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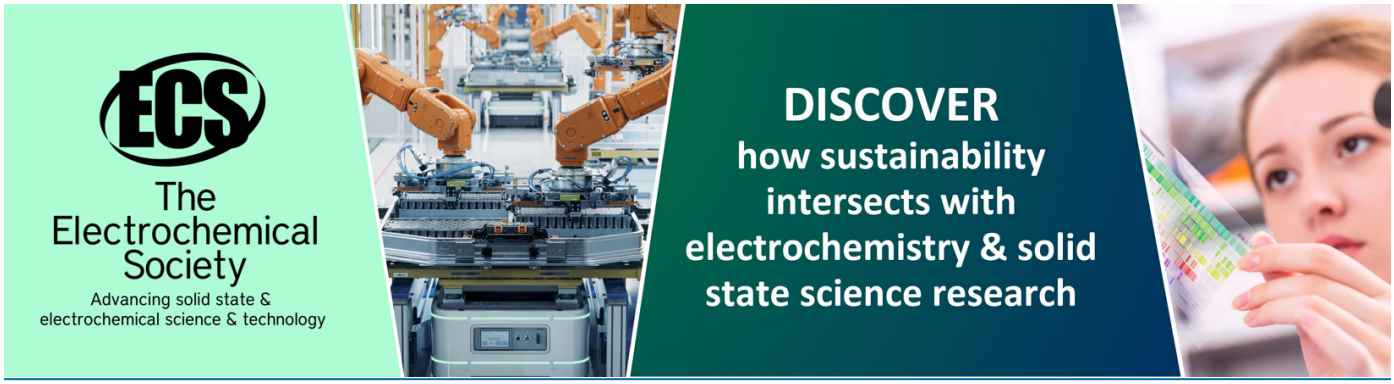

The effect of harvesting with threshold on the dynamics of prey predator model

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Abstract. This article deals with the dynamics of Lotka-Volterra prey predator population. The populations are considered as economically valuable stocks and then exploited. There is no harvesting when the densities of population are still low and the populations are harvested when the threshold value is achieved. The rate of harvesting is assumed to be an increased function and bounded. Phase portrait and linearization approach are used to analyze the behavior of the populations. There exists one equilibrium point for system without harvesting and it is a centre. The trajectories of the population oscillate around the stable equilibrium point. It is possible to find one, two, three, or none equilibrium points for model with harvesting. From the analysis we found that when the populations are not harvested then the equilibrium point becomes a centre. But when the populations are harvested with a smaller value, the equilibrium point becomes unstable spiral. When the value of harvesting rate is increased, the equilibrium point becomes either stable spiral or stable node. When the equilibrium points are unstable, the populations will meet a condition where their sizes are smaller than the threshold value and then the populations must stop being harvested.

1. Introduction

An ecosystem which is inhabited by more than one population, then ecologically there will be interaction between these populations. The most common form of interaction is predation, one population acts as a predator and the other acts as a prey. The dynamics of the growth rate of prey and predator populations based on the Lotka-Volterra system is one popular model in mathematical ecology. A prey predator population model in [1] has been studied and found that the prey and predator populations may live together for a long time when the frequency of their interaction is reduced.

The prey predator model has been widely studied in ecology and bioeconomics, see for example in [2, 3]. The population as a renewable natural resources and useful stock must be managed properly. A prey predator model with harvesting in [4] has been studied and found that it was possible to get bioeconomics equilibirium and optimal harvesting. In particular, authors in [5] studied the effect of over harvesting and drought on prey predator model and offered a strategy to prevent the population from extinction. Some prey predator models with various types of studies in harvesting have been investigated by many authors, see [6, 7, 8, 9].

In the ecology, the studies of fisheries management in the form of prey predator system with constant rate of harvesting were often considered [10]. The populations were harvested when the density of populations exceeded a certain value and no harvesting when the density of populations were still low.

In particular [11] analyzed a prey predator model with threshold continuous harvesting only for predator population. This kind of harvesting strategy is more realistic and beneficial in management of ecology.

In this article, a prey predator population model with continuous threshold harvesting is studied. The model includes one prey and one predator populations that refers to the Lotka-Volterra model. Under consideration that the populations are beneficial, then the populations are harvested with continuous threshold function. The harvesting function is still bounded above. The populations are harvested when their densities are greater than a certain value and no harvesting when the size of populations are still small. The existence and stability of interior equilibrium points of the model, with and without harvesting, are analyzed. This is important to know whether the populations will become extinct if the populations are harvested with threshold harvesting function. Phase portrait and linearization method are used to analyse the effect of harvesting.

2. The prey predator system without harvesting

Rosenzweig-MacArthur population model is given in the form of

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\phi(x)y \\ \frac{dy}{dt} &= \gamma\phi(x)y-cy\,. \end{aligned} \tag{1}$$

Variable $x = x(t)$ denotes the size of prey population and $y = y(t)$ denotes the size of predator population. Parameter r, K, γ, and c are assumed to be positive. Function $\phi(x)$ is well known as a predation response function for predator to the prey population. Such as functions have several types, for example response function Holling type I, type II, type III, and type IV. The response function Holling type I is given as $\phi(x) = \alpha x$. When the Holling type I is applied to the model (1), the model is reduced to the form of

$$\begin{aligned} \frac{dx}{dt} &= rx-bx^2-\alpha xy \\ \frac{dy}{dt} &= \beta xy-cy\,. \end{aligned} \tag{2}$$

In this article, we assume that the value of b is zero. Therefore the growth rate of prey population is assumed to grow exponentially when there is no interaction with predator population. Under this assumption, the model (2) is then reduced to the form of

$$\begin{aligned} \frac{dx}{dt} &= rx-\alpha xy \\ \frac{dy}{dt} &= \beta xy-cy\,. \end{aligned} \tag{3}$$

The nonnegative equilibrium points of model (3) are $(0,0)$ and $E_0 = \left(\frac{c}{\beta}, \frac{r}{\alpha}\right)$. It is easy to check using phase portrait analyses that the equilibrium point $(0,0)$ is unstable saddle point. The equilibrium point E_0 is stable and it is a centre. The trajectories around the equilibrium point E_0 form closed orbit. Furthermore, the equilibrium point E_0 is also globally stable in the first quadrant [12].

Example 1. Suppose that the parameter values for model (3) are given as $r = 0.5$, $\alpha = 0.0042$, $c = 0.52$, and $\beta = 0.004$ with appropriate units. It gives an interior equilibrium point $E_0 = \left(\frac{c}{\beta}, \frac{r}{\alpha}\right) = (130, 119.0476)$ and eigenvalues associates with this equilibrium point are $\pm 0.50990i$. This means that the equilibrium point is stable centre. Plot of some trajectories around the equilibrium point are given in figure 1.

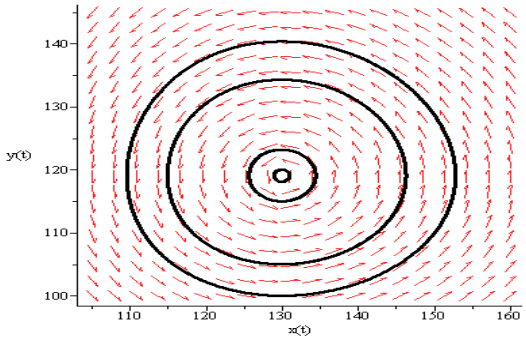


Figure 1. Some trajectories around the equilibrium point $E_0 = (130, 119.0476)$.

3. The prey predator system with threshold harvesting function

The common harvesting functions used in population dynamics are harvesting at constant rate and harvesting with constant effort. We consider a continuous threshold harvesting in the dynamics of prey predator model as proposed in [13, 14], that is

$$H(N) = \begin{cases} 0, & \text{if } N < T \\ \dfrac{h(N-T)}{h+(N-T)}, & \text{if } N \geq T \end{cases} \tag{4}$$

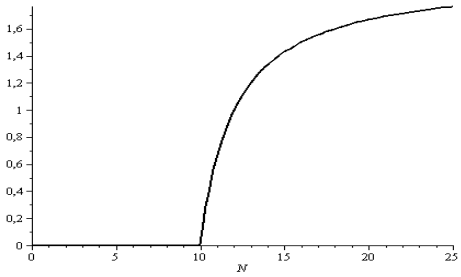


Figure 2. Graph of threshold harvesting function $H(N)$.

Harvesting function in figure 2 ecologically states that when the population size (N) is still low or less than a certain minimum value of the population which is allowed to be exploited, then the population is not harvested. But when the population size exceeds the minimum value, the population is harvested with the rate of harvesting following the increasing function and limited above when the population size is too large. Parameter h denotes the maximum value of harvesting rate and the parameter T denotes the threshold value of population that is allowed to be harvested.

The harvesting function (4) is more realistic than the harvesting function with constant rate and the constant effort of harvesting. In harvesting with constant rate, the size of harvested population per unit of time is constant and this is not relevant when the size of population is too low. In harvesting with constant effort, the rate of harvesting is proportional to the size of population and this is not relevant when the size of population is too large.

Under consideration that the dynamics of prey predator population in model (3) is useful for man, the two populations are then harvested following the threshold harvesting function (4). The considered prey predator model with threshold harvesting for both populations is as follows

$$\begin{aligned} \frac{dx}{dt} &= rx - \alpha xy - H(x) \\ \frac{dy}{dt} &= \beta xy - cy - H(y), \end{aligned} \tag{5}$$

where

$$H(x)=\begin{cases}0 & \text{if } x<T\\ \dfrac{h_1(x-T)}{h_1+(x-T)} & \text{if } x\geq T\end{cases} \text{ and } H(y)=\begin{cases}0 & \text{if } y<T\\ \dfrac{h_2(y-T)}{h_2+(y-T)} & \text{if } y\geq T.\end{cases}$$

Under conditions $x<T$ and $y<T$, the equilibrium points and their stability of model (5) are the same with the model (3). In the case of $x\geq T$ and $y\geq T$ and the value of parameter h is too small (h tends to zero), the equilibrium point $E_1=(x_*, y_*)$ for model (5) tends to the equilibrium point $E_0=\left(\dfrac{c}{\beta},\dfrac{r}{\alpha}\right)$. In this article, we focus on the case of $x\geq T$ and $y\geq T$.

The interior equilibrium points for model (5) are found by solving the system of equations $dx/dt=0$ and $dy/dt=0$ in x and y simultaneously. The considered equilibrium point must satisfy the conditions $x_*>T$ and $y_*>T$. From the simple isocline $dx/dt=0$, it follows

$$rx-\alpha xy-\frac{h_1(x-T)}{h_1+x-T}=0,$$

$$\frac{rxh_1+rx^2-rxT-\alpha xyh_1-\alpha x^2y+\alpha xyT-h_1T+h_1T}{h_1+x-T}=0,$$

$$rxh_1+rx^2-rxT-\alpha xyh_1-\alpha x^2y+\alpha xyT-h_1x+h_1T=0, \tag{6}$$

that gives

$$y=\frac{rxh_1+rx^2-rxT-h_1x+h_1T}{\alpha x(h_1+x-T)}.$$

From the simple isocline $dy/dt=0$, it follows

$$-cy+\beta xy-\frac{h_2(y-T)}{h_{21}+y-T}=0,$$

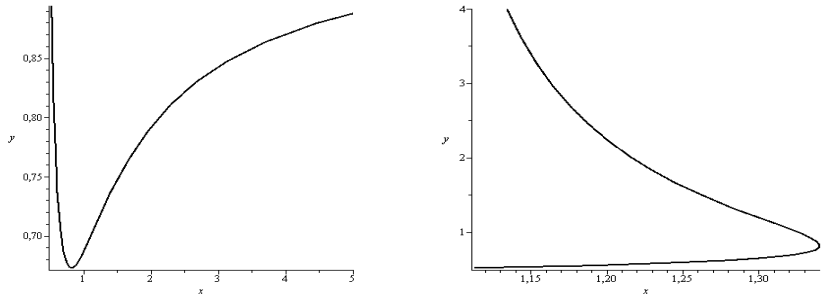
$$\frac{\beta h_2xy+\beta xy^2-\beta xyT-ch_2y-cy^2+cyT-h_2y+h_2T}{h_1+x-T}=0,$$

$$\beta h_2xy+\beta xy^2-\beta xyT-ch_2y-cy^2+cyT-h_2y+h_2T=0, \tag{7}$$

that gives

$$x=\frac{ch_2y+cy^2-cyT+h_2y-h_2T}{\beta y(h_2+y-T)}.$$

Further, the interior equilibrium points are found by solving the equations (6) and (7) which respects to x and y simultaneously.



(a) (b)

Figure 3. Simple isoclines (a) for $dx/dt=0$ and (b) for $dy/dt=0$.

From the simple isoclines, figure 3 (a) and (b), the equilibrium points are the intersection of the two isoclines. It is easy to see that it is possible to get one, two, three, or none equilibrium points and it must satisfy the conditions $x > T$ and $y > T$. The Jacobian matrix from the model (5) is written as

$$J = \begin{pmatrix} P_1 & -P_2 \\ P_3 & P_4 \end{pmatrix},$$

where $P_1 = r - \alpha y - \frac{h_1}{h_1 + x - T} + \frac{h_1(x-T)}{(h_1 + x - T)^2}$, $P_2 = \alpha x$, $P_3 = \beta y$, and $P_4 = \beta x - c - \frac{h_2}{h_2 + y - T} + \frac{h_2(y-T)}{(h_2 + y - T)^2}$. It is clear that P_2 and P_3 are always positive. The characteristic equation associates with the Jacobian matrix is given by

$$f(\lambda) = \det(\lambda I - J) = \det\begin{pmatrix} \lambda - P_1 & P_2 \\ -P_3 & \lambda - P_4 \end{pmatrix} = 0.$$

Further, we get $f(\lambda) = \lambda^2 - tr(J)\lambda + \det(J) = 0$, where $tr(J) = P_1 + P_4$ and $\det(J) = P_1P_4 + P_2P_3$. From the characteristic equation we get the eigenvalues $\lambda_{1,2} = \frac{tr(J) + \sqrt{\Delta}}{2}$, where $\Delta = (tr(J))^2 - 4\det(J)$. The stability criteria of the equilibrium point (x_*, y_*) for model (5) are as follow.

1. If $\Delta > 0$, $\det(J) > 0$, and $tr(J) < 0$, then both eigenvalues are real, negative, and different. The equilibrium point (x_*, y_*) is asymptotically stable, it is a node.
2. If $\Delta > 0$, $\det(J) < 0$, and $tr(J) > 0$, then both eigenvalues are real and opposite signs. The equilibrium point (x_*, y_*) is unstable, it is a saddle point.
3. If $\Delta < 0$ and $tr(J) > 0$, then both eigenvalues are complex number with positive real part. The equilibrium point (x_*, y_*) is unstable spiral point.
4. If $\Delta < 0$ and $\det(J) < 0$, then both eigenvalues are complex number with negative real part. The equilibrium point (x_*, y_*) is asymptotically stable, it is stable spiral point
5. If $\Delta < 0$ and $tr(J) = 0$, then both eigenvalues are complex number with zero real part. The equilibrium point (x_*, y_*) is neutrally stable, it is a centre.

Example 2. Suppose that the parameter values for model (5) are given as $r = 0.5$, $\alpha = 0.52$, $c = 0.52$, $\beta = 0.5$, and $T = 0.5$ with appropriate units. The values of h_1 and h_2 will be given in various values. The equilibrium point, eigenvalues, and stability associates with the values of h_1 and h_2 are given in table 1 below.

Table 1. Existence, eigenvalues, and stability of the equilibrium point.

No.	Values of h_1 and h_2	Equilibrium Point	Eigenvalues	Stability
1.	$h_1 = 0$, $h_2 = 0$	$(1.0400, 0.9615)$	$\pm 0.50990i$	neutrally stable, centre
2.	$h_1 = 0.02$, $h_2 = 0.02$	$(1.0812, 0.9272)$	$0.01769 \pm 0.51052i$	unstable, spiral
3.	$h_1 = 0.2$, $h_2 = 0.2$	$(1.3329, 0.7289)$	$0.00622 \pm 0.49659i$	unstable, spiral
4.	$h_1 = 0.35$, $h_2 = 0.35$	$(1.3018, 0.6016)$	$-0.18745 \pm 0.35207i$	asymptotically stable, spiral

5.	$h_1 = 0.4,\ h_2 = 0.4$	$(1.2159, 0.5557)$	$-0.30004 \pm 0.17111i$	asymptotically stable, spiral
6.	$h_1 = 0.4122,\ h_2 = 0.4122$	$(1.1864, 0.5441)$	$-0.33332 \pm 0.00661i$	asymptotically stable, spiral
7.	$h_1 = 0.41225,$ $h_2 = 0.41225$	$(1.1863, 0.5440)$	$-0.32449,\ -0.34243$	asymptotically stable, node
8.	$h_1 = 0.43,\ h_2 = 0.43$	$(1.1378, 0.5274)$	$-0.17177,\ -0.59942$	asymptotically stable, node
9.	$h_1 = 0.5,\ h_2 = 0.5$	$(0.9606, 0.4816)$	The equilibrium point does not satisfy the conditions $x > T$ and $y > T$	

From table 1 we know that when the values of $h_1 = 0$ and $h_2 = 0$, the equilibrium point $(x_*, y_*) = (1.0400, 0.9615)$ is the same with the equilibrium point $E_0 = \left(\frac{c}{\beta}, \frac{r}{\alpha}\right) = (1.0400, 0.9615)$ for model without harvesting. When the values of h_1 and h_2 are too small ($h_1 = 0.02,\ h_2 = 0.02$), the equilibrium point changes slightly and becomes unstable spiral. But when the values of h_1 and h_2 are increased ($h_1 = 0.35,\ h_2 = 0.35$), the equilibrium point also changes and becomes an asymptotically stable spiral point. There exists stability switches from neutrally stable to unstable spiral and again to asymptotically stable spiral point. There is a change in the real part of complex eigenvalues from positive becomes negative. Therefore, there exists a certain values of h_1 and h_2 so that the real part of complex eigenvalues becomes zero.

When the values of h_1 and h_2 are increased again ($h_1 = 0.4122,\ h_2 = 0.4122$), the equilibrium point remains asymptotically stable spiral point with real part of the eigenvalues is close to zero. But when the values of h_1 and h_2 are increased again ($h_1 = 0.41225,\ h_2 = 0.41225$), the equilibrium point becomes stable node, the eigenvalues are real with opposite the signs. Further, when the values of h_1 and h_2 are increased again ($h_1 = 0.5,\ h_2 = 0.5$), the equilibrium point does not exist anymore because it does not satisfy the conditions $x > T$ and $y > T$. The effect of changing the values of h_1 and h_2 and its stability through examining the real part of the eigenvalue is given in figure 4 below.

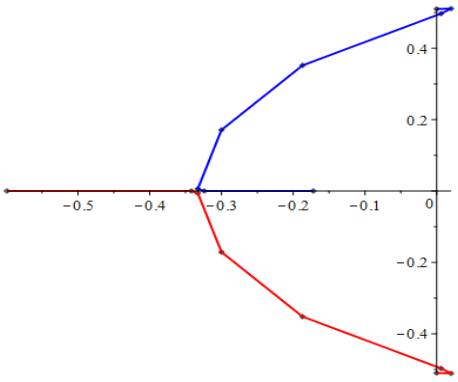


Figure 4. The effect of changing the values of h_1 and h_2 and stability of the equilibrium point.

Example 3. Suppose that the parameter values for model (5) are given as $r = 0.5$, $\alpha = 0.0052$, $c = 0.52$, $\beta = 0.005$, and $T = 10$ with appropriate units. Suppose also that the values of $h_1 = 100$ and $h_2 = 102$. The model has two interior equilibrium points, namely $E_1 = (115.2866, 10.6018)$ with eigenvalues are 0.17897 and -0.90326, and the other equilibrium point $E_2 = (219.3520, 36.8232)$ with eigenvalues are $0.07693 \pm 0.44029i$. The two equilibrium points are not stable. Plot of trajectories of populations around the equilibrium point E_1 and E_2 are given in figure 5. In this case, when the trajectories achieve the population size $x(t) < 10$ or $y(t) < 10$, then the two populations will stop being harvested.

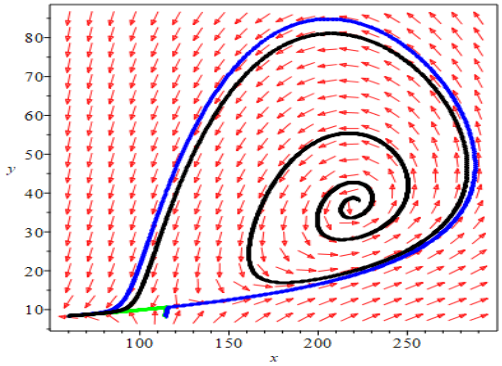


Figure 5. Behavior of the trajectories around the unstable equilibrium points.

4. Conclusions

There exists one interior equilibrium point for the prey predator model without harvesting. The equilibrium point is neutrally stable. The prey and predator populations oscillate harmonically around the equilibrium point and they can live together for a long time.

In the prey predator model with continuous threshold harvesting for both populations, it is possible to have one, two, three, or none interior equilibrium points. The existence of the equilibrium points depend on the values of parameter model, harvesting rate, and threshold value. The increasing values of h_1 and h_2 may change the stability of the equilibrium point, from neutrally stable to unstable spiral to neutrally stable again to asymptotically stable spiral to asymptotically stable node. The equilibrium point may not exist when the values of h_1 and h_2 are increased too high.

There exists a condition for the harvested prey predator model has two interior equilibrium points where the two equilibrium points are not stable, one is a saddle point and the other is an unstable spiral point. In this case the size of populations at a certain time will be less than the value of threshold harvesting. At that time the populations will stop being harvested.

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