Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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Abstract. Method of V.M. Alekseev was used to obtain the clarifications of lower estimates for the lower singular exponent and of upper estimates for the upper singular exponent of a linear system of differential equations. In contrast to the classical results of the freezing method, which assume that the derivatives of the elements of the matrix of the coefficients of the system are small, the new estimates admit that these derivatives are unbounded.

1. Introduction

Consider a linear system of differential equations.

$$x' = A(t)x, \quad x \in R^n, \quad A(t) \in R^+, \quad \| A(t) \| \le M. \tag{1}$$

It is well known that it is impossible to describe the behavior of solutions of a non-autonomous system (1) only by the eigenvalues of the matrix $A(t)$. It is necessary to introduce additional conditions on the matrix $A(t)$. In the freezing method, system (1) is written in the form $x' = A(t_0)x + \left(A(t) - A(t_0)\right)x$, the addendum $\left(A(t) - A(t_0)\right)x$ is considered as a perturbation of the linear autonomous system $x' = A(t_0)x$. In order to apply the freezing method, it is now necessary to accurately estimate the growth of the matrix $\exp(A(t_0)t)$, which can be considered as the fundamental matrix of the "frozen" system $x' = A(t_0)x$. We will assume that the matrix $A(t)$ of system (1) satisfies for any $t, s \in R^+$ to inequality

$$\| A(t) - A(s) \| \le \delta \, | t - s |^{\alpha} \tag{2}$$

with some positive constants α and δ.

The case when $\alpha > 1$ presents no interest, since from inequality (2) it follows, that $A'(t) = \mathrm{O}$ and the system (1) must be autonomous. The case when $\alpha = 1$, which under the existence of $A'(t)$ is equivalent to [1, 136] inequality $\| A'(t) \| < \delta$, was studied in the classical works on the method of freezing. We introduce the notation

$$\rho = \sup_{t \in R^+} \max_i \operatorname{Re} \lambda_i(t), \sigma = \inf_{t \in R^+} \min_i \operatorname{Re} \lambda_i(t). \tag{3}$$

Here $\lambda_i(t), i=1,...,n$ are the matrix $A(t)$ eigenvalues. The first estimates from above of the highest exponent Λ of system (1) in terms of ρ and δ in the case $\alpha=1$ were obtained in [2, 3].

These estimates were refined in [4;5]. The attainability of estimating the method of freezing by parameter δ in the case $\alpha=1$ was proved in [6] for the second-order system (1) and in [7] for the system (1) of arbitrary order. In [8] for the case when $\alpha=1$, various refinements were obtained for estimate of the upper exponent Λ and the lower exponent λ from below. For the case when $\alpha\in(0;1)$ in [9], the upper estimate of the senior exponent Λ and the lower estimate of the lower exponent λ are obtained.

$$\Lambda \le \rho + C\delta^{1/(n+\alpha)}, \quad \lambda \ge \sigma - C\delta^{1/(n+\alpha)}. \tag{4}$$

Here $C = n+\alpha-1+D\exp\left((n+\alpha-1)\delta^{1/(n+\alpha)}\right)$, constant D from estimate [10, 22]

$$\| \exp(A(u)t) \| \le D(1+t)^{n-1}\exp(\rho t), \quad u,t\in R^+. \tag{5}$$

Note that estimate (5) follows from a more accurate estimate. [1, 132]

$$\| \exp(A(u)t) \| \le \exp(\rho t)\sum_{k=0}^{n-1}\frac{(2Mt)^k}{k!}, \quad u,t\in R^+. \tag{6}$$

It is true that,

$$\| \exp(A(u)t) \| \le \exp(\max_i \operatorname{Re}\lambda_i(u)t)\sum_{k=0}^{n-1}\frac{(2t\| A(u)\|)^k}{k!} \le \exp(\rho t)\sum_{k=0}^{n-1}\frac{(2Mt)^k}{k!}$$

$$=\exp(\rho t)\left(1+\sum_{k=1}^{n-1}\frac{(2Mt)^k}{k!}\right) \le (1+t^{n-1})D\exp(\rho t) \le (1+t)^{n-1}D\exp(\rho t),\ u,t\in R^+.$$

V.M. Alekseev received the upper estimate of the upper special exponent Ω_0 of system (1)

$$\Omega_0 \le \rho + 2M\left(\frac{\delta n(n+1)}{8M^2}\right)^{\frac{1}{n+1}} \tag{7}$$

assuming that $\| A(t)-A(s) \| \le \delta |t-s|$. Using a more accurate estimate (6), with the method of V.M. Alekseev for the case

$$\| A(t)-A(s) \| \le \delta |t-s|^{\alpha}, \quad \alpha\in(0;1],$$

we clarify the constant C in the estimates (4). We will get the lower estimate for the lower special exponent ω_0 and the upper estimate of the upper special exponent Ω_0.

Since inequality holds true [10, 149]

$$-M \le \omega_0 \le \omega \le \lambda \le \Lambda \le \Omega \le \Omega_0 \le M,$$

we get similar estimates for the central exponents ω and Ω, as well as for the junior and senior Lyapunov's exponents λ and Λ.

2. The main result and examples

Theorem. Let the matrix of coefficients $A(t)$ of system (1) satisfy inequality (2) with constants $\alpha \in (0;1]$ and $\delta > 0$, the parameters ρ and ω are given by relations (3), then for the upper special exponent Ω_0 and for the lower special exponent ω_0 the estimates are valid

$$\Omega_0 \le \rho + 2M\left((2M)^{-1-\alpha} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!} \right)^{\frac{1}{n+\alpha}} \delta^{\frac{1}{n+\alpha}},$$

$$\omega_0 \ge \sigma - 2M\left((2M)^{-1-\alpha} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!} \right)^{\frac{1}{n+\alpha}} \delta^{\frac{1}{n+\alpha}}. \tag{8}$$

In the last inequalities $\Gamma(\cdot)$ is the gamma function.

Remark 1. Both estimates (8) are of interest if the constant δ is sufficiently small. If $\delta(2M)^{-1-\alpha} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!} \ge 1$, then the estimates (8) are worse than the obvious estimates $\Omega_0 \le M \le \rho + 2M$ and $\omega_0 \ge -M \ge \sigma - 2M$. We took into account that $-M \le \rho \le M$ and $-M \le \sigma \le M$.

Proof. We use the following theorem by V.M. Alekseev [1, 134].
Let

$$\| \exp(A(u)t) \| \le \eta(t) \le D\exp(\gamma t), \quad u, t \in R^{+}, \tag{9}$$

$$\| A(t) - A(s) \| \le \varphi(t-s), \tag{10}$$

and γ is such that

$$\int_0^{+\infty} e^{-\gamma\tau} \eta(\tau)\varphi(\tau) d\tau < 1. \tag{11}$$

We denote the lower bound of such γ by γ_0, then the upper special exponent Ω_0 of system (1) satisfies the inequality. The central and senior exponents of system (1) even more satisfy this inequality $\Omega_0 \le \gamma_0$.

We apply the Alekseev theorem, using as the $\eta(t)$ function which is on the right-hand side of inequality (6). To satisfy condition (9) of the theorem, it suffices to take any $\gamma > \rho$. Let condition (10) be satisfied for the function $\varphi(\tau) = \delta \,|\, \tau \,|^{\alpha}$, $\alpha \in (0;1]$. To fulfill condition (11) of the theorem, we require that

$$\int_0^{+\infty} e^{-\gamma\tau} \eta(\tau)\varphi(\tau) d\tau = \int_0^{+\infty} e^{-(\lambda-\rho)\tau} \delta \sum_{k=0}^{n-1} \frac{(2M)^k \tau^{k+\alpha}}{k!} d\tau < 1.$$

Since $\int_0^{+\infty} e^{-(\gamma-\rho)\tau} \tau^{k+\alpha} d\tau = \frac{\Gamma(k+\alpha-1)}{(\gamma-\rho)^{k+\alpha+1}}$, we come to the inequality

$$\sum_{k=0}^{n-1} \delta \frac{(2M)^k}{k!} \cdot \frac{\Gamma(k+\alpha+1)}{(\gamma-\rho)^{k+\alpha+1}} < 1,$$

where $\Gamma(\cdot)$ is a gamma-function.

For $\gamma_0 = \inf \gamma$ the following equality should be preformed

$$\sum_{k=0}^{n-1} \delta \frac{(2M)^k}{k!} \cdot \frac{\Gamma(k+\alpha+1)}{(\gamma_0-\rho)^{k+\alpha+1}} = 1.$$

It is clear that the quantity $z_0 = \dfrac{1}{\gamma_0 - \rho}$ must satisfy the equation

$$\sum_{k=1}^{n} \delta \frac{(2M)^{k-1}}{(k-1)!} \Gamma(k+\alpha) z_0^{k+\alpha} = 1,$$

which after the change of $z = 2Mz_0$ looks like

$$\frac{\delta}{(2M)^{\alpha+1}} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha) z^{k+\alpha}}{(k-1)!} = 1. \tag{12}$$

We assume that the constant δ is so small that

$$\frac{\delta}{(2M)^{\alpha+1}} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!} < 1. \tag{13}$$

Otherwise, as noted in Remark 1, estimates (8) are worse than trivial estimates.

The left side of inequality (12) monotonously increases, when $z > 0$ and when $z = 1$ satisfies inequality (13), therefore the positive root ζ_0 of equation (12) is larger than 1. For this root we have

$$1 = \frac{\delta}{(2M)^{\alpha+1}} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)\zeta_0^{k+\alpha}}{(k-1)!} < \frac{\delta\zeta_0^{n+\alpha}}{(2M)^{\alpha+1}} \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!}.$$

Out of the last inequality we have

$$\zeta_0 = 2Mz_0 > \left((2M)^{-1-\alpha} \delta \sum_{k=1}^{n} \frac{\Gamma(k+\alpha)}{(k-1)!} \right)^{-\frac{1}{n+\alpha}}.$$

Since $\gamma_0 = \rho + \dfrac{1}{z_0}$, then by the Alekseev theorem we have the first of inequalities (8).

A lower bound for the lower special exponent is easily obtained by going to the adjoint system. Let us consider the system $y' = -A^T(t)y$ associated with the system $x' = A(t)x$. If the matrix $A(t)$ satisfies inequality (2), then the matrix $-A^T(t)$ of the adjoint system also satisfies the same inequality.

The eigenvalues $\lambda_i(t)$ of the matrix $A(t)$ are related to the eigenvalues $\mu_i(t)$ of the matrix $-A^T(t)$ by equality $\lambda_i(t) = -\mu_i(t),\ i = \overline{1,n}$, so $\rho^T = \sup\limits_{t\in R^+} \max\limits_i \operatorname{Re}\mu_i(t) = \sup\limits_{t\in R^+} \max\limits_i \left(-\operatorname{Re}\lambda_i(t)\right) = -\inf\limits_{t\in R^+} \min\limits_i \operatorname{Re}\lambda_i(t) = -\sigma$.

For the upper special exponent Ω_0^T of the conjugate system, we have the estimate from above.

$$\Omega_0^T \le \rho^T + 2M\left((2M)^{-1-\alpha}\sum_{k=1}^{n}\frac{\Gamma(k+\alpha)}{(k-1)!}\right)^{\frac{1}{n+\alpha}}\delta^{\frac{1}{n+\alpha}}.$$

Since $\Omega_0^T = -\omega_0$, we go to the second inequality (8). The theorem is proved.

Remark 2. System (1) with a matrix $A(t)$ satisfying inequality (2) with $\alpha \in (0;1]$, can be correct and wrong. We remind that the concept of a correct system of differential equations was introduced by A.M. Lyapunov.

System (1) is called correct if the sum of the Lyapunov indices of this system is equal to the lower integral average of the matrix of coefficients, i.e.

$$\sum_{i=1}^{n}\lambda_i = \varliminf_{t\to+\infty} t^{-1}\int_0^t SpA(u)du.$$

Solving the problem of stability in the first approximation is easier if the linear part of the system is correct. Note that correct systems include reducible systems (and, therefore, autonomous and periodic systems) and almost reducible systems according to B.F. Bylov.

Example 1. System

$$x' = \begin{pmatrix} \delta\sin\left(t^{1/3}\right) & 1 \\ 0 & 1 \end{pmatrix} x,$$

is correct, since the elements of the main diagonal of the triangular matrix of coefficients have strict integral averages (Lyapunov regularity criterion [1,141]).

The equality $\lim\limits_{t\to+\infty} t^{-1}\int_0^t \sin\left(\tau^{1/3}\right)d\tau = 0$ is easily proved by replacing the variable $\eta = \tau^{1/3}$ and further integration in parts.

The matrix $A(t)$ of coefficients of the system satisfies the inequality

$$\| A(t) - A(s) \| \le \delta\, | t - s |^{\alpha}, \quad t, s \in R^+$$

with $\alpha = 1/3$, since the function $a_{11}(t) = \delta\sin\left(t^{1/3}\right)$ satisfies the inequality. It is true that

$$\delta\left|\sin\left(t^{1/3}\right) - \sin\left(s^{1/3}\right)\right| \le 2\delta\left|\sin\left(0.5\left(t^{1/3} - s^{1/3}\right)\right)\right| \cdot 1 \le \delta\left|t^{1/3} - s^{1/3}\right| \le \delta\left|t - s\right|^{1/3}.$$

Example 2. System

$$x' = \begin{pmatrix} 0.75\,\delta\left(\sin\ln\sqrt[3]{t+1} + 1/3\cos\ln\sqrt[3]{t+1}\right) & 1 \\ 0 & 1 \end{pmatrix} x, \quad t \in R^+,$$

is wrong because the diagonal element $a_{11}(t)$ has no strict integral average. Indeed, the limit

$$\lim_{t\to+\infty} t^{-1}\int_0^t a_{11}(\tau)d\tau = 0.75\delta \lim_{t\to+\infty}\frac{t+1}{t}\sin\ln\sqrt[3]{t+1}$$

does not exist.

It remains to be noted that the element $a_{11}(t)$ satisfies the inequality $|a_{11}(t)-a_{11}(s)|\le\delta|t-s|^{1/3}$ for any $t,s\in R^+$. We have $|a_{11}(t)-a_{11}(s)|$

$$\le 0.75\delta\left(\left|\sin\ln\sqrt[3]{t+1}-\sin\ln\sqrt[3]{s+1}\right|+1/3\left|\cos\ln\sqrt[3]{t+1}-\cos\ln\sqrt[3]{s+1}\right|\right)$$

$$\le 0.75\delta\left(8/3\left|\sin 0.5\left(\ln\sqrt[3]{t+1}-\ln\sqrt[3]{s+1}\right)\right|\right)\le\delta\left|\ln\sqrt[3]{t+1}-\ln\sqrt[3]{s+1}\right|$$

$$\le\delta\left|(t+1)^{1/3}(s+1)^{-1/3}-1\right|\le\delta|t-s|^{1/3}|s+1|^{-1/3}\le\delta|t-s|^{1/3}.$$

We used the inequalities $\ln x\le x-1,\quad x>0$ and $\left|x^{1/3}-y^{1/3}\right|\le|x-y|^{1/3}$.

3. Conclusions

Note, that in the case, when $\alpha=1$, taking into account that $\Gamma(k+1)=k!$ and $\sum_{k=1}^{n}k=0.5n(n+1)$, the first of the estimates (8) of the theorem turns into an estimate

$$\Omega_0\le\rho+2M\left(\frac{\delta n(n+1)}{8M^2}\right)^{\frac{1}{n+1}},$$

received by V.M. Alekseev [1, 137].

Estimates (8) can be converted taking into account the properties of the gamma function $\Gamma(x+1)=x\Gamma(x)$ to the form

$$\Omega_0\le\rho+2M\left((2M)^{-1-\alpha}\sum_{k=1}^{n}\frac{(k+\alpha-1)(k+\alpha-2)\cdots(\alpha+1)\alpha\Gamma(\alpha)}{(k-1)!}\right)^{\frac{1}{n+\alpha}}\delta^{\frac{1}{n+\alpha}},$$

$$\omega_0\ge\sigma-2M\left((2M)^{-1-\alpha}\sum_{k=1}^{n}\frac{(k+\alpha-1)(k+\alpha-2)\cdots(\alpha+1)\alpha\Gamma(\alpha)}{(k-1)!}\right)^{\frac{1}{n+\alpha}}\delta^{\frac{1}{n+\alpha}}.$$

For $\alpha\in(0;1)$ and very small δ the estimates obtained, the order is not higher than $\delta^{1/(n+\alpha)}$ which is better than order $\delta^{1/(n+1)}$.

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IOP Conf. Series: Journal of Physics: Conf. Series **1352** (2019) 012030 doi:10.1088/1742-6596/1352/1/012030

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