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On some pseudo-differential equations and transmutation operators

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Abstract. We describe some operators for solving model elliptic pseudo-differential equations in special canonical domains. It helps us to write a general solution of corresponding pseudo-differential equation in an explicit form. Moreover, knowing a general solution we can choose additional (possibly boundary) conditions to determine uniquely the solution. All considerations we give in Sobolev–Slobodetskii spaces.

1. Introduction

For studying pseudo-differential equations on manifolds the main difficulty is to obtain invertibility conditions for a model pseudo-differential equation in a so-called canonical domain. Since a pseudo-differential operator is defined by its symbol which depends on two variables x and ξ, we say "model operator" if its symbol does not depend on x. Further, canonical domains are distinct in dependance on a type of manifold under consideration. So, for example, if we consider a compact smooth manifold without a boundary then we deal with only one canonical domain, i.e. $\mathbf{R}^m$. The first singularity appears if the manifold has a smooth boundary then we need to add one more canonical domain, it is a half-space $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > 0\}$, because our manifold is a half-space in a neighborhood of a boundary point. The last situation was studied in details in the book [2]. But if our manifold has at least one conical point at a boundary this method of rectification of a boundary does not work, and we have next type of a singularity and next canonical domain, i.e. a cone.

This report is devoted to some studies of this case (see also [11–16]). Some other approaches one can find, for example, in [7, 8].

2. Elliptic symbols and wave factorization

We will consider the operators in the Sobolev – Slobodetskii space $H^s(\mathbf{R}^m)$ with norm

$$||u||_s^2 = \int\limits_{\mathbf{R}^m} |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi,$$

where the sign "$\sim$" over a function denotes its Fourier transform, $\tilde{u} = Fu$, and introduce the following class of symbols non-depending on spatial variable x: $\exists c_1, c_2 > 0$, such that

$$c_1 \leq |A(\xi)(1 + |\xi|)^{-\alpha}| \leq c_2, \ \xi \in \mathbf{R}^m. \tag{1}$$

The number $\alpha \in \mathbf{R}$ we call the order of pseudo-differential operator A.

It is well-known that pseudo-differential operator with symbol $A(\xi)$ satisfying (1) is a linear bounded operator acting from $H^s(\mathbf{R}^m)$ into $H^{s-\alpha}(\mathbf{R}^m)$ [2].

We are interested in studying invertibility of the operators in corresponding Sobolev – Slobodetskii spaces. Let $S(\mathbf{R}^m)$ be the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions, C be a sharp convex cone non-including a whole straight line. By definition, $H^s(C)$ consists of distributions from $H^s(\mathbf{R}^m)$ with support in $\overline{C}$. The norm in the space $H^s(C)$ is induced by the norm $H^s(\mathbf{R}^m)$. We consider the equation

$$(Au)(x) = f(x),\ x \in C, \tag{2}$$

where right-hand side f is chosen from the space $H_0^{s-\alpha}(C)$.

If $S'(\mathbf{R}^m)$ is the space of distributions over the $S(\mathbf{R}^m)$ then $S'(C)$ denotes the space of distributions from $S'(\mathbf{R}^m)$ with support in $\overline{C}$, and $H_0^s(C)$ is the space of distributions $S'(C)$, which admit continuation onto $H^s(\mathbf{R}^m)$. The norm in $H_0^s(C)$ is defined by

$$||f||_s^+ = \inf ||lf||_s,$$

where *infimum* is chosen for all possible continuations lf.

Below we will consider the symbols $A(\xi)$ satisfying the condition (1).

Let us denote by $\overset{*}{C}$ the conjugate cone

$$\overset{*}{C} = \{x \in \mathbf{R}^m : x \cdot y > 0,\ \forall y \in C\}.$$

Definition *Wave factorization of symbol $A(\xi)$ with respect to the cone C is called its representation in the form*

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined everywhere, may be except the points $\{\xi \in \mathbf{R}^m : \xi \in \partial(\overset{*}{C} \cup (-\overset{*}{C}))\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C}), T(-\overset{*}{C})$ respectively, which satisfy the estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)},\ \forall \tau \in \overset{*}{C}.$$

The number æ is called the index of wave factorization.

3. Transmutation operators and solvability

Let the boundary surface of the cone C be a function $x_m = \varphi(x')$, where $\varphi \in C^\infty(\mathbf{R}^{m-1} \setminus \{0\})$ is a homogeneous function of order 1. We will introduce the following change of variables

$$\left\{ \begin{array}{rcl} t_1 & = & x_1 \\ t_2 & = & x_2 \\ \cdots & & \\ t_{m-1} & = & x_{m-1} \\ t_m & = & x_m - \varphi(x') \end{array} \right. \tag{3}$$

and denote this operator by $T_\varphi : \mathbf{R}^m \to \mathbf{R}^m$.

Theorem *The following relation holds*

$$FT_\varphi u = V_\varphi F u,$$

where V_φ is the following operator

$$(V_\varphi \tilde{u})(\xi) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} \int\limits_{\mathbf{R}^m} e^{i(t-y)\cdot\xi} e^{i\varphi(t')\xi_m} \tilde{u}(y) dy dt.$$

Proof. Indeed,

$$(FT_\varphi u)(\xi) = \int\limits_{\mathbf{R}^m} e^{ix\cdot\xi} (T_\varphi u)(x) dx,$$

and after change of variables (3)

$$(FT_\varphi u)(\xi) = \int\limits_{\mathbf{R}^m} e^{it'\cdot\xi'} e^{i\varphi(t')\xi_m} e^{it_m\xi_m} u(t) dt,$$

and taking into account

$$u(t) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} e^{-iy\cdot t} \tilde{u}(y) dy,$$

we can write

$$(V_\varphi \tilde{u})(\xi) = \frac{1}{(2\pi)^m} \int\limits_{\mathbf{R}^m} \int\limits_{\mathbf{R}^m} e^{i(t-y)\cdot\xi} e^{i\varphi(t')\xi_m} \tilde{u}(y) dy dt$$

$\triangle$

Remark. This is very similar to a definition of Fourier integral operators [4, 9, 10].

For concrete cones it is possible to calculate such operators, but before we will give the main theorem.

To formulate this theorem we will introduce a special integral operator [11]

$$(G_m u)(x) = \lim_{\tau\to 0} \int\limits_{\mathbf{R}^m} B(x - y + i\tau) u(y) dy, \quad \tau \in \overset{*}{C},$$

where $B(z)$ is the Bochner kernel [1, 17]

$$B(z) = \int\limits_C e^{ix\cdot z} dx, \quad z = \xi + i\tau, \quad \tau \in \overset{*}{C}.$$

Theorem *Let æ $- s = n + \delta$ with $n \in \mathbf{N}$ and $|\delta| < 1/2$. A general solution of the equation (2) in Fourier image is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) Q(\xi) G_m Q^{-1}(\xi) A_{=}^{-1}(\xi) \widetilde{lf}(\xi) +$$

$$+ A_{\neq}^{-1}(\xi) V_\varphi^{-1} F \left(\sum_{k=1}^{n} c_k(x') \delta^{(k-1)}(x_m) \right),$$

where $c_k(x') \in H^{s_k}(\mathbf{R}^{m-1})$ are arbitrary functions, $s_k = s -$ æ $+ k - 1/2$, $k = 1, 2, ..., n$, lf is an arbitrary continuation of f onto $H^{s-\alpha}(\mathbf{R}^m)$, $Q(\xi)$ is an arbitrary polynomial satisfying (1) for $\alpha = n$.

The a priori estimate holds

$$||u||_s \leq C(||f||^+_{s-\alpha} + [c_k]_{s_k}),$$

where $[\cdot]_{s_k}$ *denotes* $H^{s_k}(\mathbf{R}^{m-1})$*-norm.*

Remark. It is easily verified that

$$V_\varphi^{-1} = FT_{-\varphi}$$

(see, for example, [15]).

4. Examples

We will give some calculations for the operator V_φ for two concrete cones.

4.1. A flat case

Let us consider the case $m = 2$ in details. This case admits only single sharp convex cone of the following type

$$C_+^a = \{x \in \mathbf{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\}.$$

So we have

$$(FT_\varphi u)(\xi) = \int_{-\infty}^{+\infty} e^{ia|y_1|\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1 =$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{-iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1 =$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{-iy_1(a\xi_2-\xi_1)} \hat{u}(y_1, \xi_2) dy_1,$$

where $\hat{u}(y_1, \xi_2)$ denotes one-dimensional Fourier transform on the last variable.

The last two summands are the Fourier transforms of functions

$$\chi_+(y_1)\hat{u}(y_1, \xi_2), \quad \chi_+(y_1)\hat{u}(y_1, \xi_2)$$

with respect to the first variable y_1 respectively. So we can use the following properties [2] (these are Sokhotskii formulas [3, 6])

$$\int_{-\infty}^{+\infty} \chi_+(x) e^{ix\xi} u(x) dx = \frac{1}{2}\tilde{u}(\xi) + v.p.\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta},$$

$$\int_{-\infty}^{+\infty} \chi_-(x) e^{ix\xi} u(x) dx = \frac{1}{2}\tilde{u}(\xi) - v.p.\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta}.$$

Taking into account these properties we have

$$(FT_\varphi u)(\xi) = \frac{\tilde{u}(\xi_1 + a\xi_2, \xi_2) + \tilde{u}(\xi_1 - a\xi_2, \xi_2)}{2} +$$

$$+ v.p.\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{\xi_1 + a\xi_2 - \eta} - v.p.\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{\xi_1 - a\xi_2 - \eta} \equiv (V_\varphi \tilde{u})(\xi).$$

4.2. A spatial case

There are a lot of sharp convex cones in a space, and we consider here $m = 3$ and the following cone

$$C_+^{\mathbf{a}} = \{x \in \mathbf{R}^3 : x = (x_1, x_2, x_3), x_3 > a_1|x_1| + a_2|x_2|, a_1, a_2 > 0\}.$$

For calculating the operator V_φ we evaluate

$$\int\limits_{\mathbf{R}^2} e^{i(a_1|y_1|+a_2|y_2|)\xi_3} e^{i(y_1\xi_1+y_2\xi_2)} \hat{u}(y_1, y_2, \xi_3) dy_1 dy_2 =$$

$$= \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} (\int\limits_{-\infty}^{+\infty} e^{i(a_2|y_2|\xi_3+y_2\xi_2)} \hat{u}(y_1, y_2, \xi_3) dy_2 dy_1 =$$

$$= \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} (\frac{\hat{u}(y_1, \xi_2 - a_2\xi_3, \xi_3) + \hat{u}(y_1, \xi_2 + a_2\xi_3, \xi_3)}{2} +$$

$$+ v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\hat{u}(y_1, \eta, \xi_3) d\eta}{\xi_2 + a_2\xi_3 - \eta} - v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\hat{u}(y_1, \eta, \xi_3) d\eta}{\xi_2 - a_2\xi_3 - \eta}) dy_1,$$

where $\hat{u}$ denotes the Fourier transform with respect to the two last variables.

Let us denote

$$v_1(\xi) = \frac{1}{2} \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} \hat{u}(y_1, \xi_2 - a_2\xi_3, \xi_3) dy_1,$$

$$v_2(\xi) = \frac{1}{2} \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} \hat{u}(y_1, \xi_2 + a_2\xi_3, \xi_3) dy_1,$$

$$w_1(\xi) = \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} (S_2\hat{u})(y_1, \xi_2 + a_2\xi_3, \xi_3) dy_1,$$

$$w_2(\xi) = \int\limits_{-\infty}^{+\infty} e^{i(a_1|y_1|\xi_3+y_1\xi_1)} (S_2\hat{u})(y_1, \xi_2 - a_2\xi_3, \xi_3) dy_1,$$

where

$$(S_2 u)(\xi_1, \xi_2, \xi_3) = v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{u(\xi_1, \eta, \xi_3) d\eta}{\xi_2 - \eta}.$$

Further, taking into account the fact $\hat{\hat{\hat{u}}} \equiv \tilde{u}$ and the relation

$$(S_1 u)(\xi_1, \xi_2, \xi_3) = v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{u(\tau, \xi_2, \xi_3) d\tau}{\xi_1 - \tau},$$

we obtain

$$\int\limits_{\mathbf{R}^2} e^{i(a_1|y_1|+a_2|y_2|)\xi_3} e^{i(y_1\xi_1+y_2\xi_2)} \hat{u}(y_1, y_2, \xi_3) dy_1 dy_2 =$$

$$= \frac{\tilde{u}(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3) + \tilde{u}(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3)}{4} +$$
$$+\frac{1}{2}(S_1\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3) - \frac{1}{2}(S_1\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3) +$$
$$+\frac{\tilde{u}(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) + \tilde{u}(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3)}{4} +$$
$$+\frac{1}{2}(S_1\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) - \frac{1}{2}(S_1\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) +$$
$$+\frac{(S_2\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) + (S_2\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3)}{2} +$$
$$+(S_1S_2\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) - (S_1S_2\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3, \xi_3) -$$
$$-\frac{(S_2\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3) + (S_2\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3)}{2} -$$
$$-(S_1S_2\tilde{u})(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3) + (S_1S_2\tilde{u})(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3, \xi_3).$$

Thus, we see that the operator V_φ is composed from operators S_1, S_2 and certain change of variables.

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