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Asymptotical rotating consensus algorithm with processing delay

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Abstract. In this paper, we studied some consensus algorithms for the collective rotating motions of a team of agents, which has been widely studied in different disciplines ranging from physics, networks and engineering. Both discrete and continues consensus algorithm with processing delays are investigated. There are three motion patterns determined by the information exchange topology of systems and rotation angle of rotation matrices. The asymptotic consensus appears when 0 is an simple eigenvalue of Laplacian matrix and the rotation angle is less than the critical value, and the rotating consensus achieves when the rotation angle is equal to the critical value. At this point, all agents move on circular orbits and the relative radii of orbits are equal to the relative magnitudes of the components of a right eigenvector associated with 0 eigenvalue of the non-symmetric Laplacian matrix. Finally, all agents move along logarithmic spiral curves with a fixed center when the rotation angle is larger than the critical value.

1. Introduction

As we know, a multi-agent systems consists of a number of agents who communicate with each other via some pairwise links and aims to accomplish various control objectives by local interactions of designated agents. The consensus problems derive from all agents eventually reach an agreement of interest generally determined by their initial stats, first appear in distributed computation and automata theory in computer science [1]. It is important to understand the way these subsystems manage to accomplish a collective behaviour, as such phenomena are observed in nature. These collective behaviours such as flocking, herding, and schooling have been observed in many self-organized systems including fish swimming in schools, birds flying in flocks for the purpose of enhancing the foraging success, and the flight guidance in honeybee swarms. See, for example, Vicsek, Czirok, Ben-Jacob, Cohen, and Shochet[2]; Vicsek [3]; Strogatz [4]; Couzin, Krause, Franks and Levin(2005)[5].

Many models have been introduced to appraise the emergence of consensus. Representative examples can be found in [6],[7] and [8]. The standard models for consensus dynamics in social, biological, and physical sciences assume that the dependence of a_{ij} decreases as a function of $|x_i - x_j|$, where x_i may account for opinion, position, velocity, or other attributes of agent i. For example, Motsch [9] considered the form $a_{ij} = \phi(|x_i - x_j|)$, where $\phi(\cdot)$ is a compactly supported influence function which is increasing over its support.

The starting point for our discussion is a continuous framework with delay effects, which embeds both processing delay and transmission delay describing consensus dynamics. Mathematically, we consider the discrete evolution of N agents, x_i denotes the opinion of ith agent, and each agent adjusts its opinion according to the opinion of its neighbors:

$$\frac{d}{dt}x_i(t) = \varepsilon \sum_{j\neq i} a_{ij}C(x_j(t-\tau_T-\tau_P) - x_i(t-\tau_P)), i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in \mathbb{R}^3$, $a_{ij} \geq 0$ are constants for all i, j; C denotes the 3×3 rotating matrices. τ_P is *processing delay* (i.e., the time it takes agents to process the packet data), and τ_T is *transmission delay* (i.e., the amount of time required to push the information from one agent to another). In general, it costs more time for an agent to process information than to transmit it. That is, $\tau_P > \tau_T$. To normalize the processing delay, set $t = \tau_P s$, $y_i(s) = x_i(\tau_P s)$, we have

$$\begin{aligned}\frac{d}{ds}y_i(s) &= \frac{d}{dt}x_i(\tau_P s)\times\tau_P = \varepsilon\tau_P\sum_{j\neq i} a_{ij}C[x_j(\tau_P s-\tau_T-\tau_P) - x_i(\tau_P s-\tau_P)] \\ &= \varepsilon\tau_P\sum_{j\neq i} a_{ij}C[y_j(s-1-\frac{\tau_T}{\tau_P}) - y_i(s-1)].\end{aligned}$$

Then the corresponding discretization equation with unit step size is given by following

$$R_i(n+1) = R_i(n) + \varepsilon\tau_P\sum_{j\neq i} a_{ij}C[R_j(n-1-\frac{\tau_T}{\tau_P}) - R_i(n-1)), \tag{2}$$

where $i = 1, 2, \cdots, N, R_i(n) = (r_{1i}(n), r_{2i}(n), r_{3i}(n))$, $n = 1, 2, \cdots$.

In this work, we ignore the effects of transmission delay and consider the effects of processing delay. To this end, let $\tau_P = \tau$ and $\tau_T = 0$ in (2), then we obtain the following first-order difference system with processing delay:

$$R_i(n+1) = R_i(n) + \varepsilon\tau\sum_{j\neq i} a_{ij}C(R_j(n-1) - R_i(n-1)), i = 1, 2, \cdots, N, \tag{3}$$

equipping with the initial value $R_i(0) = R_i^0$ and $R_i(1) = R_i^1$.

Letting $A = (a_{i,j})_{N\times N}$ and $D = diag(d_1, d_2, \cdots, d_N)$, then $L = (l_{i,j})_{N\times N} = I - D^{-1}A$ is the Laplacian matrix of the system (3). Also, a weighted adjacency matrix $\mathcal{A}$ introduces to a directed graph $\mathcal{G}$. In order to establish the consensus criteria with the processing delay effects, we list the following three lemmas:

Lemma 1.1 *[10] Let L be the Laplacian matrix of the directed graph $\mathcal{G}$. Then $-L$ has a simple zero eigenvalue and all other eigenvalues have negative real parts if and only if $\mathcal{G}$ has a directed spanning tree. let μ_i is eigenvalue of L for $i = 1, 2, \cdots, N$, $\mu_1 = 0$, Moreover, 0 is an eigenvalue of L with an associated right eigenvector $\mathbf{1}_N$, left eigenvector $\mathbf{p}$(nonnegative vector) satisfying $L\mathbf{1}_N = \mathbf{0}_{N\times 1}$, $\mathbf{p}^T L = \mathbf{0}_{1\times N}$ and $\mathbf{p}^T\mathbf{1}_N = 1$.*

Lemma 1.2 *[11] Given a rotation matrix $C \in \mathbb{R}^{3\times 3}$. Let $\vec{a} = [a_1, a_2, a_3]^T$ be a unit vector in the direction of rotation and let $\theta \in (0, 2\pi)$ be the rotation angle. Then eigenvalues of C are $c_1 = 1, c_2 = \mathrm{e}^{\mathbf{j}\theta}, c_3 = \mathrm{e}^{-\mathbf{j}\theta}$. If a_2, a_3 not all zero, then we may choose the right eigenvectors of C to be $\varrho_1 = \vec{a}, \varrho_2 = [a_2^2 + a_3^2, -a_1a_2 + a_3\mathbf{j}, -a_1a_3 - a_2\mathbf{j}]^T, \varrho_3 = \bar{\varrho}_2$, left eigenvectors is $\rho_1 = \varrho_1, \rho_2 = \bar{\varrho}_2/|\varrho_2|^2, \rho_3 = \bar{\varrho}_3/|\varrho_3|^2$, where $\mathbf{j} = \sqrt{-1}$ is the imaginary unit , $\bar{\cdot}$ is the conjugate of a complex number. Moreover, $\varrho_l^T\rho_l = 1$, $l = 1, 2, 3$.*

Lemma 1.3 *[12],[13] Suppose that $U \in \mathbb{R}^{p\times p}, V \in \mathbb{R}^{q\times q}$, U has the eigenvalues β_i with associated eigenvectors $f_i \in \mathbb{C}^p, i = 1, \cdots, p$,and V has the eigenvalues α_j with associated eigenvectors $g_i \in \mathbb{C}^q, i = 1, \cdots, q$, then the pq eigenvalues of $U \otimes V$ are $\beta_i\alpha_j$ with associated eigenvectors of $f_i \otimes g_i$,$i = 1, \cdots, p, j = 1, \cdots, q$.*

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2. Consensus in discrete system with processing delay

In this section, we will explore the consensus behaviours for a first-order difference system with the processing delay effects. Setting $X(n) = (R_1(n)^T, R_2(n)^T, \ldots, R_N(n)^T)^T$, then the system (3) can be written as

$$X(n+1) = X(n) - \varepsilon\tau(L \otimes C)X(n-1). \tag{4}$$

Let $\tilde{X}(n) = (X(n)^T, X(n-1)^T)^T$, then the system would be transmitted as follows:

$$\tilde{X}(n+1) = M\tilde{X}(n), \tag{5}$$

where M is a $6N \times 6N$ matrix which is given by

$$M = \begin{pmatrix} I_{3N} & -\varepsilon\tau L \otimes C \\ I_{3N} & \mathbf{0}_{3N} \end{pmatrix}.$$

On the other hand, reset $\hat{X}(n) = (R_2(n)^T - R_1(n)^T, R_3(n)^T - R_1(n)^T, \ldots, R_N(n)^T - R_1(n)^T)^T$, Then the system (3) can further be written as

$$\hat{X}(n+1) = \hat{X}(n) - \varepsilon\tau\tilde{L} \otimes C\hat{X}(n-1), \tag{6}$$

where

$$\tilde{L} = \begin{pmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{pmatrix}$$

Let $\bar{X}(n) = (\hat{X}(n)^T, \hat{X}(n-1)^T)^T$, then the system (6) can be written as follows

$$\bar{X}(n+1) = E\bar{X}(n), \tag{7}$$

where E is a $6(N-1) \times 6(N-1)$ matrix as

$$E = \begin{pmatrix} I_{3(N-1)} & -\varepsilon\tau\tilde{L} \otimes C \\ I_{3(N-1)} & \mathbf{0}_{3(N-1)} \end{pmatrix}.$$

At this stage, we require some key lemmas.

Lemma 2.1 *Let M be given in (5). Then 0 is an eigenvalue of L with algebraic multiplicity m if and only if 1 is an eigenvalues of M with algebraic multiplicity $3m$.*

Proof. Compute

$$\begin{aligned} \det(\sigma I_{6N} - M) &= \det \begin{pmatrix} (\sigma-1)I_{3N} & \varepsilon\tau L \otimes C \\ -I_{3N} & \sigma I_{3N} \end{pmatrix} \\ &= \prod_{i=1}^{N}\prod_{j=1}^{3} m_{ij}(\sigma) = 0 \end{aligned} \tag{8}$$

where

$$m_{ij}(\sigma) = \sigma^2 - \sigma + \varepsilon\tau\mu_i c_j.$$

$$\sigma_{ij,1} = \frac{1 + \sqrt{1 - 4\varepsilon\tau\mu_i c_j}}{2}, \sigma_{ij,2} = \frac{1 - \sqrt{1 - 4\varepsilon\tau\mu_i c_j}}{2}$$

for $i = 1, 2, \ldots, N, j = 1, 2, 3$. Therefore, 1 is an eigenvalues of M with algebraic multiplicity $3m$ if and only if L has a zero eigenvalue with algebraic multiplicity m.

Lemma 2.2 *The eigenvalues of the reduced Laplacian matrix $\widetilde{L}$ consist of the rest eigenvalues of Laplacian matrix L except a zero eigenvalue, Moreover, M has three more 1 and 0 eigenvalues than E , and the rest eigenvalues are the same.*

Proof. The first part of this lemma can be obtained from the proof of Lemma 1 in [14]. Now we prove the second part of this lemma. By the proof of Lemma 2.1, we get that

$$\det(\sigma I_{6N} - M) = \det\begin{pmatrix} (\sigma-1)I_{3N} & \varepsilon\tau L\otimes C \\ -I_{3N} & \sigma I_{3N} \end{pmatrix} = \prod_{i=1}^{N}\prod_{j=1}^{3}\sigma^2 - \sigma + \varepsilon\tau\mu_i c_j$$

$$\begin{aligned} \det(\sigma I_{6N-6} - E) &= \det\begin{pmatrix} (\sigma-1)I_{3(N-1)} & \varepsilon\tau \widetilde{L}\otimes C \\ -I_{3(N-1)} & \sigma I_{3(N-1)} \end{pmatrix} \\ &= \prod_{i=2}^{N}\prod_{j=1}^{3}\sigma^2 - \sigma + \varepsilon\tau\mu_i c_j \end{aligned} \tag{9}$$

This implies that M has three more eigenvalues 1 and 0 than E, and the algebraic multiplicity of the other eigenvalues is the same.

It is evident from the previous two lemmas that the system (5) achieves consensus asymptotically if and only if the system (7) is asymptotically stable.

Lemma 2.3 *If 0 is a simple eigenvalue of the matrix L, then zero is an eigenvalue of the matrix $L\bigotimes C$ with algebraic multiplicity 3, and 1 is an eigenvalue of the matrix M with algebraic multiplicity 3. Meanwhile, the right eigenvector of M associated with eigenvalue 1 is given by $\begin{pmatrix} \mathbf{1}_N^T\bigotimes\varrho_l^T & \mathbf{1}_N^T\bigotimes\varrho_l^T \end{pmatrix}^T$, and the left eigenvector given by $\begin{pmatrix} \mathbf{p}^T\bigotimes\rho_l & \mathbf{0}_N^T\bigotimes\rho_l \end{pmatrix}$, where $l = 1, 2, 3$.*

Proof By Lemma 1.3 and Lemma 2.1, it is clear that if L has a simple zero eigenvalue, then $L\bigotimes C$ has a zero eigenvalue with algebraic multiplicity 3 and the matrix M has an eigenvalue 1 with algebraic multiplicity 3.

Next, we calculate the eigenvector of the eigenvalue 1. we assume $w = \begin{pmatrix} w_a^T, & w_b^T \end{pmatrix}^T$ is the right eigenvector of M, then

$$Mw = \begin{pmatrix} I_{3N} & -\varepsilon\tau L\bigotimes C \\ I_{3N} & \mathbf{0}_{3N\times 3N} \end{pmatrix}\begin{pmatrix} w_a \\ w_b \end{pmatrix} = \begin{pmatrix} w_a \\ w_b \end{pmatrix}.$$

Thus, we have

$$\begin{cases} I_{3n}w_a - \varepsilon\tau L\bigotimes C w_b = w_a, \\ I_{3n}w_a = w_b. \end{cases}$$

So w_b is the right eigenvectors of $L\bigotimes C$ associated with the zero eigenvalue, and the right eigenvectors of M associated with the eigenvalue 1 is given by

$$\begin{pmatrix} \mathbf{1}_N^T\bigotimes\varrho_l^T, & \mathbf{1}_N^T\bigotimes\varrho_l^T \end{pmatrix}^T.$$

The left eigenvectors can found similarly.

Next, we require a criterion about the distribution of roots of the following algebraic equation with complex coefficients

$$\sigma^2 + c_1\sigma + c_2 = 0, \tag{10}$$

where $c_k = a_k + \mathbf{j}b_k$, a_k, b_k are real numbers for $k = 1, 2$ and $\mathbf{j} = \sqrt{-1}$.

Lemma 2.4 *([15]) All the roots of (10) have negative real parts if and only if* $a_1 > 0$ *and* $a_1^2 a_2 + b_2(a_1 b_1 - b_2) > 0$.

Theorem 2.1 *System (5) achieves consensus if and only if the matrix M has exactly an eigenvalue 1 with multiplicity 3 and all the other eigenvalues are stay in the unit disk. In addition, if the consensus is reached, we have*

$$\lim_{n\to\infty} \|R_i(n) - R_\infty\| = 0. \quad i = 1, 2, \cdots, N.$$

where $R_\infty = [\mathbf{p}^T r(0), \mathbf{p}^T s(0), \mathbf{p}^T t(0)]$, $\mathbf{p} = (p_1, p_2, \cdots, p_n)^T$ satisfying $\mathbf{p}^T 1_N = 1$ is the unique nonnegative left eigenvector of L associated with zero eigenvalue, $r(0) = (r_{11}(0), r_{12}(0), \cdots, r_{1N}(0))^T$, $s(0) = (r_{21}(0), r_{22}(0), \cdots, r_{2N}(0))^T, t(0) = (r_{31}(0), r_{32}(0), \cdots, r_{3N}(0))^T$.

Proof (Necessity) Noting that 1 is the eigenvalue of matrix M with algebraic multiplicity 3, by lemma 2.3, we see that the corresponding right eigenvectors associated with the eigenvalue 1 are $(\mathbf{1}_{2N} \otimes \varrho_1)$, $(\mathbf{1}_{2N} \otimes \varrho_2)$ and $(\mathbf{1}_{2N} \otimes \varrho_3)$, the corresponding left eigenvectors associated with the eigenvalue 1 are $(\mathbf{p}^T \otimes \rho_1, \mathbf{0}_N \otimes \rho_1)$, $(\mathbf{p}^T \otimes \rho_2, \mathbf{0}_N \otimes \rho_2)$ and $(\mathbf{p}^T \otimes \rho_3, \mathbf{0}_N \otimes \rho_3)$. They are obviously linear independent. So the geometric multiplicity of the eigenvalue 1 of matrix M is 3 too. There exists a nonsingular matrix $P \in \mathbb{R}^{6N\times 6N}$, such that

$$P^{-1}MP = \begin{pmatrix} 1 & 0 & 0 & 0_{6N-3}^T \\ 0 & 1 & 0 & 0_{6N-3}^T \\ 0 & 0 & 1 & 0_{6N-3}^T \\ 0_{6N-3} & 0_{6N-3} & 0_{6N-3} & \tilde{J} \end{pmatrix},$$

where $\tilde{J}$ is the diagonal matrix composed of Jordan blocks associated with the other eigenvalues of matrix M. Thus

$$M = \begin{pmatrix} \zeta_1, \zeta_2, \dots, \zeta_{6N} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mathbf{0}_{6N-3}^T \\ 0 & 1 & 0 & \mathbf{0}_{6N-3}^T \\ 0 & 0 & 1 & \mathbf{0}_{6N-3}^T \\ \mathbf{0}_{6N-3} & \mathbf{0}_{6N-3} & \mathbf{0}_{6N-3} & \tilde{J} \end{pmatrix} \begin{pmatrix} \eta_1^T \\ \eta_2^T \\ \vdots \\ \eta_{6N}^T \end{pmatrix}$$

where ζ_j and $\eta_j (j = 1, 2, \dots, 6N)$ are columns and rows of P and P^{-1}, respectively. Since the eigenvalues of matrix M satisfy $|\sigma| < 1$ except for the eigenvalue $\sigma_{1,2,3} = 1$. Thus $\lim\limits_{n\to+\infty} \tilde{J}^n = \mathbf{0}_{(6N-3)\times(6N-3)}$.

Noting that

$$\begin{aligned}
\lim_{n\to+\infty} \tilde{X}(n) &= \lim_{n\to+\infty} M^n \tilde{X}(0) = \begin{pmatrix} \mathbf{1}_{2N} \otimes \varrho_1, \mathbf{1}_{2N} \otimes \varrho_2, \mathbf{1}_{2N} \otimes \varrho_3, \dots \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 & \mathbf{0}_{6N-3}^T \\ 0 & 1 & 0 & \mathbf{0}_{6N-3}^T \\ 0 & 0 & 1 & \mathbf{0}_{6N-3}^T \\ \mathbf{0}_{6N-3} & \mathbf{0}_{6N-3} & \mathbf{0}_{6N-3} & \lim_{n\to+\infty} \tilde{J}^n \end{pmatrix} \begin{pmatrix} \mathbf{p}^T \otimes \rho_1 & \mathbf{0}_N^T \otimes \rho_1 \\ \mathbf{p}^T \otimes \rho_2 & \mathbf{0}_N^T \otimes \rho_2 \\ \mathbf{p}^T \otimes \rho_3 & \mathbf{0}_N^T \otimes \rho_3 \\ & \vdots \end{pmatrix} \tilde{X}(0) \\
&= \sum_{i=1}^{3} ((\mathbf{1}_N^T \ \ \mathbf{1}_N^T)^T \otimes \varrho_i)((\mathbf{p}^T \ \ \mathbf{0}_N^T) \otimes \rho_i)\tilde{X}(0) \\
&= \begin{pmatrix} \mathbf{1}_N \mathbf{p}^T & \mathbf{1}_N \mathbf{0}_N^T \\ \mathbf{1}_N \mathbf{p}^T & \mathbf{1}_N \mathbf{0}_N^T \end{pmatrix} \otimes \sum_{i=1}^{3} \varrho_i \rho_i \tilde{X}(0) \\
&= \begin{pmatrix} \mathbf{1}_N \mathbf{p}^T & \mathbf{1}_N \mathbf{0}_N^T \\ \mathbf{1}_N \mathbf{p}^T & \mathbf{1}_N \mathbf{0}_N^T \end{pmatrix} \otimes I_3 \tilde{X}(0).
\end{aligned}$$

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Thus

$$\lim_{n\to+\infty} X(n) = \mathbf{1}_N \mathbf{p}^T \otimes I_3 X(0)$$

we have $\lim_{n\to\infty} \|R_i(n) - R_\infty\| = 0$ for all i, where $R_\infty = [\mathbf{p}^T r(0), \mathbf{p}^T s(0), \mathbf{p}^T t(0)]$, $r(0) = (r_{11}(0), r_{12}(0), \cdots, r_{1N}(0))^T$, $s(0) = (r_{21}(0), r_{22}(0), \cdots, r_{2N}(0))^T$, $t(0) = (r_{31}(0), r_{32}(0), \cdots, r_{3N}(0))^T$.

(Sufficiency) Suppose to the contrary, if the matrix M has exactly an eigenvalue 1 with multiplicity 3 and all the other eigenvalues are stay in the unit disk is not satisfied, then by Lemma 2.1, the multiplicity of 1 eigenvalue in M is at least 3 since L has a zero eigenvalue at least. Hence, there are three cases needed to be discussed:

Case I: The multiplicity of 1 eigenvalue in M is 3, and there exists at least an eigenvalue which is not in the unit disk;

Case II: The multiplicity of 1 eigenvalue in M is more than 3, and the rest eigenvalues are in the unit disk;

Case III: The multiplicity of 1 eigenvalue in M is more than 3, and there exists at least an eigenvalue which is not in the unit disk.

For Case I, by Lemma 2.2, if M has an eigenvalue which is not in the unit circle, then E also has an eigenvalue which is not in the unit circle. Therefore, the stability of system (7) cannot be achieved, which means that the consensus of system (5) cannot be achieved. Similarly, we can prove Case II and Case III. This completes the proof.

Theorem 2.2 *System (5) achieves consensus asymptotically if and only if the digraph $\mathcal{G}$ has a directed spanning tree, and*

$$\theta < \theta^c = \min_{i=2,3,\cdots,N}\{\arccos(\varepsilon\tau|\mu_i|) - \arg\mu_i, \arccos\frac{3\varepsilon\tau|\mu_i| - \varepsilon^3\tau^3|\mu_i|^3}{2} - \arg\mu_i\}. \tag{11}$$

Proof (Sufficiency) It follows from Theorem 2.1 that if the system (5) achieves asymptotical consensus, then 1 is an eigenvalue of matrix M with algebraic multiplicity three and the all other eigenvalues are inside the unit disk. By Lemma 2.1 and Lemma 1.1, matrix L has a simple zero eigenvalue, which implies that $\mathcal{G}$ has a directed spanning tree.

Meanwhile, considering the characteristic equation (9), by applying the bilinear transformation $s = \frac{\sigma+1}{\sigma-1}$ to m_{ij}, we get a series of new polynomials

$$f_i(s) = (s-1)^2((\frac{s+1}{s-1})^2 - \frac{s+1}{s-1} + \varepsilon\tau\mu_i c_j) = s^2\varepsilon\tau\mu_i c_j + 2s(1-\varepsilon\tau\mu_i c_j) + 2 + \varepsilon\tau\mu_i c_j.$$

Define $\gamma_i(s)$ $(i = 2, 3, \cdots, N)$ as

$$\gamma_i(s) = \frac{f_i(s)}{\varepsilon\tau\mu_i c_j} = s^2 + (\frac{2}{\iota} - 2)s + (\frac{2}{\iota} + 1), \tag{12}$$

where $\iota = \varepsilon\tau\mu_i c_j$.

Noting that the properties of bilinear function, we see that all roots of (9) are inside the unit disk if and only if all roots of $\gamma_i(s) = 0$ have negative real parts for $i = 2, 3, \cdots, N$. Let $a_1 + b_1\mathbf{j} = \frac{2}{\iota} - 2, a_2 + b_2\mathbf{j} = \frac{2}{\iota} + 1$, then we see that $b_1 = b_2, a_1 = a_2 - 3$. It follows from Lemma 2.4 that all roots of $\gamma_i(s) = 0$ have negative real parts if and only if

$$a_1 > 0, a_1^2(a_1 + 3) + a_1 b_1^2 - b_1^2 > 0.$$

Noting the fact that

$$a_1 = \Re(\frac{2}{\varepsilon\tau\mu_i c_j} - 2) = \Re(\frac{2\overline{\mu_i c_j}}{\varepsilon\tau|\mu_i|^2} - 2) = \frac{2\cos(\theta + \arg\mu_i)}{\varepsilon\tau|\mu_i|} - 2 > 0$$

if and only if

$$\cos(\theta + \arg \mu_i) > \varepsilon\tau|\mu_i|,$$

we see that $\theta < \arccos(\varepsilon\tau|\mu_i|) - \arg \mu_i$, for all $i = 2, 3, \cdots, N$.

Also, by direct calculation, we get

$$a_1^2 + b_1^2 = (\frac{2}{\iota} - 2)(\frac{2}{\bar{\iota}} - 2) = 4(\frac{1}{|\iota|^2} + 1 - \frac{2\Re(\iota)}{|\iota|^2})$$

$$= 4(\frac{1}{\varepsilon^2\tau^2|\mu_i|^2} + 1 - \frac{2\cos(\theta + \arg \mu_i)}{\varepsilon\tau|\mu_i|})$$

and

$$a_1^2 = 4(\frac{\cos(\theta + \arg \mu_i)}{\varepsilon\tau|\mu_i|} - 1)^2.$$

It follows from

$$a_1^2(a_1 + 3) + a_1 b_1^2 - b_1^2 > 0$$

that

$$(a_1 - 1)(a_1^2 + b_1^2) + 4a_1^2 > 0.$$

This implies that

$$\theta < \arccos \frac{3\varepsilon\tau|\mu_i| - \varepsilon^3\tau^3|\mu_i|^3}{2} - \arg \mu_i, i = 2, 3, \cdots, N.$$

Thus

$$\theta < \theta^c = \min_{i=2,3,\cdots,N}\{\arccos(\varepsilon\tau|\mu_i|) - \arg \mu_i, \arccos \frac{3\varepsilon\tau|\mu_i| - \varepsilon^3\tau^3|\mu_i|^3}{2} - \arg \mu_i\}.$$

Hence the sufficiency.

(Necessity)if θ satisfy (11), we have that all the roots of (9) stay inside the unit disk for each $i = 2, 3, \cdots, N$. It implies that the eigenvalues of M are inside the unit disk except eigenvalue 1. Since the digraph $\mathcal{G}$ contains a directed spanning tree, we have that the Laplacian matrix L has a simple zero-eigenvalue. By Lemma 2.2, 1 is not the eigenvalue of matrix E, but 1 is the eigenvalue of matrix M with algebraic multiplicity three. By Theorem 2.1, system (5) achieves consensus asymptotically. This completes the proof of Theorem 2.2.

Remark 2.1 *By theorem 2.1 We can see that the consensus value is only determined by* $X(0)$ *and the topological structure of the system, it has nothing to do with the value of* $X(-1)$*, the delay* τ *and the iterative step* ε*, this is the same as our experimental results, the results of the experiment of the system (5) with Laplacian matrix* (13) *shown as table* 1*(when* $\tau = 3, \varepsilon = 0.01, \theta^c = 54.1102$*):*

Remark 2.2 *By Theorem 2.2 we can see that the delay* τ*, the iterative step* ε *and the directed graph of the system play a decisive role in the critical rotation angle* θ^c*. this fact matches with our experimental results for system* (5) *with Laplacian matrix* (13) *shown as table 2:*

Table 1. Examples for consensus value with different initial value.

X(0)	X(-1)	consensus value
(30 3 5 2 8 8 2 2 2 4 3 9)	(4 2 9 10 4 1 3 4 6 3 6 7)	(8.5435 4.5326 6.3370)
(30 3 5 2 8 8 2 2 2 4 3 9)	(0 0 0 0 0 0 0 0 0 0 0 0)	(8.5435 4.5326 6.3370)
(30 3 5 2 8 8 2 2 2 4 3 9)	(1 2 3 4 5 6 7 8 9 10 11 12)	(8.5435 4.5326 6.3370)
(3 3 5 2 8 8 2 2 2 4 3 9)	(1 2 3 4 5 6 7 8 9 10 11 12)	(2.6739 4.5326 6.3370)
(30 4 5 2 8 8 2 2 2 4 3 9)	(1 2 3 4 5 6 7 8 9 10 11 12)	(8.5435 4.7500 6.3370)

Table 2. Examples for θ^c with different initial value,τ, ε.

X(0)	τ	ε	θ^c
(3 3 5 2 8 8 2 2 2 4 3 9)	3	0.01	54.1102
(3 3 5 2 8 8 2 2 2 4 3 9)	3	0.03	21.7472
(3 3 5 2 8 8 2 2 2 4 3 9)	1	0.01	64.7168
(3 3 5 2 8 8 2 2 2 4 3 9)	1	0.03	54.1102
(2 3 5 2 8 2 2 2 2 4 3 9)	1	0.03	54.1102
(2 3 5 2 8 2 2 2 2 4 3 9)	1	0.03	54.1102

3. Numerical Simulation

In this section, the main conclusions of the article are verified by numerical simulation. In the numerical simulation, we assume $N = 4$ and consider the Laplacian matrix

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3 & -1 & -2 \\ -1 & -4 & 5 & 0 \\ -1 & 0 & -3 & 4 \end{pmatrix} \tag{13}$$

and all its eigenvalues are $\mu_1 = 0, \mu_2 = 5.7869 + 2.1051\mathbf{j}, \mu_3 = 5.7869 - 2.1051\mathbf{j}$ and $\mu_4 = 2.4262$. After simple calculations, we can get $\mathbf{p} = (0.2174 \;\; 0.3478 \;\; 0.2065 \;\; 0.2283)^T$ is a non-negative left eigenvector of L associated with eigenvalue 0 with $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, Take $\varepsilon = 0.01$ and initial values selected as

$$r(0) = (2, 2, 2, 4)^T, s(0) = (3, 8, 2, 3)^T, t(0) = (5, 2, 2, 9)^T.$$

by Theorem 2.1 and Theorem 2.2 the consensus value is $(2.4565 \;\; 4.5326 \;\; 4.2500)$, $\theta^c = 0.9444\, rad$ $(54.1102\ degree)$. Fig. 1 confirms the correctness of our theoretical results.

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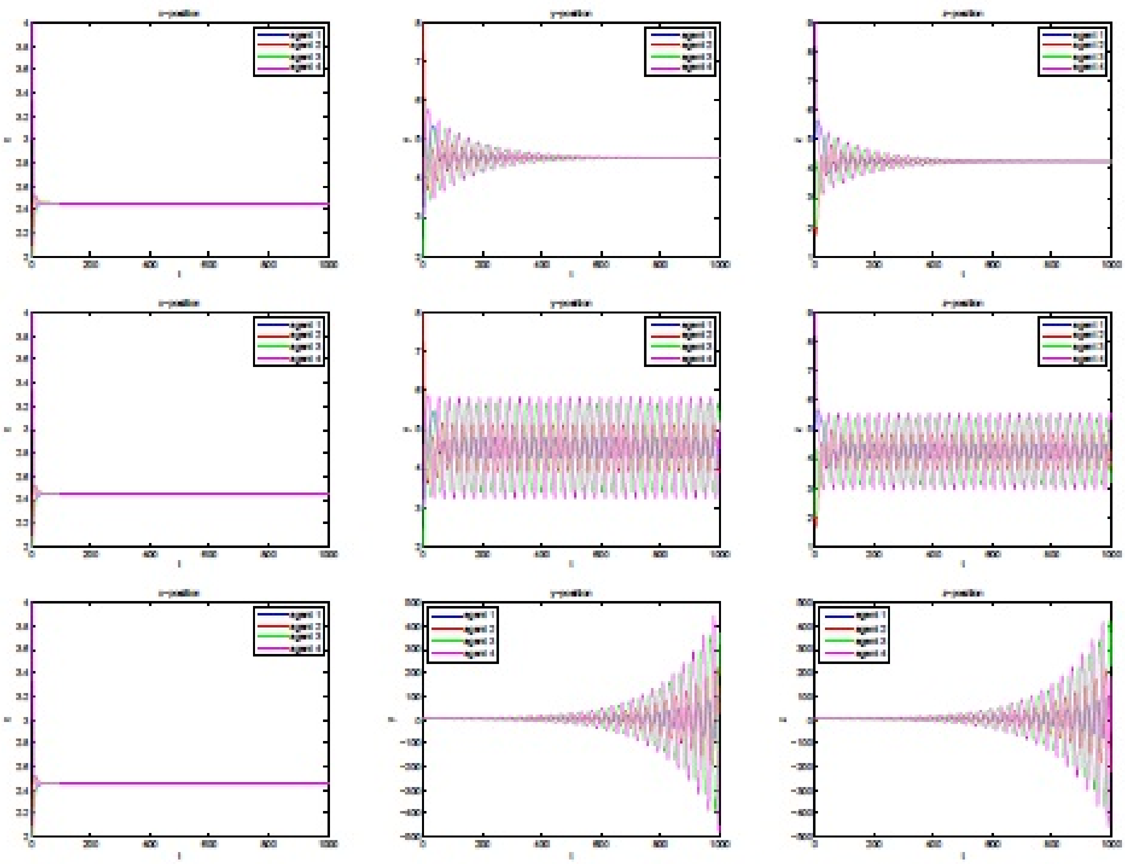

Figure 1. x, y, z direction trajectory of 4 agents, The 3 subgraph of the first lines with $\theta = \theta^c - \frac{\pi}{90} = 0.9095$ rad, (52.1102 degree), the second lines with $\theta = \theta^c = 0.9444$ rad (54.1102 degree), the third lines with $\theta = \theta^c + \frac{\pi}{90} = 0.9793$ rad (56.1102 degree).