Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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IOP Conf. Series: Journal of Physics: Conf. Series **1334** (2019) 012015 doi:10.1088/1742-6596/1334/1/012015

Elimination of the boundary condition singularity. A new approach to solving a system of nonlinear two-dimensional singularly perturbed differential equations

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Abstract. A new method of equation formation for boundary layer series has been suggested. It enables to eliminate boundary condition singularity, to construct a complete asymptotics for problem solving and to substantiate it for the system of singularly perturbed elliptic equations given that Dirichlet boundary condition for one function and Neyman boundary condition for the other have been provided in the case of degenerate equation multiply root. Solution expansion was carried out on fractional degrees of a small parameter. Asymptotics contains two different types of boundary series. First type coefficients of boundary series exponentially decrease, coefficients of the second type have a different structure, a special benchmark function emerges in order to evaluate them. The boundary layer is divided into a few zones with different ways of proceeding for each zone. The constructed boundary layer series describe solution for all zones of boundary layer. The problem is solved without using splicing method.

A new approach to equations construction for boundary layer function has been proposed because the existing methods didn't allow to construct even initial approximation of asymptotics.

1. Problem description and solving

Let us consider the following system of elliptic equations

$$\begin{aligned} \varepsilon^2 \Delta u &= F(u, v, x, \varepsilon)\big|_{x\in\Omega}, \\ \varepsilon \Delta v &= f(u, v, x, \varepsilon)\big|_{x\in\Omega} \end{aligned} \tag{1}$$

with boundary conditions different for each function

$$\begin{aligned} u(x, \varepsilon) &= u^0(x)\big|_{x\in\partial\Omega}, \\ \frac{\partial v}{\partial n}(x, \varepsilon) &= v^0(x)\Big|_{x\in\partial\Omega}. \end{aligned} \tag{2}$$

Hereafter x is two-dimensional variable $x = (x_1, x_2)$, ε is a small parameter, function F has nonlinear dependence on u.

Solution asymptotics of problem (1), (2) with rapid and slow equations has the following structure $u = \bar{u} + \Pi u + Pu, \qquad v = \bar{v} + \Pi v + Pv$. Here $\bar{u}$, $\bar{v}$ are regular parts of asymptotics, Πu, Πv are simple boundary layer series, Pu, Pv are multizone boundary layer series. The asymptotics is described by local coordinates (r, l) in vicinity of the boundary $\partial\Omega$. Two boundary layer variables emerge near $\partial\Omega$: $\rho = r/\sqrt{\varepsilon}$, $\zeta = r/\varepsilon$. Boundary layer series are expanded in powers of $\varepsilon^{1/4}$ $\left(Pu\,(\zeta, l, \varepsilon) = \sum\limits_{i=0}^{\infty} \varepsilon^{i/4} P_i u\,(\zeta, l)\right.$, etc.).

The order of constructing of boundary layer functions is of importance:

$$P_i u \to P_i v \to \Pi_i v \to \Pi_i u$$

($i = 0, 1, 2...$, $P_i u$ are coefficients of Pu series and similarly indicated coefficients of another series $Pv, \Pi v, \Pi u$), since all asymptotics terms are couldn't be found consistently by means of the order previously used.

Moreover boundary conditions for functions $\Pi_i v$ contain singularity:

$$\begin{aligned}
\frac{\partial \Pi_0 v}{\partial \rho}(0, l) &= v^0(l) - \frac{\partial \bar{v}_0}{\partial r}(0, l) - \frac{1}{\sqrt{\varepsilon}} \frac{\partial P_0 v}{\partial \zeta}(0, l), \\
\left.\frac{\partial \Pi_i v}{\partial \rho}(0, l)\right|_{i=1,2,\ldots} &= \begin{cases} -\dfrac{\partial \bar{v}_{i/2}}{\partial r}(0, l) - \dfrac{1}{\sqrt{\varepsilon}} \dfrac{\partial P_i v}{\partial \zeta}(0, l), & i \text{ is even number}, \\ -\dfrac{1}{\sqrt{\varepsilon}} \dfrac{\partial P_i v}{\partial \zeta}(0, l), & i \text{ is odd number}, \end{cases} \\
\Pi_i v(\infty, l)\big|_{i=0,1,2,\ldots} &= 0.
\end{aligned} \tag{3}$$

Some difficulties also arise in constructing equations for boundary layer functions. In case of constructing equations conventionally (for example, [8], [9]) boundary functions $P_i u$ have already got power dependence of variables x_1, x_2. This obstacle doesn't allow us to construct and substantiate the asymptotic of problem (1),(2) solution. Functions $P_i u, i = 0, 1, 2...$ could be obtained by means of the method analogous to the previously used one for researching one singularly perturbed equation with degenerate equation's multiple root ([1]–[7]) using lemma [3] which proves that $P_i u$ partial derivatives of any order also have the same multizoned structure for any number $i = 0, 1, 2....$ It should be noted that functions $P_i v, i = 0, 1, 2...$ must have a 3 zoned structure and both series Pu, Pv coefficients must be evaluated with benchmark estimating function $P_\kappa(\zeta) = \sqrt{\varepsilon} \frac{\exp\left(-\varepsilon^{1/4}\kappa\zeta\right)}{\left(1+\varepsilon^{1/4} - \exp\left(-\varepsilon^{1/4}\kappa\zeta\right)\right)^2}$, $\zeta \geq 0$ which is needed for constructing asymptotics. In this case the boundary condition (3) is singular. Indeed the equation for function $P_i v(\zeta, l)$ will look as

$$\frac{\partial^2 P_i v}{\partial \zeta^2} = s_i(\zeta, l)\Big|_{\zeta \geq 0,\ 0 \leq l \leq l_0}, \tag{4}$$

where $s_i(\zeta, l)$ will be expressed through boundary functions $P_j u, P_j v, P_i u$ $(j < i,)$ found previously. s_i will be estimated as $|s_i(\zeta, l)| \leq c({P_\kappa}^2(\zeta) + \sqrt{\varepsilon} P_\kappa(\zeta))$ or $|s_i(\zeta, l)| \leq c \cdot P_\kappa(\zeta)$ respectively, where $P_\kappa(\zeta)$ is a benchmark estimating function. Then

$$\left| \frac{1}{\sqrt{\varepsilon}} \frac{\partial P_i v}{\partial \zeta}(0, l) \right| = \frac{1}{\sqrt{\varepsilon}} \left| \int\limits_{\infty}^{0} s_i(t, l)\, dt \right| = O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

i.e. function $\Pi_i v$ boundary condition (3) remains singular in case of using previously known methods for constructing equations of boundary layer functions.

Applying the new method proposed in this paper to forming functions $P_i v(\zeta, l)$ $(i = 0, 1, 2...)$ equations enables us to get rid of boundary condition singularity and to construct full asymptotics of problem (1), (2) solution with its substantiation.

The equations for functions $P_i v(\zeta, l)$ are formed as follows

$$\frac{\partial^2 P_i v}{\partial \zeta^2} = \sqrt{\varepsilon} s_i(\zeta, l, \varepsilon)\bigg|_{\zeta \geq 0,\ 0 \leq l \leq l_0}, \tag{5}$$

where $s_i(\zeta, l)$ is expressed through previously found functions $P_j u, P_j v, P_i u,\ j < i$, and is estimated as $|s_i(\zeta, l)| \leq c \cdot P_\kappa(\zeta)$. The right hand side of equation (5) must have the power of small parameter ε two steps of expanding more then the left hand side (if Pv is expanded in powers of $\varepsilon^{1/4}$, then right hand side of equation for function $P_i v$ is multiplied by $\varepsilon^{1/4+1/4}$ more).

The method in question enables us to get the needed estimation $|P_i v(\zeta, l)| \leq c \cdot P_\kappa(\zeta)$, $\zeta \geq 0$, $0 \leq l \leq l_0$ for the function $P_i v(\zeta, l)$ itself and the estimation for its derivative as well

$$\left| \frac{1}{\sqrt{\varepsilon}} \frac{\partial P_i v}{\partial \zeta}(0, l) \right| \leq c. \tag{6}$$

In this case the boundary conditions (3) for functions $\Pi_i v$ become regular and the functions $\Pi_i v$ themselves are found in implicit form. Functions $\Pi_i u$ are expressed through these with respective numbers. Both functions $\Pi_i u, \Pi_i v, i = 0, 1, 2...$ have exponential assessment of the kind $|\Pi_i u(\rho, l)| \leq c \cdot \exp(-\kappa\rho)$, $|\Pi_i v(\rho, l)| \leq c \cdot \exp(-\kappa\rho)$, $\rho \geq 0$, $0 \leq l \leq l_0$. The complete problem (1), (2) solution asymptotics of any order has been constructed.

Let us remark that asymptotics behaviour and method of its construction for the system (1) depends on the kind of boundary conditions significantly. The system of one-dimensional equations analogous to the considered ones (1) with boundary conditions for both functions derivatives has been comsidered in paper [6]. Then all boundary layer functions are exponentially assessed in a standard way. The equations for boundary layer functions are formed conventionally through equating the summands with the same power of a small parameter in the eigen equation.

The constructed asymptotics of the problem (1), (2) solution has been substantiated with the help of the proved theorem.

Theorem

For sufficiently small ε problem (1), (2) has solution $u(x, \varepsilon)$, $v(x, \varepsilon)$, and functions

$$\begin{aligned} U_n(x, \varepsilon) &= \sum_{i=0}^{n} \varepsilon^{i/2} \bar{u}_i(x) + \sqrt{\varepsilon} \sum_{i=0}^{2n} \varepsilon^{i/4} \Pi_i u(\rho, l) + \sum_{i=0}^{2n} \varepsilon^{i/4} P_i u(\zeta, l), \\ V_n(x, \varepsilon) &= \sum_{i=0}^{n} \varepsilon^{i/2} \bar{v}_i(x) + \sqrt{\varepsilon} \sum_{i=0}^{2n} \varepsilon^{i/4} \Pi_i v(\rho, l) + \sqrt{\varepsilon} \sum_{i=0}^{2n} \varepsilon^{i/4} P_i v(\zeta, l). \end{aligned}$$

are found to be uniform in $\bar{\Omega}$ asymptotic approximation with the accuracy of the order $O(\varepsilon^{\frac{n}{2}+\frac{1}{4}})$, i. e. the equations

$$\begin{aligned} u(x, \varepsilon) &= U_n(x, \varepsilon) + O(\varepsilon^{\frac{n}{2}+\frac{1}{4}}), \\ v(x, \varepsilon) &= V_n(x, \varepsilon) + O(\varepsilon^{\frac{n}{2}+\frac{1}{4}}) \end{aligned}$$

hold true for any $n = 0, 1, 2, ..., x \in \bar{\Omega}$.

The asymptotics constructed here was justified by method of differential inequalities [10]. Upper and lower solutions contain function T of three-zoned type ($|T(\zeta)| \le cP_\kappa(\zeta), \zeta \ge 0$):

$$\begin{aligned}
\underline{U}(x,\varepsilon) &= U_n(x,\varepsilon) - \varepsilon^{n/2-1/4}\widetilde{\alpha}(x),\\
\underline{V}(x,\varepsilon) &= V_n(x,\varepsilon) - \varepsilon^{n/2-1/4}\widetilde{\beta}(x) - \varepsilon^{n/2+1/4}z(r,\varepsilon) + \varepsilon^{n/2+1/4}T(\zeta),\\
\bar{U}(x,\varepsilon) &= U_n(x,\varepsilon) + \varepsilon^{n/2-1/4}\widetilde{\alpha}(x),\\
\bar{V}(x,\varepsilon) &= V_n(x,\varepsilon) + \varepsilon^{n/2-1/4}\widetilde{\beta}(x) + \varepsilon^{n/2+1/4}z(r,\varepsilon) - \varepsilon^{n/2+1/4}T(\zeta).
\end{aligned} \tag{7}$$

The suggested method of constructing $P_i v$ equations and order of finding asymptotics coefficients allowed to get all summands of upper and low solutions (7), to satisfy all demands for upper and low solutions (demands for operator acting upper and lower solutions, for boundary condition of upper and low solutions, etc.) and to prove the Theorem on the existence and asymptotic expansion of solutions.

2. Conclusion

Finally, a new approach enabling to eliminate singularities in boundary conditions for boundary layer functions has been proposed here. This way also permits to construct full asymptotics of original problem (1), (2) solution by using new method of forming $P_i v$ equations without obstacles which appear in conventional approach and previously used ones. The method in question allows to substantiate the constructed asymptotics and to prove solution's existence for the system of singularly perturbed two-dimensional equations (1).

Acknowledgments
The research was supported by the Russian Foundation for Basic Research (project no. 18-31-00402).

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