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The simplest oscillating solutions of nonlocal nonlinear models

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Abstract. In their recent works, Ablowitz and Musslimani proposed a new type of integrable nonlinear equations – nonlocal analogues of the nonlinear Schrödinger equation, the modified Korteweg-de Vries equation, and other nonlinear differential equations. In subsequent works, numerous researchers constructed the simplest soliton and rational solutions of these equations. In this paper, we construct the simplest oscillating solutions of some of the integrable nonlocal nonlinear differential equations associated to the nonlinear Schrödinger equation.

1. Introduction

Research in this area began with the work of Ablowitz and Musslimani [2], where the authors modified the Lax pair for the nonlinear Schrödinger equation. The authors also found [2] a one-soliton solution of the nonlocal nonlinear Schrödinger equation by the inverse scattering problem. Then they continued their research on this topic in works [1, 3-5, 10], the intermediate results of which are published in [5], where 16 types of integrable nonlinear nonlocal differential equations are considered and analyzed.
Research on this topic has attracted other researchers who have published at least 15 papers (see, for example, [6-8, 11-13, 16-20, 22, 23, 25]) on the subject in 2018 alone. In these works, as a rule, the authors construct soliton or rational solutions by methods of Darboux, Hirota transformation or self-similar substitution. Also, a large number of papers on this subject have been published in previous years, since each of the authors applied his method to several nonlocal equations proposed in [5] (note that the paper [5] was previously published in 2016 as a preprint arXiv:1610.02594). There is no doubt that research on this subject will continue (see, for example, [9, 21, 24]).
We remind that the equations of the AKNS hierarchy have the form

$$p_{tk} = -i^k H_{k+1}(p,q),\ q_{tk} = -i^k G_{k+1}(p,q) \text{ or } p_{tk} + i^k H_{k+1}(p,q) = 0,\ q_{tk} + (-i)^k H_{k+1}(p,q) = 0, \quad (1)$$

where the functions H_k and G_k satisfy the following equations [14, 15]

$$H_1(p,q) = -p_x,\ G_1(p,q) = -q_x,\ (F_k(p,q))_x = -pG_k(p,q) - qH_k(p,q),\ H_{k+1}(p,q) = 2pF_k(p,q) + (H_k(p,q))_x,\ G_{k+1}(p,q) = -2qF_k(p,q) - (G_k(p,q))_x. \quad (2)$$

In particular,

$$\begin{gathered} F_1(p,q) = pq,\ H_2(p,q) = 2p^2q - p_{xx}, \\ G_2(p,q) = -2q^2p + q_{xx},\ F_2(p,q) = p_xq - pq_x, \\ H_3(p,q) = 6pqp_x - p_{xxx},\ G_3(p,q) = 6pqq_x - q_{xxx}, \\ F_3(p,q) = pq_{xx} + qp_{xx} - p_xq_x - 3p^2q^2, \end{gathered} \quad (3)$$

$$H_4(p,q) = -6p^3q^2 + 6qp_x^2 + 4pp_xq_x + 8pqp_{xx} + 2p^2q_{xx} - p_{xxxx},$$
$$G_4(p,q) = 6p^2q^3 - 6pq_x^2 - 4qp_xq_x - 8pqq_{xx} - 2q^2p_{xx} - q_{xxxx},$$
$$F_4(p,q) = -6pq^2p_x + 6p^2qq_x - q_xp_{xx} + p_xq_{xx} + qp_{xxx} - pq_{xxx}.$$

It is easy to show that the functions $F_k(p,q)$, $H_k(p,q)$ and $G_k(p,q)$ have the following properties $F_k(q,p) = (-1)^{k-1}F_k(p,q)$, $F_k(-p,-q) = F_k(p,q)$, $G_{k+1}(p,q) = (-1)^kH_{k+1}(q,p)$, $H_{k+1}(-p,-q) = -H_{k+1}(p,q)$

And

$$\begin{aligned} F_k\left(p|_{x=-x}, q|_{x=-x}\right) &= (-1)^{k-1} F_k(p,q)|_{x=-x}, \\ G_k\left(p|_{x=-x}, q|_{x=-x}\right) &= (-1)^{k} G_k(p,q)|_{x=-x}, \\ H_k\left(p|_{x=-x}, q|_{x=-x}\right) &= (-1)^{k} H_k(p,q)|_{x=-x}. \end{aligned} \tag{4}$$

Since properties (3), (4) of equations (1) depend on the equation number, then reductions of equations of the AKNS hierarchy depend on the equation number as well. In particular, equations (1) except for general reductions

$$q(x,t_n) = \sigma p^*(x,t_n),$$
$$q(x,t_n) = \sigma p^*(-x, -t_n),$$

where $\sigma = \pm 1$, the following reductions are allowed

$$q(x,t_{2n-1}) = \sigma p(x, -t_{2n-1}), \tag{5a}$$

$$q(x,t_{2n}) = \sigma p(x, t_{2n}), \tag{5b}$$

$$q(x,t_{2n-1}) = \sigma p^*(-x, t_{2n-1}), \tag{5c}$$

$$q(x,t_{2n}) = \sigma p^*(-x, -t_{2n}). \tag{5d}$$

Naturally, the mixed equations [15] also allow reductions (5).

Equations of the form

$$p_t + \sum_{k\geq 1} i^{2k-1}\gamma_k H_{2k}(p,q) = 0$$

allow reductions (5a), (5c), and equations

$$p_t + \sum_{k\geq 1} i^{2k}\gamma_k H_{2k+1}(p,q) = 0$$

allow reductions (5b), (5d).

In this paper, we consider a class of relatively simple solutions of odd equations of the AKNS hierarchy. The solutions considered by us can be used to obtain more complex solutions using the Darboux transformation.

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2. Modulated plane wave

Recall that the algebraic-geometric solution of equations of the AKNS hierarchy for all values of t_j satisfies the stationary mixed equation (see, for example, [15])

$$H_g(p,q) + \sum_{k=1}^{g-1} C_k H_{g-k}(p,q) = C_g p,$$

where g is the genus of the corresponding spectral curve, C_j are some constants. Using the method described in [15], one can find the equation of the spectral curve corresponding to a given particular stationary solution.

Suggested $g = 2n - 1$ and

$$p(x,0) = f(x) \text{ and } q(x,0) = \sigma f(x)$$

or

$$p(x,0) = f(x) \text{ and } q(x,0) = \sigma f^*(-x)$$

from equation (6) we find the function $f(x)$. Then adding the dependence on t by formulas

$$p(x,t_{2n-1}) = f(x)e^{ibt_{2n-1}},\ q(x,t_{2n-1}) = \sigma f(x)e^{-ibt_{2n-1}} \tag{6}$$

or

$$p(x,t_{2n-1}) = f(x)e^{ibt_{2n-1}},\ q(x,t_{2n-1}) = \sigma f^*(-x)e^{-ibt_{2n-1}}, \tag{7}$$

we obtain the solution of the mixed equation of the AKNS hierarchy with a reduction (5a) or (5c).

Further, applying the Darboux transformation to the found solutions, it is possible to obtain more complex solutions of the considered mixed equations of the AKNS hierarchy. Naturally, each of the reductions corresponds to its Darboux transformation. Therefore, from the same solution satisfying both reductions, it is possible to obtain further different solutions that will satisfy only one reduction.

For example, making in the coupled nonlinear Schrödinger equation:

$$\begin{cases} ip_{t_1} + p_{xx} - 2p^2q = 0, \\ -iq_{t_1} + q_{xx} - 2q^2p = 0, \end{cases} \tag{8}$$

replacing (8), we obtain

$$f_{xx} = 2\sigma f^3 + bf. \tag{9}$$

Multiplying equation (9) by $2f_x$ and integrating, we obtain

$$(f_x)^2 = \sigma f^4 + bf^2 + c, \tag{10}$$

where c is the integration constant. In general, equation (10) is the equation of an elliptic function. Depending on which elliptic Jacobi function is used, one of three solutions of equations (8) with reduction (5a) is obtained. If the solution of the equation (10) satisfies the condition

$$f^*(-x) \equiv f(x),$$

then the functions (6) are the solution of equations (8) with reduction (5c).

3. Elliptic solution I

If $f(x) = A\mathrm{sn}[a(x-x_0);k^2]$ then

$$A = \frac{ak}{\sqrt{\sigma}},\ b = -a^2(k^2+1),\ c = \frac{a^4k^2}{\sigma}$$

and

$$p = \frac{ake^{-ia^2(k^2+1)t}}{\sqrt{\sigma}}\mathrm{sn}[a(x-x_0);k^2],$$

$$q = \sqrt{\sigma}ake^{ia^2(k^2+1)t}\,\mathrm{sn}[a(x-x_0);k^2]. \tag{11}$$

Figure 1 shows the real part of the function $p(x,t)$ (11) when $\sigma = 1$, $a = 3$, $k = 0{,}8$

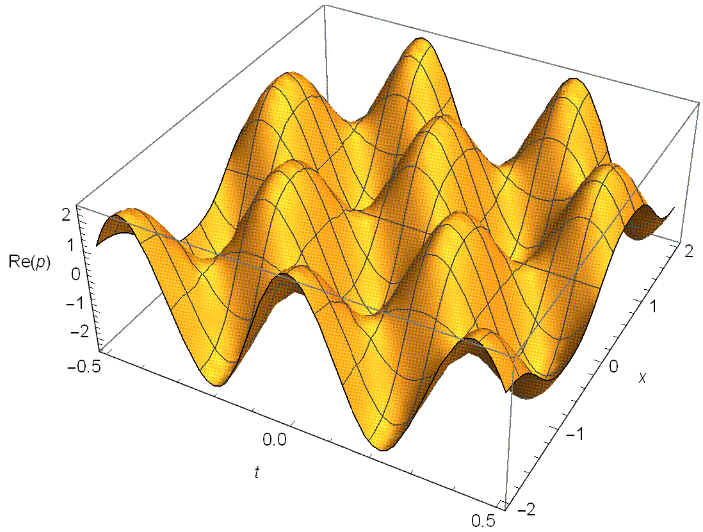

Figure 1. The real part of the function $p(x,t)$ (12) when σ = 1, a = 3, k = 0,8.

For $a \in$ R, $0 < k < 1$, $x_0 = 0$, $\sigma = -1$ the condition is satisfied $f^*(-x) \equiv f(x)$.

Therefore, for given parameter values, solution (11) satisfies equations (8) with reduction (5c).

4. Elliptic solution II

If $f(x) = A\mathrm{cn}[a(x-x_0);k^2]$ then

$$A = \frac{iak}{\sqrt{\sigma}},\ b = a^2(2k^2-1),\ c = \frac{a^4k^2(k^2-1)}{\sigma}$$

and

$$p = \frac{iake^{ia^2(2k^2-1)t}}{\sqrt{\sigma}}\mathrm{cn}[a(x-x_0);k^2],$$

$$q = i\sqrt{\sigma}ake^{-ia^2(2k^2-1)t}\,\mathrm{cn}[a(x-x_0);k^2]. \tag{12}$$

Figure 2 shows the real part of the function $p(x,t)$ (12) when σ = 1, a = 3, k = 0,8

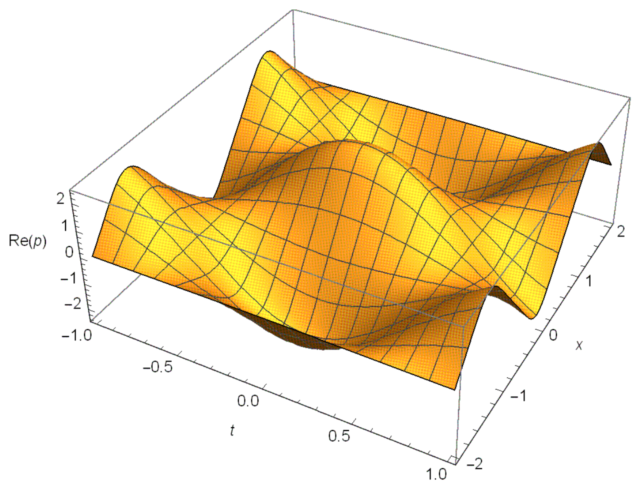

Figure 2. The real part of the function $p(x,t)$ (12) when σ = 1, a = 3, k = 0,8.

Solution (12) for $a \in$ R, $0 < k < 1$, $x_0 = 0$, $\sigma = -1$ also satisfies equations (8) with reduction (5c), since in this case the condition is also satisfied

$$f^*(-x) \equiv f(x).$$

5. Elliptic solution III

If $f(x) = A\mathrm{dn}[a(x - x_0);k^2]$ then

$$A = \frac{ia}{\sqrt{\sigma}},\quad b = a^2(2 - k^2),\quad c = \frac{a^4(1-k^2)}{\sigma}$$

and

$$p = \frac{iae^{ia^2(2-k^2)t}}{\sqrt{\sigma}}\mathrm{dn}[a(x-x_0);k^2],$$

$$q = i\sqrt{\sigma}ae^{-ia^2(2-k^2)t}\,\mathrm{dn}[a(x-x_0);k^2]. \tag{13}$$

Figure 3 shows the real part of the function $p(x,t)$ (13) when $\sigma = 1$, $a = 3$, $k = 0{,}8$

Solution (13) for $a \in$ R, $0 < k < 1$, $x_0 = 0$, $\sigma = -1$ also satisfies equations (8) with reduction (5c), since in this case the condition is also satisfied

$$f^*(-x) \equiv f(x).$$

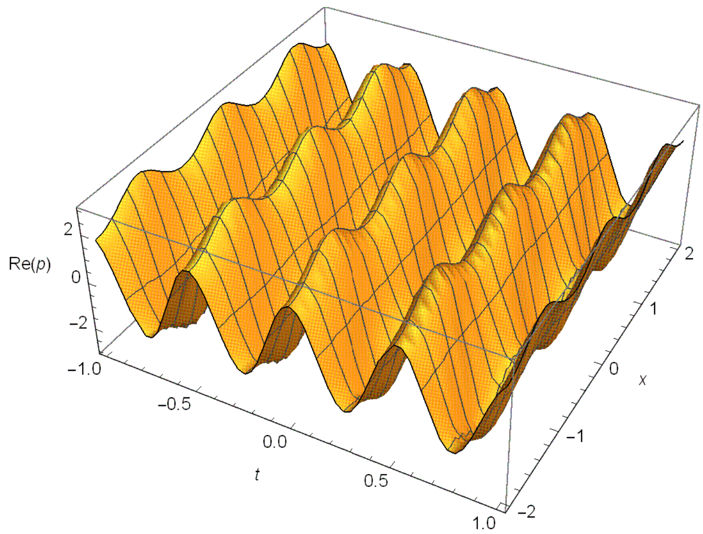

Figure 3. The real part of the function $p(x,t)$ (13) when $\sigma = 1$, $a = 3$, $k = 0{,}8$.

6. Solutions in elementary functions

However, there are two special cases where equation (11) has a solution in elementary functions.

In the first of them the constant $c = 0$ and equation (11) is reduced to the simplest differential equation of the first order

$$\frac{df}{dx} = f\sqrt{b + \sigma f^2}.$$

Dividing the variables and integrating, we have

$$x = \int \frac{df}{f\sqrt{b+\sigma f^2}}.$$

Calculating the integral standing on the right side and expressing the function f, we obtain, depending on the sign of b, three different cases.

1. If $b = 0$, then

$$f(x) = \frac{1}{(x-x_0)\sqrt{\sigma}}$$

and

$$p = \frac{1}{(x-x_0)\sqrt{\sigma}},\quad q = \frac{\sqrt{\sigma}}{(x-x_0)}. \tag{14}$$

2. If $b = -a^2 < 0$, then

$$f(x) = \frac{a}{\sqrt{\sigma}\sin[a(x-x_0)]}$$

and

$$p = \frac{ae^{-ia^2t}}{\sqrt{\sigma}\sin[a(x-x_0)]},\quad q = \frac{\sqrt{\sigma}ae^{ia^2t}}{\sin[a(x-x_0)]}. \tag{15}$$

3. If $b = a^2 > 0$, then

$$f(x) = \frac{ia}{\sqrt{\sigma}\cosh[a(x-x_0)]}$$

and

$$p = \frac{iae^{ia^2t}}{\sqrt{\sigma}\cosh[a(x-x_0)]},\quad q = \frac{i\sqrt{\sigma}ae^{-ia^2t}}{\cosh[a(x-x_0)]}. \tag{16}$$

Figure 4 shows the real part of the function $p(x,t)$ (16) when σ = 1, a = 3

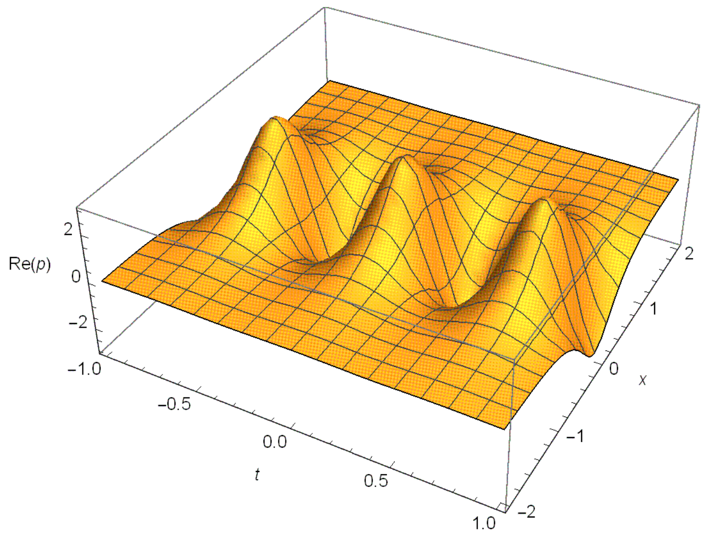

Figure 4. The real part of the function $p(x,t)$ (16) when σ = 1, a = 3.

It is easy to see that for $a \in$ R, $x_0 = 0$, $\sigma = -1$, the solution (16) also satisfies equations (8) with reduction (5c).

In the second special case

$$c = \frac{b^2}{4\sigma} \neq 0$$

and equation (10) is also reduced to the first order differential equation

$$\frac{df}{dx} = \sqrt{\sigma} f^2 + \frac{b}{2\sqrt{\sigma}} \text{ or } \frac{d(\sqrt{\sigma} f)}{dx} = \left(\sqrt{\sigma} f\right)^2 + \frac{b}{2}.$$

By introducing a new variable $\tilde{f} = \sqrt{\sigma} f$, separating the variables and integrating, we have

$$x = \int \frac{d\tilde{f}}{\left(\tilde{f}^2 + \frac{b}{2}\right)}.$$

Calculating the integral standing on the right side and expressing the function h, we obtain, depending on the sign of b two different cases

1. If $b = 2a^2 > 0$, then

$$f(x) = \frac{a}{\sqrt{\sigma}} \operatorname{ctg}[a(x - x_0)]$$

and

$$p = \frac{ae^{2ia^2 t}}{\sqrt{\sigma}} \operatorname{ctg}[a(x - x_0)], \ q = \sqrt{\sigma} ae^{-2ia^2 t} \operatorname{ctg}[a(x - x_0)]. \tag{17}$$

2. If $b = -2a^2 > 0$, then

$$f(x) = \frac{a}{\sqrt{\sigma}} \tanh[a(x - x_0)]$$

and

$$p = \frac{ae^{-2ia^2 t}}{\sqrt{\sigma}} \tanh[a(x - x_0)], \ q = \sqrt{\sigma} ae^{2ia^2 t} \tanh[a(x - x_0)]. \tag{18}$$

Figure 5 shows the real part of the function $p(x,t)$ (18) when σ = 1, a = 3

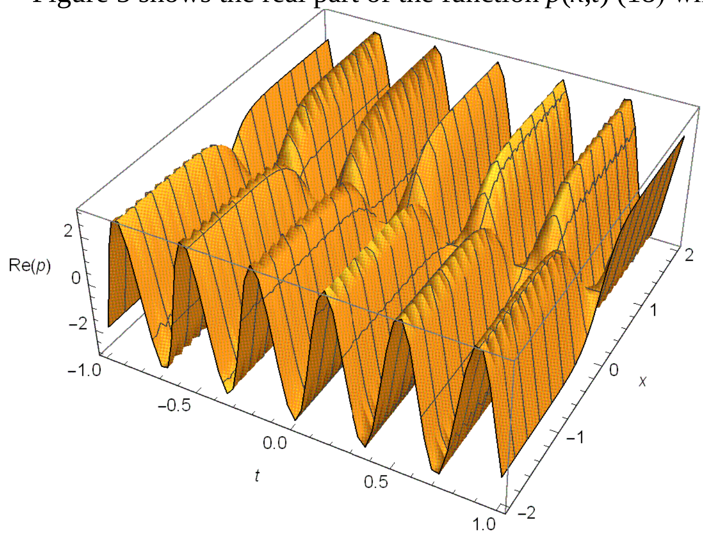

Figure 5. The real part of the function $p(x,t)$ (18) when σ = 1, a = 3.

It is easy to check that when $a \in$ R, $x_0 = 0$, $\sigma = -1$, the solution (18) also satisfies equations (8) with reduction (5c).

7. Conclusion

In conclusion, we note that the solutions found above satisfy the subsequent odd equations of the AKNS hierarchy, but with a different coefficient before the variable *t*. In particular, substituting (6) into equation

$$ip_{t_3}+H_4(p,q)=0$$

and simplifying, we obtain (for $b = b_3$).

$$-b_3f - 6\sigma^2f^5 + 10\sigma ff_x^2 + 10\sigma f^2f_{xx} - f_{xxxx} = 0. \quad (19)$$

Suppose that the function $f(x)$ satisfies equations (9) and (10).
Then

$$f_{xxx} = 6\sigma f^2f_x + bf_x$$

and

$$f_{xxxx} = 12\sigma ff_x^2 + 6\sigma f^2f_{xx} + bf_{xx}$$

or

$$f_{xxxx} = 24\sigma^2f^5 + 20\sigma bf^3 + 12\sigma cf + b^2f. \quad (20)$$

Substituting (20), (9) and (10) into (19) and simplifying, we obtain

$$b_3 = -b^2 - 2\sigma c.$$

Naturally, the Darboux transformations of the solutions found will also satisfy all odd equations of the AKNS hierarchy with corresponding reductions and their mixed forms.

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