0.03 & 1.0 & -0.0 & 0.14 & 0.0 & 0.0 \\ 0.0 & -0.0 & 1.0 & -0.02 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.09 & -0.01 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.02 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix},$$

$$Q_{2\,0}^3 = \begin{pmatrix} 1.01 & 0.0 & -0.0 & -0.03 & 0.0 & 0.0 \\ -0.03 & 0.99 & 0.0 & 0.14 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & -0.02 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix},$$

$$R_{1\,0}^3 = \begin{pmatrix} 2.39 & -0.99 & 4.39 & 0.0 & 0.0 & 0.0 \\ -0.92 & 1.61 & 0.33 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}.$$

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Number $p2$ of (3) equals 0. Conditions imposed on function $x^3(t)$ and functions $z^m(t) : m = \overline{0,3}$ are expressed by the formula below.

$$x_i^{j\,3} = \widetilde{X}_{j,3} x_i^{0\,0} + \widetilde{U}^0_{j,3} u_i^{0\,0},\ j = \overline{0,4},\ i = \overline{0,l},$$

$$z_i^{j\,s} = \overline{X}_{j,s} x_i^{0\,0} + \overline{U}^k_{j,s} u_i^{0\,0},\ s = \overline{0,3},\ j = \overline{0,s},\ i = \overline{0,l}.$$

There is an example of some matrices:

$$\widetilde{X}_{0,3} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.44 & 0.09 & -0.01 & -0.14 & 1.0 & 0.0 \\ -0.09 & -0.02 & 0.0 & 0.01 & 0.0 & 1.0 \end{pmatrix},$$

$$\widetilde{X}_{1,3} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.03 & -0.01 & 0.0 & 0.14 & 0.0 & 0.0 \\ 1.01 & 0.21 & -0.02 & -0.03 & 0.0 & 0.0 \end{pmatrix},$$

$$\widetilde{X}_{2,3} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.08 & 0.13 & -0.01 & 0.0 & 0.0 \\ -0.06 & -2.26 & 0.29 & 0.32 & 0.0 & 0.0 \end{pmatrix},$$

$$\widetilde{U}^0_{3,3} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.03 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.01 & -0.36 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}.$$

Here basic functions $\varphi(t)$ of (14) and $\varphi_{z_0}(t)$, $\varphi_{z_1}(t)$, $\varphi_{z_2}(t)$ of (15) have the following form

$$\varphi(t) = \left(\frac{sin(0.1t)}{(t-60)^2}, \frac{sin(0.2t)}{(t-60)^4}, \frac{cos(0.2t)}{(t-60)^3}, \frac{cos(0.1t)}{(t-60)}, \frac{cos(0.3t)}{(t-60)^5}, \frac{cos(0.3t)}{(t-60)^6}, \frac{sin(0.4t)}{(t-60)^8}, \frac{cos(0.4t)}{(t-60)^7} \right),$$

$$\varphi_{z_0}(t) = \left(\frac{cos(0.1t)}{(t-60.0)} \quad \frac{sin(0.1t)}{(t-60.0)^2} \right),\ \varphi_{z_1}(t) = \left(\frac{sin(0.2t)}{(t-60.0)^4} \quad \frac{cos(0.2t)}{(t-60.0)^3} \quad \frac{cos(0.1t)}{(t-60.0)} \quad \frac{sin(0.1t)}{(t-60.0)^2} \right),$$

$$\varphi_{z_2}(t) = \left(\frac{cos(0.1t)}{(t-60.0)} \quad \frac{sin(0.2t)}{(t-60.0)^4} \quad \frac{cos(0.2t)}{(t-60.0)^3} \quad \frac{sin(0.1t)}{(t-60.0)^2} \quad \frac{cos(0.3t)}{(t-60.0)^5} \quad \frac{sin(0.3t)}{(t-60.0)^6} \right).$$

Matrix F of (14) and $F_{z_0},\ F_{z_1},\ F_{z_2}$ of (15) are of the following form

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$$F = \begin{pmatrix} 3\cdot 10^{-4} & -1.9\cdot 10^{-14} & 3.3\cdot 10^{-6} & -0.01 & 0 & 0 & 0 & 0 \\ 0.0005 & 1.5\cdot 10^{-13} & 10^{-5} & 0.01 & 0 & 0 & 0 & 0 \\ 3.5\cdot 10^{-5} & -1\cdot 10^{-14} & 1.6\cdot 10^{-6} & 0.001 & 0 & 0 & 0 & 0 \\ 2.4\cdot 10^{-6} & 2.1\cdot 10^{-14} & -1.6\cdot 10^{-6} & 0.002 & 0 & 0 & 0 & 0 \\ -8.3\cdot 10^{-7} & 0.0 & 5.7\cdot 10^{-8} & 10^{-3} & 0 & 0 & 0 & 0 \\ -6.1\cdot 10^{-6} & -2.1\cdot 10^{-14} & -6.8\cdot 10^{-7} & 2.3\cdot 10^{-5} & 0 & 0 & 0 & 0 \\ -3.9\cdot 10^{-7} & 2\cdot 10^{-15} & -5.9\cdot 10^{-8} & -6.8\cdot 10^{-6} & 0 & 0 & 0 & 0 \\ -6.7\cdot 10^{-7} & -1.3\cdot 10^{-14} & -4.1\cdot 10^{-8} & -2.1\cdot 10^{-5} & 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 1.5\cdot 10^{-7} & 8.4\cdot 10^{-13} & 3\cdot 10^{-9} & 9\cdot 10^{-12} \\ 0 & 0 & 0 & 0 & -3\cdot 10^{-7} & 8.8\cdot 10^{-13} & -9.3\cdot 10^{-9} & -7\cdot 10^{-11} \\ 0 & 0 & 0 & 0 & -1.7\cdot 10^{-9} & -2.7\cdot 10^{-13} & 5\cdot 10^{-10} & -2\cdot 10^{-11} \\ 0 & 0 & 0 & 0 & -8.1\cdot 10^{-8} & 2.6\cdot 10^{-12} & -2\cdot 10^{-9} & 6\cdot 10^{-11} \\ 0 & 0 & 0 & 0 & -6.7\cdot 10^{-9} & -2.2\cdot 10^{-13} & -2\cdot 10^{-10} & -5\cdot 10^{-12} \\ 0 & 0 & 0 & 0 & -2.1\cdot 10^{-9} & 7\cdot 10^{-13} & 4.6\cdot 10^{-10} & 3\cdot 10^{-11} \\ 0 & 0 & 0 & 0 & -10^{-9} & -1.7\cdot 10^{-14} & -9\cdot 10^{-11} & 5\cdot 10^{-13} \\ 0 & 0 & 0 & 0 & 3.1\cdot 10^{-9} & -2\cdot 10^{-13} & 3\cdot 10^{-10} & 2\cdot 10^{-12} \end{pmatrix},$$

$$F_{z_0} = \begin{pmatrix} -0.010806 & 0.000337 \\ 0.010404 & 0.000568 \end{pmatrix},\ F_{z_1} = \begin{pmatrix} -0.010806 & 1.45\cdot 10^{-7} & 0.000337 & 3\cdot 10^{-6} \\ 0.010404 & -2.95\cdot 10^{-7} & 0.000568 & 1\cdot 10^{-5} \\ 0.001467 & -1.67\cdot 10^{-9} & 3.5\cdot 10^{-5} & 2\cdot 10^{-6} \\ 0.002533 & -8.06\cdot 10^{-8} & 2\cdot 10^{-6} & -2\cdot 10^{-6} \end{pmatrix},$$

$$F_{z_2} = \begin{pmatrix} 0.000 & 3.33\cdot 10^{-6} & -0.01 & 1.45\cdot 10^{-7} & 3.17\cdot 10^{-9} & 9\cdot 10^{-12} \\ 0.0005 & 1.02\cdot 10^{-5} & 0.01 & -2.96\cdot 10^{-7} & -9.38\cdot 10^{-9} & -6.8\cdot 10^{-11} \\ 3.51\cdot 10^{-5} & 1.65\cdot 10^{-6} & 0.001 & -1.67\cdot 10^{-9} & 4.52\cdot 10^{-10} & -1.8\cdot 10^{-11} \\ 2.4\cdot 10^{-6} & -1.59\cdot 10^{-6} & 0.002 & -8.06\cdot 10^{-8} & -1.99\cdot 10^{-9} & 6\cdot 10^{-11} \\ -8.29\cdot 10^{-7} & 5.73\cdot 10^{-8} & 0.0001 & -6.78\cdot 10^{-9} & -2.2\cdot 10^{-10} & -5.2\cdot 10^{-12} \\ -6.152\cdot 10^{-6} & -6.86\cdot 10^{-7} & 2.26\cdot 10^{-5} & -2.08\cdot 10^{-9} & 4.63\cdot 10^{-10} & 2.5\cdot 10^{-11} \end{pmatrix}.$$

It is required to solve the task to find a minimum limit for control function norm $u(t)$ if the system is imposed with the following conditions at two points $t_0 = 10$, $t_1 = 20$ [11]:

$$x_1(10) = 13.0,\ x_2(10) = 11.0,\ x_3(10) = 8.0,\ x_4(10) = 10.0,\ x_5(10) = 12.0,\ x_6(10) = 7.0,$$

$$x_1(20) = x_2(20) = x_3(20) = ... = x_6(20) = 0.0. u_1(10) = u_2(10) = u_3(10) = ... = u_6(10) = 0.0.,$$
$$u_1(20) = u_2(20) = u_3(20) = ... = u_6(20) = 0.0.$$

Then val, val_{z_0}, val_{z_1}, val_{z_2} are recalculated with the use of (14) and (15). One can get

$$val = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 17.2336648 & 5.6606009 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0108233 & 14.8664882 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.8606289 & -20.0979801 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.9104164 & 6.9242199 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix},$$

$$val_{z_0} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}, val_{z_1} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix},$$

$$val_{z_2} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 10^{-15} & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}.$$

From (14) $x^3(t)$ is determined and kernel functions $z^m(t) : m = \overline{0,3}$ are determined using formula (15). Further, according to formula (9), prior state and control functions are found. Then, using the cascade decomposition method, twelve new solutions of system (16) are determined that satisfy zero boundary conditions of type (2), (3) when $p2 = 1$. These functions will not be identical zeros since conditions are imposed on the second derivative of each of the control functions, while one of these conditions is equal to 1 and the others are zeros. After that a solution to problem (16) is found that satisfies the above conditions with a minimum norm of control function [9]. A great time gain is obtained because only matrices val, val_{z_0}, val_{z_1}, val_{z_2} are required to be re-calculated. The total time to find the described functions using a Python program and the conventional cascade decomposition method is 62 seconds and 7.8 s with the use of the disclosed method. Time to find the minimum control function norm is 58 s.

5. Conclusion

Thus, the use of the disclosed methods allows to simplify and accelerate the search of the desired functions $x(t)$ and $u(t)$ because when changing conditions (2)–(3), it is sufficient to use them in formula (10) and after calculating the conditions imposed on $x^p(t)$ and kernel functions $z^m(t) : m = \overline{0,p}$, find $x^p(t)$ by (14) and $z^m(t) : m = \overline{0,p}$ by (15) with further applying them in (9). This leads to a high time gain in comparison with the conventional cascade decomposition method. For example, when searching the minimum limit for the norm of a control function, which is a solution to system (16), the gain is double with the use of the Python program.

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