

PAPER • OPEN ACCESS

Fermat's Principle and Hamilton's Principle: Does a least action take a least time for happening?

To cite this article: M Anderson *et al* 2020 *J. Phys.: Conf. Ser.* **1467** 012038

View the [article online](#) for updates and enhancements.

You may also like

- [High-Order Hamilton's Principle and the Hamilton's Principle of High-Order Lagrangian Function](#)
Zhao Hong-Xia and Ma Shan-Jun
- [Fermat's principle, Huygens' principle, Hamilton's optics and sailing strategy](#)
J C Kimball and Harold Story
- [Radiation reaction as a non-conservative force](#)
Sandeep Aashish and Asrarul Haque



DISCOVER
how sustainability
intersects with
electrochemistry & solid
state science research

Fermat's Principle and Hamilton's Principle: Does a least action take a least time for happening?

M Anderson¹, M Hadi^{2*}, and U A Deta³

^{1,2}Department of Mathematics, Universiti Brunei Darussalam, Negara Brunei Darussalam

²Research Center for Physics, Indonesia Institute of Science (LIPI), Puspiptek, Serpong, Indonesia

³Department of Physics, Universitas Negeri Surabaya, Ketintang, Surabaya, Indonesia

*itpm.id@gmail.com

Abstract. We explore deeper and analyse in more detail Fermat's and Hamilton's principles. We try to address some questions: Is it possible to have S negative? Is Hamilton's principle always valid for entire path of the system? Is there a relation between Fermat's principle and Hamilton's principle? We assume analogy with Hamilton's principle, is Fermat's principle always valid for entire path of the system? Does a least action take a least time for happening?

1. Fermat's Principle

Nature behaves lazy, a minimum effort [1, 2]. The minimum principle was encountered for the first time in optics. Pierre de Fermat (1601-1665) postulated that, no matter to what kind of reflection or refraction a ray is subjected. Probably Abu Ali al-Hassan ibn al-Haytham/Alhacen (965-1040), the writer of Kitab al-Manathir/Book of Optics, is the first one who expresses an early version of the principle of least time and expand this principle to both reflection and refraction [3]. It travels from one point, r_1 , to another, r_2 , in such a way as to make the time taken a minimum [4, 5, 6]. In rare cases, it can be maximum [6, 7, 8, 9]. Or, in more accurate statement: it is stationary [10, 12]. Fermat extended what Hero of Alexandria (10 AD - 70 AD) had discovered a similar law for the particular case where the ray of light is reflected by a mirror [4, 13]. Refer to Hero, the light travels in such a way that it goes to the mirror and to the other point in the shortest possible distance. It was this that inspired Fermat to suggest to himself that perhaps refraction operated on a similar basis. But for refraction, light obviously does not use the path of shortest distance, so Fermat tried the idea that it takes the shortest time [14].

Mathematically, Fermat's principle can be written as [10].

$$T = \int_{r_1}^{r_2} dt = \text{stationary} \quad (1)$$

The points r_1 and r_2 are two fixed points in space [6]. In general, being stationary (being a minimum, maximum or inflection) is a property of a functional (that is, a function of one or more functions). It does not mean physically that light travels through a minimum, maximum or inflection point, although one could loosely say that light travels along the minimizing path, maximizing path or inflection path.



2. Hamilton's Principle

Hamilton's principle (1834) or principle of least action can be stated as: the motion of the system from fixed time t_1 to fixed time t_2 is such that the line integral (called the action or the action integral) [15].

$$S = \int_{t_1}^{t_2} \mathcal{L}[x(t), \dot{x}(t), t] dt \quad (2)$$

where S is a functional of the function or path $x(t)$, $\mathcal{L}$ is a Lagrangian system and $\dot{x}(t) = dx/dt$, has a stationary value for the actual path of the motion. Or

$$\delta S = 0 \quad (3)$$

δ notation means variation, i.e. virtual and infinitesimal change [13]. $\delta S = 0$ is a necessary condition for S to have a minimum value. This means that $\delta S = 0$ can be because of S has a maximum value or inflection.

The point of Hamilton's principle is that it identifies paths $x = x_0(t)$ that satisfy certain sets of differential equations with the paths that minimize a particular functional S . Because, all possible paths "near" the actual path $x = x_0(t)$ must give values for S that are larger than the value given by $x = x_0(t)$. That is, the action on nearby paths must be larger than the action on the actual path.

When minimizing the action, the limits on the integral are fixed points in both space and time: $(r_1; t_1)$ and $(r_2; t_2)$. The velocity $V = |r_2 - r_1|/|t_2 - t_1|$ on the actual path (which we know must have constant velocity), is also fixed, and the value of the action on the actual path is

$$S_0 = \int_{t_1}^{t_2} \mathcal{L} dt = \frac{1}{2} m V^2 (t_2 - t_1) \quad (4)$$

again.

A path that is close to the actual path $x(t) = x_1 + V(t_2 - t_1)$ would have the form

$$x(t) = x_1 + V(t_2 - t_1) + ka(t) \quad (5)$$

Where a is any function satisfying $a(t_1) = a(t_2) = 0$, and k is a small parameter. So

$$[\dot{x}(t)]^2 = [V + k\dot{a}(t)]^2 = V^2 + 2Vk\dot{a}(t) + k^2[\dot{a}(t)]^2 \quad (6)$$

On the nearby path, and the action on the nearby path is

$$S(a, k) = \int_{t_1}^{t_2} \frac{1}{2} m [\dot{x}(t)]^2 dt = \frac{1}{2} m V^2 (t_2 - t_1) + Vkm \int_{t_1}^{t_2} \dot{a}(t) dt + \frac{1}{2} k^2 m \int_{t_1}^{t_2} [\dot{a}(t)]^2 dt$$

$$S(a, k) = S_0 + Vkm[a(t_2) - a(t_1)] + \frac{1}{2} k^2 m \int_{t_1}^{t_2} [\dot{a}(t)]^2 dt = S_0 + S_1 + S_2$$

$$S(a, k) = S_0 + S_2 \quad (7)$$

As $a(t_2) = a(t_1) = 0$. This means that

$$S_2 = m \int_{t_1}^{t_2} [\dot{a}(t)]^2 dt \geq 0 \quad (8)$$

and the action is a minimum on the actual path, as required.

Is it possible to have e.g. a negative value, instead 0, for δS ? If we can find a nearby path $x = x_0(t) + \delta x(t)$ which has δS negative, then S has a smaller value on this new path than it does on the original path, $x = x_0(t)$. So, S obviously does not have a minimum value on $x = x_0(t)$. Hamilton's principle identifies solutions of the equation of motion with paths that minimize S . If the path $x = x_0(t)$ does not minimize S (meaning that δS is negative for some nearby path) then the path $x = x_0(t)$ does not satisfy the equation of motion. So, it is not possible to have a negative value for δS , instead 0, for S to have a minimum value.

3. Is Hamilton's Principle always valid for entire path of the system?

Hamilton's principle is not always valid for entire path of the system, but only for any sufficiently short segment of the path [16]. This is because there do exist mechanical systems, usually with periodic or oscillating behaviour, which have the property that the solutions (to the equation of motion) do not necessarily minimize the action S if they extend beyond half an oscillation period. An example of this is simple harmonic motion of a oscillator.

Suppose that Lagrangian system is

$$\mathcal{L} = \left(\frac{dx}{dt}\right)^2 - x^2 \quad (9)$$

The Euler-Lagrange equation of motion is

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x} \quad (10)$$

Substituting eq. (9) into (10), we obtain the equation of motion of the oscillator

$$\frac{d^2 x}{dt^2} + x = 0 \quad (11)$$

The solution of the equation of motion of the oscillator is [10,11]

$$x(t) = A \cos t + B \sin t \quad (12)$$

Where A and B are constants.

Suppose we want to use Hamilton's principle to find the path with boundary conditions $x(0) = 0$ and $x(T) = 1$, where $T > 0$. The solution of the equation of motion with these boundary conditions has $A = 0$ and $B = 1 = \sin T$. The value of the action is

$$S = \int_0^T (B^2 \cos^2 t - B^2 \sin^2 t) dt = B^2 \sin T \cos T = \cot T \quad (13)$$

It turns out that $\cot T$ is the minimum possible value for S provided that $T < \pi$ (and π is half the period of the oscillator, which is 2π). Note that $\cot T \rightarrow -\infty$ as $T \rightarrow \pi$ from below, but $\cot T \rightarrow +\infty$ as $T \rightarrow \pi$ from above. That is, $\cot T$ jumps from $-\infty$ to $+\infty$ at $T = \pi$, and so it is unlikely to minimize S for $T > \pi$.

If we consider other possible paths from $x(0) = 0$ to $x(T) = 1$, one of the simplest is $x(t) = t/T$. This does not satisfy the equation of motion, and the corresponding value of the action is

$$S = \int_0^T \left(\frac{1}{T^2} - \frac{t^2}{T^2} \right) dt = \frac{1}{T} - \frac{T}{3} \quad (14)$$

This value of S is larger than $\cot T$ if $0 < T < \pi$ (as expected from the principle of least action), but is smaller than $\cot T$ for $T > \pi$ (up to just less than 2π , where $\cot T \rightarrow -\infty$ again). So, Hamilton's principle does not work for solutions to the equation of motion with $T > \pi$.

4. Fermat's Principle and Hamilton's principle: Is there a relation?

Physically, Fermat's principle and Hamilton's principle are not equivalent because the boundary conditions (i.e. the limits on the integrals) are quite different. As we stated previously, in Fermat's principle, the limits r_1 and r_2 on the integral are two fixed points in space [6]. We can write them as $r_1 = (x_1; y_1; z_1)$ and $r_2 = (x_2; y_2; z_2)$. If we represent a general path from r_1 to r_2 in the form $x = x(z), y = y(z)$ where $x(z_1) = x_1, x(z_2) = x_2, y(z_1) = y_1$ and $y(z_2) = y_2$, then the element of distance along the path is [6]

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + 1} dz \quad (15)$$

Where $\dot{x} = dx/dz, \dot{y} = dy/dz$ and so

$$v = \frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2 + 1} \frac{dz}{dt} \quad (16)$$

Solving this last equation for dt gives

$$dt = \sqrt{\dot{x}^2 + \dot{y}^2 + 1} \frac{dz}{v} \quad (17)$$

And if we define the refractive index, n , to be

$$n(x, y, z) = \frac{1}{v} \quad (18)$$

Where v is a velocity of light in a non-vacuum, the eq. (17) becomes

$$dt = n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + 1} dz \quad (19)$$

So, Fermat's principle, eq. (1) $\int_{r_1}^{r_2} dt = \text{stationary}$, becomes

$$\int_{z_1}^{z_2} n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + 1} dz = \text{stationary} \quad (20)$$

At this point, Fermat's principle is now mathematically equivalent to Hamilton's principle, and we can identify the function

$$\mathcal{L}(x, y, \dot{x}, \dot{y}, z) = n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + 1} \quad (21)$$

As the equivalent of a Lagrangian.

In the case of Fermat's principle, the (functional) transit time can be stated as

$$T = \int_{r_1}^{r_2} dt = \frac{1}{c} \int_{r_1}^{r_2} dr = \frac{1}{c} |r_2 - r_1| \quad (22)$$

is stationary, where c is velocity of light in vacuum, $dr = (dx; dy; dz)$. Obviously, minimising the transit time is equivalent to minimising the trajectory, $|r_2 - r_1|$, (because c is constant), which means that the light ray will trace out a straight line in vacuum space. In the case of Hamilton's principle, the action of a single particle with mass, m , in a vacuum (no interaction) is

$$S = \frac{1}{2} \int_{t_1}^{t_2} m \left(\frac{dr}{dt} \right) \left(\frac{dr}{dt} \right) dt \quad (23)$$

Table 1. Mathematical form of Fermat's principle

Fermat's Principle
$\int_{r_1}^{r_2} dt = \int_{r_1}^{r_2} n(x, y, z) ds = \text{stationary}$
$\int_{z_1}^{z_2} \mathcal{L}[x(z), y(z), \dot{x}(z), \dot{y}(z), z] dz = \text{stationary}$

Table 2. Mathematical form of Hamilton's principle

Hamilton's Principle
$\int_{t_1}^{t_2} \mathcal{L}[x(t), y(t), \dot{x}(t), \dot{y}(t), t] dt = \text{stationary}$

where the limits on the integral are now the initial and final fixed times, t_1 and t_2 , and it is understood that $r_1 = r(t_1)$ and $r_2 = r(t_2)$. In vacuum, this action is minimized because there is no potential energy or interaction. We can state that

$$m \frac{d^2 r}{dt^2} = 0 \quad (24)$$

and so the solution is

$$r(t) = r_1 + V(t_2 - t_1) \rightarrow (t_2 - t_1) = \frac{1}{V} (r_2 - r_1) \quad (25)$$

where $V = (r_2 - r_1)/(t_2 - t_1)$, i.e. all possible velocity.

The two solutions, (22), (25), are obviously equivalent if $|V| = c$, but many other velocities are possible in the least action case. This means that Fermat's principle applies only to light rays in the geometrical optics limit and assumes that the velocity of light (or equivalently the refractive index relative to the vacuum) is known at all points in space-time. Hamilton's principle applies to both particles and field, and it assumes nothing about speeds of propagation. It just requires a knowledge of kinetic and potential energies of the particle or field. In the simplest case, i.e. motion in a vacuum space-time, the two principles do give the same solution: motion in a straight line.

We see from Table 1 and Table 2 that Fermat's principle takes a stationary value for a function of a length coordinate and Hamilton's principle takes a stationary value for functions of time. Or, Fermat's principle takes a minimum value for the transit time with the endpoints, r_1, r_2 , fixed in space, whereas Hamilton's principle takes a minimum value for the action with the endpoints, $(r_1, t_1), (r_2, t_2)$ fixed in both space and time.

Using the same argument for Hamilton's principle which we stated earlier, Fermat's principle is valid only for a sufficiently short segment of the path. A "path" in both cases (Hamilton and Fermat) is a parametric representation of the position of the particle in space: $(x; y; z) = (X(s); Y(s); Z(s))$, where s is some parameter. In both principles the end points of the position of the particle in space, r_1, r_2 , are fixed. Because the start time and the finish time in Hamilton's principle are also fixed, so we can use t in place of the parameter s . But in Fermat's principle, the transit time is what is being minimized, so the finish time is not fixed, and it is often more convenient to use, say, z as the parameter s . Then $(x; y; z) = [X(s); Y(s); z]$, and Fermat's principle turns into the integral of a Lagrangian function, as we saw in Table 1.

5. Does a least action take a least time for happening?

The initial and final fixed times, t_1 and t_2 , are assumed to be given in a least action problem. So, the transit time cannot be minimised: it has a fixed value $t_2 - t_1$. Even if we were to consider a slightly different type of problem, where r_1 and r_2 are fixed, but the transit time is not fixed, and we are asked to calculate the total action, S , for particles travelling from r_1 to r_2 in a straight line (vacuum) at all possible speeds, V , it is not true that the smallest time will correspond to the smallest value of S . In fact, in terms of V , we must have

$$t_2 - t_1 = \frac{|r_2 - r_1|}{V} \quad (26)$$

so from eqs. (23), (26), because of velocity $dr=dt = V$ is constant, then we obtain

$$S = \frac{1}{2} m V^2 (t_2 - t_1) = \frac{1}{2} m V^2 \left(\frac{|r_2 - r_1|}{V} \right) = \frac{1}{2} m V |r_2 - r_1| \quad (27)$$

and the action is smallest when $V = 0$, which corresponds of course, from eq. (26), to an infinite transit time $t_2 - t_1$.

6. What is the relevance of Fermat's and Hamilton's principles with the frontier of applied physics?

The main contribution of Fermat's and Hamilton's principles is they led to development of calculus of variations. Mathematically, we speak of an extremum problem whenever the largest or smallest possible value of a quantity is involved. For the solution of finding such an extremum (maximum/minimum) problem of a definite integral, a special branch of mathematics, called the calculus of variations, has been developed [13].

In optimal control theory, Lev Pontryagin developed the relationship between the theory of optimal processes and the classical calculus of variations. He showed that the optimal problem is a generalization of the problem of Lagrange in the calculus of variations [21]. In dynamic programming, Richard Bellman developed the functional-equation technique to provide a new approach to some classical problems in the calculus of variations [22]. Some examples of "practical" application of Fermat's and Hamilton's principles can be found here [23, 24, 25, 26].

7. Conclusions

Mathematically, Fermat's principle is equivalent with Hamilton's principle. Fermat's principle minimizes the transit time with the limits on the integral are fixed points in space: r_1 , r_2 , whereas Hamilton's principle minimizes the action with the limits on the integral are fixed points in both space and time: (r_1, t_1) , (r_2, t_2) .

Fermat's and Hamilton's principles are not always valid for entire path of the system, but only for any sufficiently short segment of the path. In the simplest case, i.e. motion in a vacuum space-time, the two principles do give the same solution: motion in a straight line, since a straight line is the shortest connection between any two points. The main contribution of Fermat's and Hamilton's principles is they led to development of calculus of variations. Nature behaves stationary, it takes biggest time for smallest action.

Acknowledgement

Thank to Richard T. R. Hutagalung, Andri Husein, Suharyo Sumowidagdo, Hani N. Santosa, Kesetaraan Massa dan Energi, for fruitful discussions. Thank to Referees for reviewing this manuscript. For beloved ones: Ibu, Juwita and Aliya.

References

- [1] Cho Y 2010 *Lecture on Fundamental Theoretical Physics*, UNIST, unpublished.
- [2] Rojo A G 2005 *Am. J. Phys.* **73** 831
- [3] Mihas P 2008 *Sci. Educ.* **17** 751
- [4] Yourgrau W and Mandelstam S 1960 *Variational Principles in Dynamics and Quantum Theory* (London: Pitman Publishing Corporation)

- [5] Annala A 2011 *Mon. Not. R. Astron. Soc.* **416** 2944.
- [6] Marcuse D 1982 *Light Transmission Optics* (New York: Van Nostrand Reinhold Company).
- [7] Luneberg R K 1964 *Mathematical Theory of Optics* (California: University of California Press)
- [8] Kaushik B K and Sukheeja B D 1984 *Phys. Educ.* **19** 90.
- [9] Halliday D and Resnick R 1980 *Physics (Part II)* (New York: John Wiley & Sons).
- [10] Boas M L 2006 *Mathematical Methods in the Physical Sciences* (New York: John Wiley & Sons).
- [11] Taylor J R 2005 *Classical Mechanics* (USA: University Science Books)
- [12] Mak S-y 1986 *Phys. Educ.* **21** 365.
- [13] Lanczos C 1970 *The Variational Principles of Mechanics* (Canada: University of Toronto Press)
- [14] Feynman R P, Leighton R B, and Sands M 2010 *The Feynman Lectures on Physics, Vol.1* https://www.feynmanlectures.caltech.edu/I_toc.html
- [15] Goldstein H, Poole C, and Safko J 2000 *Classical Mechanics* (San Francisco: Addison Wesley).
- [16] Landau L D and Lifshitz E M 2000 *Mechanics* (USA: Butterworth-Heinemann).
- [17] Harper C 1995 *Introduction to Mathematical Physics* (USA: Prentice Hall).
- [18] Lakshminarayanan V, Ghatak A K, and Thyagarajan K 2002 *Lagrangian Optics* (Berlin: Springer).
- [19] Born M and Wolf E 1999 *Principles of Optics* (UK: Cambridge University Press).
- [20] Chaves J 2015 *Introduction to Nonimaging Optics* (Florida: CRC Press).
- [21] Pontryagin L S, Boltyanskii V G, Gamkrelidze R V, Mishchenko E F, and Pontryagin L S 1986 *Selected Works, Volume IV: The Mathematical Theory of Optimal Processes* (UK: Gordon and Breach Science Publisher S.A.)
- [22] Bellman R 1954 *Proc. Natl. Acad. Sci.* **40** 231.
- [23] Petrov A G and Shkundin S Z 2018 *Doklady Phys.* **63** 38
- [24] Hua Y-C and Guo Z-Y 2017 *Int. J. Heat and Mass Transf.* **105** 697.
- [25] Connolly G D, Lowe M J S, Temple J A G, and Rokhlin S I 2009 *Proc. R. Soc. London Ser. A* **465** 3401.
- [26] De Sapio V, Khatib O, and Delp S 2008 *Multibody Syst Dyn* **19** 303.