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The method of differentiable functional in a problem of lightning discharge localization and time moment determination

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Abstract. In this article, the problem of detection of lightning discharge time and location within the area of the Republic of Kazakhstan is considered. The mathematical model offered in this article is based on a method of timepoints recording by mutually spaced detectors of the network receiving and identifying radio signals, which are generated by cloud to ground lightning discharges (the TOA method – time-of-arrival). According to this method the system of the equations of model is constructed by comparison of distance from a lightning discharge point to stations with the same distance passed by a radio signal taking into account the moment of its records at stations. For solution finding the system of the equations is reducing to continuously differentiable functional with respect to geographical coordinates and the time moment of lightning discharge. The conditions which performing the reached minimum of functional with a practical accuracy provides determination of coordinates and time moment of lightning discharge are considered. The test calculations with estimated accuracy depending on an error of the time moments of lightning discharge detected at the stations are performed.

1. Introduction

The known TOA method [1–7] of detection of time moments of radio signals arrival from lightning discharges by antennas of lightning detection network allows to create the set of equations, comparing distance from stations to the lightning with distance which passes the lightning radio signal at the speed of light to stations.

In work Lozbin et.al. [8] it is shown that for three stations network this set of equations considered on spherical model of Earth has two exact solutions. If the network includes more than three stations, then it is possible to divide the set of equations into groups by threes, to find solutions for each of groups and to select the unique solution of set, which will determine coordinates and lightning impulse time moment.

However, such result takes place ideally when time moments measures precisely. The measuring time moments of radio signals recorded from stations contain errors in reality. The origin of errors are rather various. First, technical parameters of antennas, condition of the atmosphere and landscape aspects on the radio impulse way and random electromagnetic noise.

Existence of errors in time moments measurement leads to the fact, that solutions of three equations sets are always different. For values of coordinates and time moment of a lightning from sets of threes the subset of close solutions is chosen and then, from this sets average values are chosen.

In this article, the solution of set of equations comes to a problem of minimization of the non-negative differentiable functional depending on parameters of the lightning discharge. The sets of equations solutions are coinciding with functional minimum in case of set of equations consistency. Previously, in a model the area of observation of lightning activity and detection stations positions are limited. For formulas simplification the orthogonal transformation of geographical reference system is executed.

Minimization of functional by the method of coordinate descent is executed. In conclusion of this article the comparative analysis of results accuracy with the similar results presented in Lozbin et.al. [8] is provided.

2. Mathematical model

The set of the equations of mathematical model is formed in an assumption of sphericity of the Earth's surface.

Let's select on the Earth's surface a spherical segment S_0 (Figure 1) with the center in point $P_0(\varphi_0,\theta_0)$ (φ_0 and θ_0 – is a longitude and latitude respectively, -180º < $\varphi_0 \leq$ 180º, -90º $\leq \theta_0 \leq$ 90º) and radius of surface R_0, km.

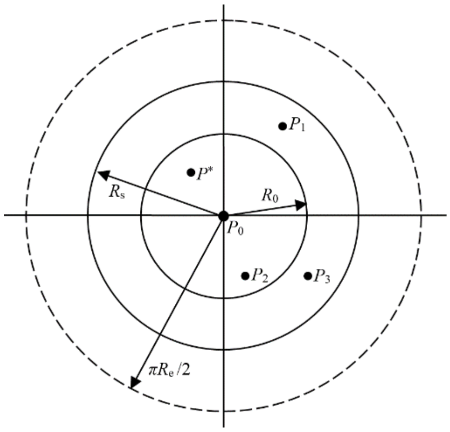

Figure 1. Schematic view on the Earth's surface with center point P_0 of the observing area S_0 (P_1, P_2, P_3, ... – points of the measuring stations, P^*– point of the lightning discharge).

The S_0 area is determine a region of lighting activity observation for parameters of lightning discharges calculation. Let us suppose that measuring stations are locate randomly in Earth's points $P_i(\varphi_i,\theta_i)$, $i=1,2,\ldots,N_s$ with long offset R_s from center P_0. For R_0 and R_s the formula is:

$$R_0 \leq R_s < \frac{\pi R_e}{2}, \tag{1}$$

where, R_e – Radius of the Earth.

Thus, mathematical model assumes, that measuring stations can be inside or outside controlled area. Arise from Eq.(1), maximal distance of lightning discharge d_{max} inside controlled area from any station is fulfil a condition:

$$d_{\max} < R_0 + R_s < \pi R_e. \tag{2}$$

Let in some time moment t^* in point $P^*(\varphi^*,\theta^*)$ the lightning discharge was happened. The radio signal from this discharge was detected on the stations in time moments t_i. As lightning discharge precedes its detection, then

$$t^* \leq t_i, \qquad i = 1,2,\ldots,N_s. \tag{3}$$

The equality in Eq. (3) can be only in case of lightning discharged just in this station.

It is known that the long-wave radio signal from lightning to stations passes with velocity of light along the Earth's surface on geodetic lines. On the sphere smaller arches of big circles which in spherical geometry are called right lines are geodetic. As a result we come to the following identical equations:

$$P_iP^* = c(t_i - t^*),\ i = 1,2,\ldots,N_s, \tag{4}$$

where, P_iP^* – straight-line distance from lightning discharge point to measuring station.

Let P(φ, θ) and t – are required point and time of lightning discharge. Based on identical Eq. (4) for calculation we will write the set of the equations:

$$P_iP = c(t_i - t), \qquad i = 1,2,\ldots,N_s. \tag{5}$$

Add unit radius-vectors $\mathbf{r}_i = \{x_i, y_i, z_i\}$ of the *i*- measuring station and unit radius-vector $\mathbf{r} = \{x, y, z\}$ of the target lightning discharge in geocentric Greenwich coordinate system. The distance from the station to a point of lightning discharge is equal:

$$P_iP = R_e\Lambda_i, \tag{6}$$

where, R_e – average Earth's radius, Λ_i – an angle (smallest) between vectors $\mathbf{r}_i$ and $\mathbf{r}$. By definition of a scalar product of unit vectors:

$$\Lambda_i = \arccos(\mathbf{r}_i\mathbf{r}) = \arccos(x_ix + y_iy + z_iz), \quad 0 \le \Lambda_i \le \pi. \tag{7}$$

As components of the vectors $\mathbf{r}_i$ and $\mathbf{r}$ are coupled to spherical coordinates (latitude and longitude) by the equations:

$$x_i = \cos\varphi_i \cos\theta_i; y_i = \sin\varphi_i \cos\theta_i; z_i = \sin\theta_i., \quad i = \overline{1, N_s}. \tag{8}$$

$$x = \cos\varphi\cos\theta, \quad y = \sin\varphi\cos\theta, \quad z = \sin\theta, \tag{9}$$

where, $-\pi < \varphi, \varphi_i \le \pi, \quad -\frac{\pi}{2} \le \theta, \theta_i \le \frac{\pi}{2}$, then, the set of Eq. (5) take the form:

$$R_e \arccos(x_i \cos\varphi\cos\theta + y_i \sin\varphi\cos\theta + z_i \sin\theta) = c(t_i - t),\ i = 1,2,\ldots,N_s. \tag{10}$$

The acceptable values of a lightning location in model are the points *P(φ,θ)* satisfying to inequation $P_0P \le R_0$, from which follows:

$$\arccos(\mathbf{r}_0\mathbf{r}) \le \frac{R_0}{R_e}, \tag{11}$$

where, $\mathbf{r}_0 = \{x_0, y_0, z_0\}$, $x_0 = \cos\varphi_0 \cos\theta_0; y_0 = \sin\varphi_0 \cos\theta_i; z_0 = \sin\theta_0$.

Let's define the minimum and maximum of time moments of lightning detection:

$$t_{\min} = \min_i t_i,\ t_{\max} = \max_i t_i. \tag{12}$$

Because of restriction to the area of observation and to observing stations location the maximum distance from station to the point of lightning discharge does not exceed the value R_0+R_s , i.e.:

$$c(t_{\max} - t) \le R_0 + R_s. \tag{13}$$

From Eq.(13) we can find the lower bound for the lightning discharge time moment:

$$t_{\max} - \frac{R_0 + R_s}{c} \le t. \tag{14}$$

Taking to account Eq. (3), we obtain an admissible interval of values of time moments of lightning discharge in mathematical model:

$$t_{\max} - \frac{R_0 + R_s}{c} \le t \le t_{\min}. \tag{15}$$

3. Geographical coordinates and time scale transformation

For simplification of calculation it is feasible orthogonal transformation of the geocentric Greenwich coordinate system, having transferred its pole to the center of the area of observation. Transformation is carried out by two consecutive turns by means of transfer matrix:

$$Q_\varphi = \begin{pmatrix} \cos\varphi_0 & \sin\varphi_0 & 0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Q_\theta = \begin{pmatrix} \sin\theta_0 & 0 & -\cos\theta_0 \\ 0 & 1 & 0 \\ \cos\theta_0 & 0 & \sin\theta_0 \end{pmatrix}.$$

The matrix of final transfer will be:

$$Q = \begin{pmatrix} \cos\varphi_0 \sin\theta_0 & \sin\varphi_0 \sin\theta_0 & -\cos\theta_0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \\ \cos\varphi_0 \cos\theta_0 & \sin\varphi_0 \cos\theta_0 & \sin\theta_0 \end{pmatrix}. \tag{16}$$

As a result of transformation (16) the positions of stations will be set by vectors:

$$\tilde{\mathbf{r}}_i = \left(Q\mathbf{r}_i^T\right)^T = \mathbf{r}_i Q^T, \tag{17}$$

where, $Q = Q_\theta Q_\varphi$, T - operation of matrix and vectors transformation.
New spherical coordinates of stations are defined from equations:

$$\tilde{\theta}_i = \arcsin \tilde{z}_i, \quad \cos\tilde{\varphi}_i = \frac{\tilde{x}_i}{\cos\tilde{\theta}_i}, \quad \sin\tilde{\varphi}_i = \frac{\tilde{y}_i}{\cos\tilde{\theta}_i}. \tag{18}$$

Let's change also a time scale, by setting a reference point for a time moment $t_{\min}$. In that case, the time moments of lightning discharges detection will be equal in a new scale:

$$\tilde{t}_i = t_i - t_{\min}, \tag{19}$$

and time moment of discharge will be:

$$\tilde{t} = t - t_{\min}. \tag{20}$$

As at orthogonal transformation the distance and corners are not change, and transformation of time is linear, the equations (10) keep the form:

$$R_e \arccos\left[\tilde{x}_i\left(\cos\tilde{\varphi} + \tilde{y}_i \sin\tilde{\varphi}\right)\cos\tilde{\theta} + \tilde{z}_i \sin\tilde{\theta}\right] = c(\tilde{t}_i - \tilde{t}), \quad i = 1,2,\ldots,N_s, \tag{21}$$

The Eq. (11) and (15) for admissible values of coordinates (in radians) and the time moment of discharge will take a form:

$$-\pi < \tilde{\varphi} \le \pi, \qquad \frac{\pi}{2} - \frac{R_0}{R_e} \le \tilde{\theta} \le \frac{\pi}{2}, \tag{22}$$

$$(t_{\max} - t_{\min}) - \frac{R_0 + R_s}{c} \le \tilde{t} \le 0. \tag{23}$$

4. Transformation of set of equations and functional construction

Believing that:

$$\alpha_i = \frac{c}{R_e}\tilde{t}_i, \qquad \alpha = \frac{c}{R_e}\tilde{t}, \tag{24}$$

let's write Eq. (21) as:

$$\arccos\left[\tilde{x}_i\left(\cos\tilde{\varphi} + \tilde{y}_i \sin\tilde{\varphi}\right)\cos\tilde{\theta} + \tilde{z}_i \sin\tilde{\theta}\right] = \alpha_i - \alpha, \quad i = 1,2,\ldots,N_s. \tag{25}$$

From Eq. (23) we will find a range of admissible values α:

$$\frac{c(t_{\max}-t_{\min})-(R_0+R_s)}{R_e}\leq\alpha\leq 0 \quad . \tag{26}$$

Calculating a cosine from the left and right parts of the Eq. (25), we will receive the set of the equations:

$$\tilde{x}_i\left(\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}=\cos(\alpha_i-\alpha),\quad i=1,2,\ldots,N_s\,. \tag{27}$$

The sets of the Eq. (25) and (27) are equivalent. Really, the left part of the Eq.(27) represents a scalar product of two single vectors. Therefore, its values belong to a segment [-1,1] which coincides with a range of definition of the inverse cosine. From Eq. (2), (24) and (26) we find that $-\pi<\alpha\leq 0$. On this interval $\cos\alpha$ function is also biunique with the argument α. Thus, the calculation of the arccosine function from the left and right parts of the Eq. (25) leads to the set of the Eq. (21).
Based on Eq. (27) we can construct a functional:

$$G(\alpha,\tilde{\varphi},\tilde{\theta})=\sum_{i=1}^{N_s}\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}-\cos(\alpha_i-\alpha)\right]^2\,. \tag{28}$$

The functional (28) is not negative and become zero only on solutions of system (27). Fairly and a converse: if at some values $(\alpha,\tilde{\varphi},\tilde{\theta})$ the functional $G(\alpha,\tilde{\varphi},\tilde{\theta})$ is equal to zero, then values $(\alpha,\tilde{\varphi},\tilde{\theta})$ are the solution of system (27). Thus, the solution of the set of Eq. (27) is comes to a problem of minimization of functional (28). The problem of minimization is solved by the iterative sequence construction:

$$(\alpha^k,\tilde{\varphi}^k,\tilde{\theta}^k),\quad k=0,1,2,\ldots, \tag{29}$$

on which, corresponding sequence $G(\alpha^k,\tilde{\varphi}^k,\tilde{\theta}^k)$ is tending to zero.

The essential mathematical properties of functional (28) are its differentiability and periodicity with the 2π period on each of parameters. The problem of minimization becomes complicated because of the several local minimum (as usual) of the functional (28).

If the lightning location network consists of three stations, then, as shown in Lozbin et.al. [8], the set of the equations has two solutions, that is the corresponding functional (28) has two minimum points with zero value. However, in case of more than three stations in network, the set (27) usually can have only one solution, and functional (28) – the single minimum point with zero value.

Considering that time moments t_i measures at the stations with some error, the functional (28) is tending to zero only randomly. Therefore, rules are imposed on a problem of sequence construction (29):

1) the sequence has to converge to point located in the field of observation;
2) the iterative process has to have criterion which provides achievement of functional minimum;
3) the value of functional in a minimum point has to correspond to the solution of set of the equations (27) with the required error.

When these rules performing at some k step of iterative process the point $(\alpha^k,\tilde{\varphi}^k,\tilde{\theta}^k)$ is accepted as solution (t^*,φ^*,θ^*) of set (27). Otherwise, the absence of a lightning is define.

5. Algorithm of functional (25) minimization

Process of functional minimization consists in construction of the sequence $(\alpha^k,\tilde{\varphi}^k,\tilde{\theta}^k)$ where the functional is tends to minimum value. The following sequence of approximations on each step of the minimizing cycle is providing:

$$(\alpha^k,\tilde{\varphi}^k,\tilde{\theta}^k)\to(\alpha^{k+1},\tilde{\varphi}^k,\tilde{\theta}^k)\to(\alpha^{k+1},\tilde{\varphi}^{k+1},\tilde{\theta}^k)\to(\alpha^{k+1},\tilde{\varphi}^{k+1},\tilde{\theta}^{k+1}) \tag{30}$$

Each step of a cycle comes to the end with criteria of a stop of calculations checking.

As initial values of parameters of lightning before an entrance to a cycle we believe that: $\alpha^0=0$, $\tilde{\varphi}^0=0$, $\tilde{\theta}^0=\frac{\pi}{2}-\frac{1}{2}\mathrm{arctg}\frac{R_0}{R_e}$.

From equations $\frac{\partial G(\alpha,\tilde{\varphi}^k,\tilde{\theta}^k)}{\partial\alpha}=0$, $\frac{\partial G(\alpha^{k+1},\tilde{\varphi},\tilde{\theta}^k)}{\partial\tilde{\varphi}}=0$ and $\frac{\partial G(\alpha^{k+1},\tilde{\varphi}^{k+1},\tilde{\theta})}{\partial\tilde{\theta}}=0$ subsequently find the values α^{k+1}, $\tilde{\varphi}^{k+1}$ and $\tilde{\theta}^{k+1}$.

From formulas for derivatives:

$$\frac{\partial G(\alpha,\tilde{\varphi},\tilde{\theta})}{\partial\alpha}=\sum_{i=1}^{N}\left[\sin 2\alpha_i\cos 2\alpha-\cos 2\alpha_i\sin 2\alpha\right]+$$

$$+\sum_{i=1}^{N_s}\left[2\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\theta+\tilde{z}_i\sin\tilde{\theta}\right]\left[-\sin\alpha_i\cos\alpha+\cos\alpha_i\sin\alpha\right]\right], \quad (31)$$

$$\frac{\partial G(\alpha,\tilde{\varphi},\tilde{\theta})}{\partial\tilde{\varphi}}=2\cos\tilde{\theta}\sum_{i=1}^{N_s}\left(-\tilde{x}_i\sin\tilde{\varphi}+\tilde{y}_i\cos\tilde{\varphi}\right)\times$$

$$\times\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}-\cos(\alpha_i-\alpha)\right], \quad (32)$$

$$\frac{\partial G(\alpha,\tilde{\varphi},\tilde{\theta})}{\partial\tilde{\theta}}=2\sum_{i=1}^{N_s}\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}-\cos(\alpha_i-\alpha)\right]\times$$

$$\times\left[-\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\sin\tilde{\theta}+\tilde{z}_i\cos\tilde{\theta}\right], \quad (33)$$

we draw three trigonometrical equations for indeterminate α, $\tilde{\varphi}$ and $\tilde{\theta}$:

1) $A_\alpha\cos 2\alpha+B_\alpha\sin 2\alpha+C_\alpha\cos\alpha+D_\alpha\sin\alpha=0$, $-\pi<\alpha\le 0$, (34)

where, $A_\alpha=\sum_{i=1}^{N}\sin 2\alpha_i$, $B_\alpha=-\sum_{i=1}^{N}\cos 2\alpha_i$,

$$C_\alpha=-2\sum_{i=1}^{N}\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}\right]\sin\alpha_i,$$

$$D_\alpha=2\sum_{i=1}^{N}\left[\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\tilde{\theta}+\tilde{z}_i\sin\tilde{\theta}\right]\cos\alpha_i.$$

2) $A_\varphi\cos 2\tilde{\varphi}+B_\varphi\sin 2\tilde{\varphi}+C_\varphi\cos\tilde{\varphi}+D_\varphi\sin\tilde{\varphi}=0$, $-\pi<\tilde{\varphi}\le\pi$, (35)

where, $A_\varphi=2\cos\tilde{\theta}\sum_{i=1}^{N}\tilde{x}_i\tilde{y}_i$, $B_\varphi=\cos\tilde{\theta}\sum_{i=1}^{N}(\tilde{y}_i^2-\tilde{x}_i^2)$,

$C_\varphi=2\sum_{i=1}^{N}\tilde{y}_i\left(\tilde{z}_i\sin\tilde{\theta}-\cos(\alpha-\alpha_i)\right)$, $D_\varphi=2\sum_{i=1}^{N}\tilde{x}_i\left(-\tilde{z}_i\sin\tilde{\theta}+\cos(\alpha-\alpha_i)\right)$.

3) $A_\theta\cos 2\tilde{\theta}+B_\theta\sin 2\tilde{\theta}+C_\theta\cos\tilde{\theta}+D_\theta\sin\tilde{\theta}=0$, $-\frac{\pi}{2}\le\tilde{\theta}\le\frac{\pi}{2}$, (36)

where, $A_\theta=2\sum_{i=1}^{N}\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\tilde{z}_i$, $B_\theta=\sum_{i=1}^{N}\left[\tilde{z}_i^2-\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)^2\right]$,

$C_\theta=-2\sum_{i=1}^{N}\tilde{z}_i\cos\left(\alpha-\alpha_i\right)$, $D_\theta=2\sum_{i=1}^{N}\left(\tilde{x}_i\cos\tilde{\varphi}+\tilde{y}_i\sin\tilde{\varphi}\right)\cos\left(\alpha-\alpha_i\right)$.

As it is noted above, the functional (28) is differentiable periodic function with 2π period on each of variables α, $\tilde{\varphi}$ and $\tilde{\theta}$. Follows from function continuity and periodicity that, at the fixed value of two variables the quantity of minima on the third variable coincides on a half-interval *(-π, π]* with quantity of maxima. Therefore, the functional (28) has, at least, one minimum and one maximum for each of variables α, $\tilde{\varphi}$ and $\tilde{\theta}$. Thus, the Eq. (34), (35) and (36) have either two, or four real roots on the period.

The case of three real roots is theoretically possible. However, in this case one of roots it is necessary is either repeated or an inflection point in which the extremum is absent.

The Eq. (34)–(36) solution can be passed directly, proceeding from their trigonometrical form or these equations can be transform to the algebraic equations by universal substitution $u = \mathrm{tg}\dfrac{\nu}{2}$, $\nu = \alpha, \tilde{\varphi}, \tilde{\theta}$:

$$\frac{(A_\nu - C_\nu)u^4 + (-4B_\nu + 2D_\nu)u^3 - 6A_\nu u^2 + (4B_\nu + 2D_\nu)u + (A_\nu + C_\nu)}{\left(1+u^2\right)^2} = 0 . \quad (37)$$

After calculation of a set of roots the roots corresponding to functional minima and also the range of their definition (22), (26) are allocated. If on any two variables the minima of functional are absent, then it is considered that the signal is received from a lightning out of observing area.

Iterative process (30) is ending on condition of performance an inequality:

$$\frac{\left|\tilde{\mathbf{r}}^{k+1} - \tilde{\mathbf{r}}^k\right| + \left|\alpha^{k+1} - \alpha^k\right|}{\sqrt{G(\alpha^k, \tilde{\varphi}^k, \tilde{\theta}^k)/N_s}} < \varepsilon_1 , \quad (38)$$

where, $\left|\tilde{\mathbf{r}}^{k+1} - \tilde{\mathbf{r}}^k\right| = \sqrt{(\tilde{x}^{k+1} - \tilde{x}^k)^2 + (\tilde{y}^{k+1} - \tilde{y}^k)^2 + (\tilde{z}^{k+1} - \tilde{z}^k)^2}$.

On the single sphere the module of a difference of closely radius vectors, is approximately equal to arch length between the ends of these vectors.

The numerator of the left expression in inequality (38) represents the sum of increments of lightning parameters on *(k+1)* step of an iterative cycle. The expression in a denominator is a mean square deviation from zero of differences of the left and right parts of the equations of system (27).

The chosen form of a condition (38) of interruption of iterations does not depend on the number of measurement stations. In a number of computing experiments, performed at various combinations of quantity and position of measuring stations and also various positions of a lightning and errors up to 1 µs of measurement of the time moments of lightning discharge, it is established that accuracy $\varepsilon_1 = 10^{-7}$ when calculating is sufficient for definition of a global minimum of functional $G(\alpha, \tilde{\varphi}, \tilde{\theta})$.

When performing a condition (38) on some step of iterative cycle the last values of variables are accepted as approximate solution of set of equations (27), that is believe that:

$$\tilde{\varphi}^* = \tilde{\varphi}^{k+1}, \quad \tilde{\theta}^* = \tilde{\theta}^{k+1}, \alpha^* = \alpha^{k+1} . \quad (39)$$

6. The lightning parameters recovery and estimation

Believing that $\tilde{\mathbf{r}}^* = \left\{\cos\tilde{\varphi}^*\cos\tilde{\theta}^*, \sin\tilde{\varphi}^*\cos\tilde{\theta}^*, \sin\tilde{\theta}^*\right\}$ and performing the return transformation to the initial coordinate system $\mathbf{r}^* = \tilde{\mathbf{r}}^* Q$, we can find a single vector of a point of lightning discharge **r***={x*, y*, z*} in initial coordinate system. Then we calculate geographical coordinates of the corresponding point on the Earth's surface:

$$\theta^* = \arcsin z^* , \quad \cos\varphi^* = \frac{\tilde{x}_i}{\cos\theta^*} , \quad \sin\varphi^* = \frac{y^*}{\cos\theta^*} . \quad (40)$$

7. Results

For this method visualization we will consider the test case of lightning activity monitoring on the area of Republic of Kazakhstan showed in Lozbin et.al. [8]. Data on placement of measuring stations are provided in table 1.

Table 1. Coordinates of measuring stations.

Station No	Longitude	Latitude	Name of the station
1	76.92848	43.25654	Almaty
2	78.36667	45.01667	Taldykorgan
3	77.06304	43.86681	Kapshagay
4	71.36667	42.90000	Taraz
5	74.99500	46.84810	Balkhash
6	73.76139	43.59833	Shu

For the computational experiments for determination of reliability of calculation methods providing randomly three points of the lightning discharges located near the Kazakhstan cities of Astana, Aktau and Zaysan are chosen. For these points exact calculation of time moments of lightning detection on each of measuring stations is performed. Zero is taken as lightning discharge moment in each case. Results of these calculations are given in table 2.

Table 2. The test data of lightning discharges including distance from stations and time of radio signals detection at the stations.

Station No	Lightning coordinates					
	71°E, 51°N (Astana)		51°E., 44°N (Aktau)		85°E, 47°N (Zaysan)	
	Lightning timemoment, s	Station to lightning distance, km	Lightning timemoment, s	Station to lightning distance, km	Lightning timemoment, s	Station to lightning distance, km
1	0.003236008550	970.1310	0.006938195622	2080.0186	0.002525797123	757.2149
2	0.002872390603	861.1210	0.007214464807	2162.8421	0.001859757582	557.5413
3	0.003049609604	914.2500	0.006932949958	2078.4460	0.002368726661	710.1264
4	0.003005907395	901.1484	0.005485512347	1644.5152	0.003881926243	1163.7722
5	0.001821388535	546.0385	0.006309817053	1891.6355	0.002533458619	759.5118
6	0.002831421721	848.8389	0.006076166255	1821.5887	0.003188416712	955.8633

The problem of computational experiments in this work and in Lozbin et.al. [8] consists in determination of quality of the offered methods for the single lightning discharge parameters calculation depending on measurements errors and distance to the measuring stations. For this purpose, in the exact time moments of lightning detection showed in table 2, the corrections using their rounding up to 9, 8, 7 and 6 decimal are performed, that corresponds to 1ns, 10ns, 100ns and 1ms lightning detection accuracy. The results of calculations performed by method of triangles Lozbin et.al. [8] and by a functional method are given in tables 3 and 4.

The comparison of methods is performed on values of expression

$$f(\alpha^*, \tilde{\varphi}^*, \tilde{\theta}^*) = \sqrt{F(\alpha^*, \tilde{\varphi}^*, \tilde{\theta}^*) / N_s} \,, \tag{41}$$

where,

$$F(\alpha, \tilde{\varphi}, \tilde{\theta}) = \sum_{i=1}^{N_s} \left[\arccos\left(\tilde{x}_i \cos\tilde{\varphi} + \tilde{y}_i \sin\tilde{\varphi} \right) \cos\tilde{\theta} + \tilde{z}_i \sin\tilde{\theta} - (\alpha_i - \alpha) \right]^2 \,, \tag{42}$$

is constructed at the equations of system (25).

Functional (42) is more intrinsic for set of Eq. (25) solution. But, the difference of $G(\alpha, \tilde{\varphi}, \tilde{\theta})$ functional, is that its derivatives are discontinuous, and this is considerably complicates search of its global minimum.

Table 3. The results of comparison of lightning coordinates calculations using method of triangles and functional method for 3 test lightning discharges.

Lightning time moment measurement accuracy, ns	Deviation from the exact position, km					
	Astana		Aktau		Zaysan	
	Method of triangles	Functional method	Method of triangles	Functional method	Method of triangles	Functional method
1	0.0017	0.0019	0.0130	0.0128	0.0020	0.0011
10	0.0019	0.0044	0.0137	0.0537	0.0076	0.0091
100	0.0733	0.0891	1.8692	0.5365	0.2685	0.2500
1000	58.5256	0.4960	25.4276	21.1487	2.0128	0.7278

Table 4. The results of comparison of lightning time moments calculations using method of triangles and functional method for 3 test lightning discharges.

Lightning time moment measurement accuracy, ns	Deviation from the exact time moment, μs					
	Astana		Aktau		Zaysan	
	Method of triangles	Functional method	Method of triangles	Functional method	Method of triangles	Functional method
1	-0.005	-0.0058	-0.043	-0.042	-0.006	0.004
10	-0.003	0.001470	-0.040	0.179	-0.022	-0.027
100	0.229	0.279269	-6.195	-1.784	0.879	0.817
1000	348.181	-1.656893	84.374	70.214	6.787	1.884

As we can see in table, the results of both methods coincide with practical accuracy on errors of time moments measurements up to 100 ns. At the error of measurements increasing the functional method gives more reliable results, than a method of triangles. This statement can be explained, in particular, with the fact that final values of lightning parameters by a method of triangles are obtained by means of averaging of calculation results for a considerable set of separate triangles in which dispersion rise sharply with increasing an error of measurements of time moments of lightning detection.

8. Conclusion

Conclusion follows from a series of computational experiments that the functional method is more stable to the error of time moments detection comparison with method of triangles. However, the method of coordinate descent used at minimization of functional $G(\alpha,\tilde{\varphi},\tilde{\theta})$ is rather slow.

Minimization of functional can be optimized, however, it is hardly possible to reach operation speed of a direct method of triangles. Therefore, it is possible to recommend the following algorithm. At the first stage of calculation of lightning parameters to apply a method of triangles, then, to specify result by method of minimization of functional $G(\alpha,\tilde{\varphi},\tilde{\theta})$.

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References

[1] Dowden R L, Brundell J B and Rodger C J 2002 VLF lightning location by time of group arrival (TOGA) at multiple sites *Journal of Atmospheric and Solar-Terrestrial Physics* **64** 817–830

[2] Saeed Vahabi-Mashak, Zulkurnain Abdul-Malek, Kamyar Mehranzamir, Hadi Nabipour-Afrouzi, Behnam Salimi and Chin-Leong Wooi 2015 Modeling of Time of Arrival Method for Lightning Locating Systems *Advances in Meteorolgy* Vol **2015** Article ID 870290 p 12

[3] Hu Z X, Zhao W G and Zhu H P 2010 A source location algorithm of lightning detection networks in China *Ann. Geophys.* **28** 1981–1991

IOP Conf. Series: Journal of Physics: Conf. Series **1352** (2019) 012049 doi:10.1088/1742-6596/1352/1/012049

[4] Hans Dieter Betz, Ulrich Schumann and Pierre Laroche 2008 *Lightning: Principles, Instruments and Applications: Review of Modern Lightning Research* (Springer Science & Business Media) p 641
[5] Vernon Cooray 2014 *An Introduction to Lightning* (Springer) p 386
[6] Rodger C J, Brundell J B and Dowden R L 2005 Location accuracy of VLF World-Wide Lightning Location(WWLL) network: Post-algorithm upgrade *Annales Geophysicae* **23** 277–290
[7] Rakov V A 2016 *Fundamentals of Lightning* (Cambridge University Press) p 248
[8] Lozbin A Yu, Shpadi Yu R and Inchin A S 2016 Triangles Technique for Time and Location Finding of the Lightning Discharge in Spherical Model of the Earth *Journal of Geoscience and Environment Protection* **4** 125–135