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Simple analytical solutions of one-equation modelling for steady/decaying homogeneous turbulence sensitized to small strain

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Abstract. This study presents a simple mathematical solution describing the turbulent kinetic energy in steady or decaying homogeneous turbulence sensitized to small strain. Small strain is found to have little effect on the anisotropy of decaying homogeneous turbulence, which is consistent with a previous experiment. The effect of small strain on dissipation in the governing equations of turbulent kinetic energy is examined using the standard k-ε model. Two temporal profiles, namely a constant profile and a linear profile, with small strain are used. The steady homogeneous turbulence is maintained by linear forcing, as previously reported. This study mainly focuses on the effects of small strain on the turbulence time scale. When the magnitude of the small strain is small, the effects on the turbulence time scale can be negligibly small, especially for a constant profile. Based on this result, the governing equations for steady or decaying homogeneous turbulence sensitized to small strain can be reduced to simple linear equations. The analytical solution of these equations, a simple exponential function, is the same for the two types of homogeneous turbulence.

1. Introduction

The decay exponent n of turbulent kinetic energy [1] is one of the fundamental quantities describing the characteristics of homogeneous turbulence. The decay law describing the turbulent kinetic energy in decaying homogeneous turbulence includes the decay exponent n. The value of the decay exponent is an essential engineering quantity that constitutes one model constant in the turbulence model [1]. The value of the decay exponent can be slightly larger than unity [2-7]. As shown in previous studies (e.g., [4,5]), the decay exponent of homogeneous turbulence at low to moderate Reynolds numbers may be sensitive to experimental conditions.

One of the factors that can affect decaying homogeneous turbulence is the small strain of mean flow. This small strain corresponds to the small acceleration of the free stream in a flow in a wind tunnel. The small strain of the mean flow has been used to improve the isotropy of grid-generated turbulence [2,4]. The decay exponent of the decay law that describes the velocity fluctuation intensity of grid-generated turbulence can be slightly different among the components of the velocity fluctuation intensity [2]. These small differences can be reduced by using the small strain of the mean flow [2]. The smaller strain of the mean flow, referred to as small strain in this study, has recently been experimentally investigated [6]. In contrast to the strain in previous studies [2,4,8-10], this small

strain was found to hardly affect the anisotropy of decaying homogeneous turbulence [6]. In addition, it slightly reduced the turbulent kinetic energy.

The effects of small strain have recently been investigated using a theoretical analysis based on numerical analysis [7]. This analysis derived an approximate solution representing the effects of small strain using the standard k-ε model [1]. Specifically, the relative effects of small strain on turbulent energy and its dissipation were derived as a simple mathematical solution. The present study uses this approximate solution to validate the negligible effect of wind tunnel blockage on grid-generated turbulence [6]. The approximate accuracy of the simple mathematical solution was improved in subsequent studies [11,12]. A previous study [12] found that the observed effects depend on the temporal profile of the small strain. These previous studies considered only decaying homogeneous turbulence sensitized to small strain. However, homogeneous turbulence also exists in a steady state. Therefore, the fundamental characteristics of steady homogeneous turbulence sensitized to small strain should be investigated and compared with those of decaying homogeneous turbulence.

The present study investigates the difference between decaying and steady homogeneous turbulence sensitized to small strain. The focus is on the turbulent kinetic energy in these types of homogeneous turbulence. Based on previous studies, simple mathematical solutions describing the effects of small strain on these kinds of homogeneous turbulence are derived. The effects of the temporal profile of small strain are also examined. To produce steady homogeneous turbulence, linear forcing [11-13] is used in this study.

2. Methods

2.1. Governing equation for decaying homogeneous turbulence

Homogeneous turbulence without forcing decays temporally. The turbulent kinetic energy in decaying homogeneous turbulence sensitized to small strain is first examined. The following equation governs the turbulent kinetic energy in decaying homogeneous turbulence [1]:

$$\mathrm{d}k/\mathrm{d}t' = P - \varepsilon. \tag{1}$$

Here, k is the turbulent kinetic energy, defined as $k = (1/2)\,(\langle u^2 \rangle + \langle v^2 \rangle + \langle w^2 \rangle)$, where u, v, and w are velocity fluctuations in the streamwise, transverse, and spanwise directions, respectively, and $\langle\ \rangle$ denotes the ensemble average. Also, $t' = t'_r - t'_o$, where t'_r and t'_o are the actual time and the virtual origin of the time, respectively. In the governing equation, ε is the dissipation of the turbulent kinetic energy and P is the production term due to the small strain, defined as $P = (1/2)\,(-2\langle u^2 \rangle + \langle v^2 \rangle + \langle w^2 \rangle)$ $(\mathrm{d}U/\mathrm{d}x)$, where U and x are the streamwise mean velocity and the streamwise direction, respectively. By using anisotropy a [2,7,8], the production term P can be expressed in the following form:

$$P = P_o\,(\mathrm{d}U/\mathrm{d}x)\,k(t'), \text{ where } P_o = 2\,(a-1)/(1+2a). \tag{2}$$

Here, a is a constant since small strain barely affects the anisotropy of homogeneous turbulence [6,7].

In decaying homogeneous turbulence, the turbulent kinetic energy and its dissipation may be known functions of time. The following decay laws can respectively describe the turbulent kinetic energy and its dissipation: $k = k_o(t'/t'_c)^{-n}$ and $\varepsilon = \varepsilon_o(t'/t'_c)^{-(n+1)}$ [1]. Here, $t = t'/t'_c$ and $t'_c = M/U_o$, where U_o and M are the streamwise inflow velocity and the mesh size of a turbulence-generating grid, respectively [7]. k_o and ε_o in the decay laws are decay coefficients, and ε_o is given as follows: $\varepsilon_o = n\,U_o\,k_o/M$. Exponent n in the decay law is the decay exponent and has been examined in previous studies [2-6]. In the present study, the effects of small strain on turbulent kinetic energy and its dissipation are described based on these decay laws. This study uses two influence functions, defined in equation (3), to describe the effects of small strain on these turbulence quantities [11,12].

$$k(t) = f(t)\,k_o\,t^{-n} \text{ and } \varepsilon(t) = g(t)\,\varepsilon_o\,t^{-(n+1)}. \tag{3}$$

Here, $f(t) = g(t) = 1$ in decaying homogeneous turbulence, regardless of the small strain. By using these influence functions, the governing equation of the turbulent kinetic energy can be expressed as

$$\mathrm{d}f/\mathrm{d}t = P_o S(t)\, f(t) + (n/t)\, (f(t) - g(t)), \text{ where } S(t) = (dU/dx)/(U_o/M). \tag{4}$$

Note that the above governing equation does not include the decay coefficients. In this study, the influence function $f(t)$ depends on only P_o, $S(t)$, and n.

2.2. Governing equation for steady homogeneous turbulence

Homogeneous turbulence can be maintained to be steady by using an appropriate forcing term in the governing equation. The forcing term required for generating steady homogeneous turbulence has been investigated in previous studies, which often used spectral forcing schemes. The present study adopts a simple forcing scheme, which is applied in physical space, in the analysis. Lundgren [13] proposed a linear forcing scheme, where the components of the forcing terms are taken to be proportional to those of the velocity vector. This linear forcing scheme was adopted by Rosales and Meneveau [14] in a numerical simulation. Studies have recently been conducted on this linear forcing scheme [15]. With the linear forcing scheme, the equation of turbulent kinetic energy without the production term due to small strain is given as follows [11]:

$$\mathrm{d}k_s/\mathrm{d}t' = -\varepsilon_s + 2Q\, k_s. \tag{5}$$

Here, k_s and ε_s are the turbulent kinetic energy and its dissipation in steady homogeneous turbulence, respectively. Q is a constant, defined as follows [11]:

$$Q = \varepsilon_s/(2\, k_s). \tag{6}$$

The value of Q can be considered to be a constant as a mean value even if the turbulent kinetic energy and its dissipation fluctuate temporally.

In previous analyses on decaying homogeneous turbulence sensitized to small strain [7,11,12], influence functions were used. They were defined based on the turbulent kinetic energy and its dissipation in decaying homogeneous turbulence without the production term. Similarly, this study introduces influence functions for steady homogeneous turbulence sensitized to small strain. The influence functions of turbulent kinetic energy k and its dissipation ε for steady homogeneous turbulence sensitized to small strain are defined as follows:

$$k = f(t)\, k_s \text{ and } \varepsilon = g(t)\, \varepsilon_s. \tag{7}$$

By using the defined influence functions and the definition of Q in equation (6), the following governing equation of the influence functions of the turbulent kinetic energy for steady homogeneous turbulence sensitized to small strain is derived:

$$\mathrm{d}f/\mathrm{d}t = P_o\, S(t)\, f(t) + 2Q\, (f(t) - g(t)). \tag{8}$$

The governing equation for steady homogeneous turbulence sensitized to small strain is similar to that for decaying homogeneous turbulence, with only the second term being different.

2.3. Numerical conditions

Small strain $S(t)$ is included in the governing equations. This study used two temporal profiles for small strain $S(t)$ since the temporal profile may affect turbulence statistics [12]. The first temporal profile is a constant profile, given as follows: $S(t) = S_o$. The constant profile is the most straightforward profile for small strain. When the small strain is a positive constant, the streamwise mean velocity linearly increases. A constant profile has been used in studies on the effects of small strain [7,11]. The second temporal profile is a linear profile, given as follows: $S(t) = d_t S\, t$. For the linear profile, $d_t S$ corresponds to the linear gradient of $S(t)$. A linear profile was used in a previous study [12]. The values of S_o and $d_t S$ are required in the present numerical simulation. The present study uses the following relation between $d_t S$ and S_o:
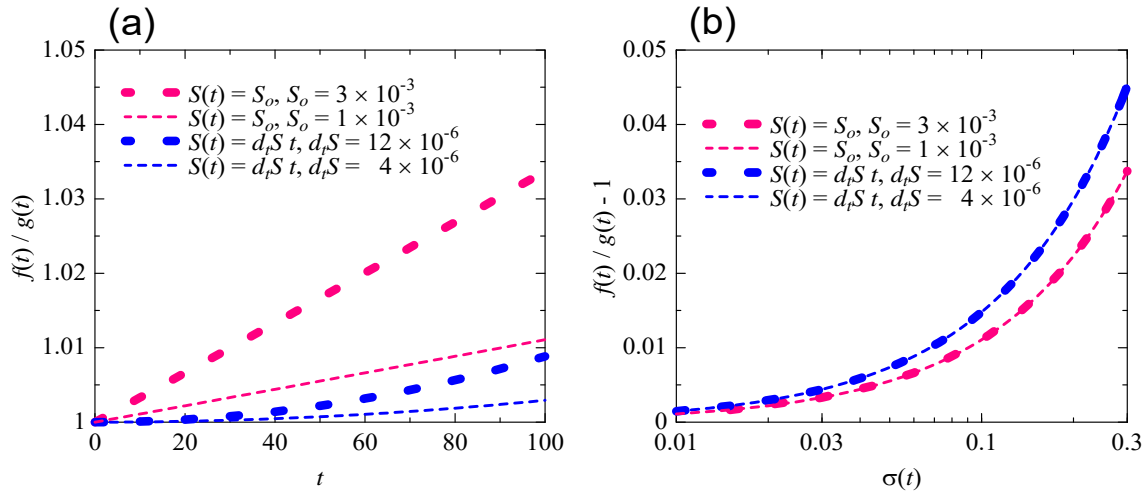
Figure 1. (a) Relative effects of small strain on the turbulence time scale in decaying homogeneous turbulence as a function of time t. (b) Deviation of the relative effects from unity as a function of total strain $\sigma(t)$. By using the total strain, the relative effects on the time scale can collapse. The deviation of the relative effects from unity can be considered to be small for a small magnitude of the total strain.

$$I_{So} = S_o\, t_l \text{ and } I_{(dtS)t} = (1/2)\, d_t S\, t^2, \text{ therefore } d_t S = 2\, S_o/t_l. \qquad (9)$$

Here, I_{So} and $I_{(dtS)}$ are temporal integrals of $S(t) = S_o$ and $S(t) = d_t S\, t$, respectively, for the integral range of 0 to t_l, where t_l is the integral limit of t.

The numerical conditions were as follows. The magnitude of the small strain based on the constant profile, S_o, was set to 0.003 or 0.001. The values of the acceleration parameter [16], which were derived from the values of S_o with the use a mesh Reynolds number of 10^4 [2-4,6,7], were smaller than or equal to those used in a previous study [14]. Using equation (9) for the linear profile, the value of $d_t S$ was set to 0.000012 or 0.00004, where t_l was set to 500 based on a previous study [12]. The magnitude of the production term due to small strain was set to -1/2 based on the values of anisotropy in previous experiments on grid-generated turbulence [2-4]. The influence function $g(t)$ of the dissipation in the decaying homogeneous turbulence sensitized to small strain was calculated using the governing equation of the standard k-ε model, which is a two-dimensional model [1]. This study adopted the following equation used in previous studies [7,11,12]:

$$\mathrm{d}g/\mathrm{d}t = C_{\varepsilon 1}\, P_o\, S(t)\, g(t) + (n + 1)\, (g(t)/f(t))\, (f(t) - g(t))/t. \qquad (10)$$

The model constant $C_{\varepsilon 1}$ in this equation was set to $C_{\varepsilon 1} = 1.44$, which is one of the most widely used values. The decay exponent n was set to 1.2 based on a previous study [5]. The governing equation of the influence functions based on the two-equation model is numerically solved using the standard finite difference method. The governing equation was integrated with respect to time using the fourth-order Runge-Kutta method. The time step was $dt = 0.1$. This study verified that the numerical analysis had sufficient accuracy. It should be noted that this study does not implement the standard k-ε model as a one-equation model.

3. Results

3.1. Turbulence time scale
The effects of small strain on the turbulence time scale are examined. In decaying homogeneous turbulence not affected by small strain, the turbulence time scale is given as follows: $k(t)/\varepsilon(t) = (U_o/M)$

(n/t). By using the influence functions of the turbulent kinetic energy and its dissipation, the turbulence time scale affected by small strain is given as follows:

$$k(t)/\varepsilon(t) = (U_o/M)\ (n/t)\ [f(t)/g(t)]. \tag{11}$$

As shown in the above equation, $f(t)/g(t)$ can be considered to be an influence function in decaying homogeneous turbulence sensitized to small strain. Figure 1(a) shows the influence function of the turbulence time scale in the observed decaying homogeneous turbulence. As shown in the figure, positive small strain increased the influence function of the turbulence time scale. The deviation of the influence function of the turbulence time scale from unity decreases with decreasing small strain. As shown in the figure, the value of the influence function of the turbulence time scale may depend on the magnitude of the small strain as well as time t.

Figure 1(a) shows that the value of the influence function of the turbulence time scale depends on the profile of the small strain as well as the magnitude of the small strain. To examine the difference in the influence function produced by the type of profile of the small strain, the total strain, defined in equation (12), was used in this study.

$$\sigma(t) = S_o\ t \text{ for } S(t) = S_o \text{ and } \sigma(t) = (1/2)\ d_tS\ t^2 \text{ for } S(t) = d_tS\ t. \tag{12}$$

Figure 1(b) shows profiles of $f(t)/g(t) - 1$ as a function of total strain. As shown in the figure, when using the total strain, the profile of the influence function does not depend on the magnitude of the small strain. Therefore, when using the total strain, the effects of the small strain profile on the influence function can be observed. The effects of small strain with the linear function are slightly larger than those of small strain with the constant function. The effects of small strain decrease with decreasing magnitude of the total strain. Even when the total strain $\sigma(t) = 0.1$, the relative effects of small strain on the turbulence time scale were smaller than 1.5%.

3.2. Derivation of analytical solutions

Figure 1(b) shows that in the region of small $\sigma(t)$, which could be caused by the small magnitude of the small strain, the effects of small strain on the turbulence time scale may be sufficiently small. Based on this result, the governing equation (equation (4)), which describes the influence functions of the turbulent kinetic energy in decaying homogeneous turbulence sensitized to small strain, can be rewritten as the following equation:

$$df/dt = P_oS(t)\,f(t) + (n/t)\,g(t)\,(f(t)/g(t) - 1). \tag{13}$$

Note that the above derived governing equation includes the deviation of the relative effects of the influence function $f(t)/g(t) - 1$ in the second term. Based on the results shown in Figure 1(b), with the use of $g(t) \sim 1$, the magnitude of the second term can be assumed to be sufficiently small because of the sufficiently small value of $f(t)/g(t) - 1$. By using this assumption, the above governing equation can be rewritten as the following simple equation:

$$df/dt = P_oS(t)\,f(t). \tag{14}$$

An analytical solution of the above equation can be derived as follows:

$$f(t) = \exp(P_oS(t)). \tag{15}$$

Therefore, the influence function of the turbulent kinetic energy could be a simple exponential function when the small strain barely affects the turbulence time scale.

By focusing on the influence function on the turbulence time scale, the governing equation, which describes the influence function of the turbulent kinetic energy in steady homogeneous turbulence sensitized to small strain, can be rewritten as the following equation:

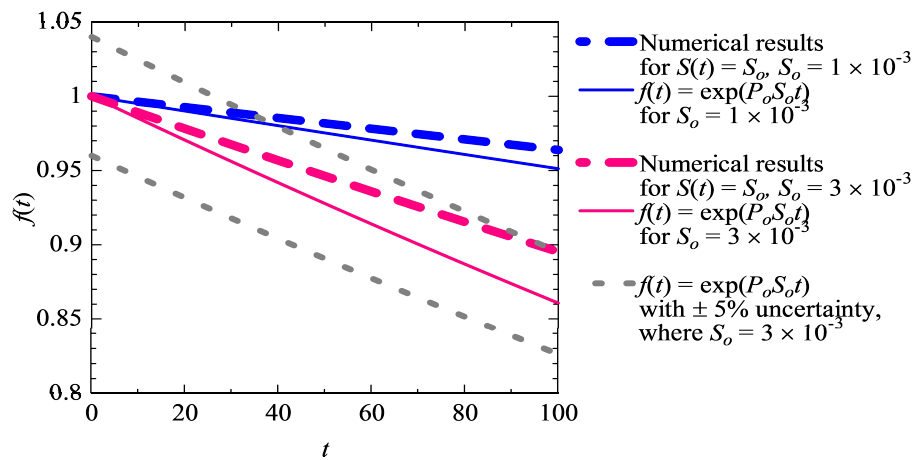$$df/dt = P_o\ S(t)\,f(t) + 2Q\,g(t)\,(f(t)/g(t) - 1). \tag{16}$$

Figure 2. Comparison of the simple solutions derived from one-equation modelling with numerical results from two-equation modelling. As shown, the absolute uncertainty of the derived simple solutions is smaller than 0.05 in the range of $t = 0$ to 100, where the value of t is nearly equal to x/M.

Similar to the governing equation of decaying homogeneous turbulence sensitized to small strain, in this study, the turbulence time scale in steady homogeneous turbulence sensitized to small strain was assumed to be sufficiently small in the region of small $\sigma(t)$. Based on this assumption, the above governing equation was rewritten as follows by the negligible second term:

$$\mathrm{d}f/\mathrm{d}t = P_o\, S(t)\, f(t). \tag{17}$$

An analytical solution of the above equation is given as follows:

$$f(t) = \exp(P_o S(t)). \tag{18}$$

It should be noted that the analytical solutions, which were derived using sufficiently small effects on the turbulence time scale, are the same for steady and decaying homogeneous turbulence. This indicates that the sensitivities of steady and decaying homogeneous turbulence to small strain are the same when the small strain is sufficiently small.

4. Discussion and Summary

4.1. Discussion
This study investigated decaying and steady homogeneous turbulence sensitized to small strain using a one-equation model. Simple analytical solutions were derived; they were found to be the same for both decaying and steady homogeneous turbulence sensitized to small strain. Previous studies [7,11,12] investigated only decaying homogeneous turbulence sensitized to small strain. The present study considered both steady and decaying homogeneous turbulence. The two-equation model adopted in previous studies should be used in the future to investigate decaying and steady homogeneous turbulence sensitized to small strain. For instance, Figure 2 compares the analytical solution derived in this study, which represents the influence function of the turbulent kinetic energy in decaying homogeneous turbulence sensitized to small strain, to the numerical solution obtained using the two-equation model. As shown in the figure, the analytical solution obtained using the one-equation model may not agree with the numerical solution obtained using the two-equation model. Specifically, in the range of $t = 0$ to 100, the maximum relative deviation of the analytical solution is 5%, which is larger than the uncertainty of the pitot tube used in a wind tunnel experiment.

This study used constant and linear temporal profiles of small strain. The effects of small strain based on these temporal profiles on turbulent kinetic energy and dissipation can be observed when the

total strain is used (equation (12)). As shown in Figure 1, the effects of small strain on the turbulence time scale in decaying homogeneous turbulence differ slightly between the constant and linear profiles, with the latter producing slightly larger effects. Therefore, the assumption of a negligible effect on the turbulence time scale used in this study is more accurate for the constant profile. Although the turbulence time scale is more sensitive to small strain with the linear profile, the order of magnitude of the effect of small strain based on the linear profile is the same as that of the effect based on the constant profile. Therefore, the temporal profile of the small strain may not significantly affect the accuracy of the assumption of a negligible effect of small strain on the turbulence time scale. In this study, a one-equation model was used; however, to investigate steady and decaying homogeneous turbulence sensitized to small strain, the application of a two-equation model may be necessary. When the effect of small strain on dissipation is investigated using a two-equation model, differences in the temporal profile of the small strain should be considered.

As shown in Figure 1(b), the relative effects can collapse when the strain itself is used. In this study, the rate of strain was used. The physical reason why strain should be a variable, rather than time, is unclear. However, the physical reason why the strain should be noticed is rather clear. As shown in this study, the analytical solutions of the influence function derived by analytically solving the governing equations contain strain itself as a variable. As shown in the numerical analysis results of the governing equations based on the two-equation model, when the magnitude of the strain rate is sufficiently small, the influence of small strain on the time scale is negligible (Figure 1(b)). If the time scale is not affected by the small strain, the governing equations of the two-equation model governing equations result in that of the one-equation model. An analytical solution of the equations based on one-equation modeling was derived here.

4.2. Summary
This study modeled the effect of small strain on steady and decaying homogeneous turbulence using a simple mathematical solution. The effects of small strain on the turbulence time scale in two types of homogeneous turbulence were investigated. When the total strain was small, the magnitude of the small strain was also small, and the relative effects of small strain on the turbulence time scale were less than about 1.5%. When the relative effects on the turbulence time scale are small, the governing equations, which can be rewritten as simple linear differential equations, are the same for the two types of homogeneous turbulence. The influence function of the turbulent kinetic energy was derived as a simple exponential function.

The present study considered the effects of the temporal profile of strain rate on decaying and steady homogeneous turbulence. The effects of strain rate with constant and linear profiles on the turbulence time scale in decaying and steady homogeneous turbulence were investigated. A previous study that used both constant and linear profiles did not consider the effects on the time scale in decaying homogeneous turbulence. The present study showed that the effect of small strain rate on the turbulence time scale is significant in turbulence modeling. In addition, this study investigated steady homogeneous turbulence sensitized to small strain.

An exact solution based on one-equation modelling could be useful for addressing the effect of small strain on homogeneous turbulence. Also, one-equation models are widely applied in engineering applications. The main result obtained in this study is that when the one-equation model is applied, the relative effect of small strain on turbulence kinetic energy is consistent between steady and decaying homogeneous turbulence. The relative effect can be described by a simple exponential function. Here, the assumption made to allow the use of the one-equation model holds when the strain rate is sufficiently small. When the one-equation model is applied, the relative effects are equivalent between steady and decaying homogeneous turbulence. If the effects of the small strain rate on the turbulence time scale are not negligible, there may be a significant difference in the relative effects between one- and two-equation models. In the future, a two-equation model will be used to investigate the effect of small strain on steady turbulence.

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