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Modules and their endomorphism rings

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Abstract. Let R and S be a pair of equivalent rings. We will prove that if R has property that a pair of modules over R are isomorphic if and only if their endomorphism rings are isomorphic than S has same property, i.e. a pair of modules over S are isomorphic if and only if their endomorphism rings are isomorphic.

1. Introduction

Let M and N are modules over a ring R and set of all R-homomorphisms from M to N is written $\mathrm{Hom}_R(M,N)$. Then $\mathrm{Hom}_R(M,N)$ is Abelian group over addition of mapping. Moreover, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M,M)$ is a ring over addition and composition of mapping, and called an endomorphism ring of M. In general, if two modules are isomorphic then their endomorphism rings are isomorphic, but the converse is not true. The Baer-Kaplansky theorem states that two torsion groups are isomorphic if and only if their endomorphism rings are isomorphic (see [1–3]). May and Taubassy [4] investigated the relationship between torsion subgroups of two abelian groups with their endomorphism rings are isomorphic. Using the method used by Kaplansky [1], Wolfson [5] works on a torsion-free module over a discrete valuation ring. The theorem obtained is "If M and N modules are torsion-free of the complete discrete valuation ring R, then each ring isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ is induced by the module isomorphism between M and N". Wolfson divided the theorem into two cases, namely for M not divisible and M divisible. For the first case the fact used is a torsion-free module and not divisible over a complete discrete valuation ring R has a direct summand that is isomorphic with R. In the second case the fact used is a torsion-free module and divisible over the complete discrete valuation ring R is the vector space over the division field of R. On a modules over complete valuation domains May [6] got a theorem like the Baer-Kaplansky theorem and proved for reduced modules which are neither torsion nor torsion-free.

Ivanov [7] got same result with the Baer-Kaplansky theorem, i.e. the triangular matrix ring has a class of modules which have same property with torsion group in Baer-Kaplansky Theorem. To get this result, Ivanov explores further the properties of the torsion group. The indecomposable p-group is isomorphic to $\mathbb{Z}(p^n)$ or $\mathbb{Z}(p^\infty)$. The ring $\mathrm{End}_R(\mathbb{Z}(p^n))$ is isomorphic to the ring $\mathbb{Z}/(p^n)$ and the ring $\mathrm{End}_R(\mathbb{Z}(p^\infty))$ is isomorphic to the p-adic integer ring (see [2] and [1]). Therefore, every ring isomorphism between two endomorphism rings of the torsion group always "preserve indecomposable direct summand". Ivanov called this isomorphism as an IP-isomorphism. For a class of modules which

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have a decomposition $M = \bigoplus_{i \in I} M_i$ with property that every indecomposable direct summand of M is contained in the sum of finite number of the M_i (we say that M has the finite embedding property), an IP-isomorphism between endomorphism rings will give isomorphism between modules. Using fact that the ring $\mathrm{End}_R\left(\mathbb{Z}(p^\infty)\right)$ is isomorphic to the p-adic integer ring, Breaz [8] also got a class of modules over principal ideal domain satisfied the Baer-Kaplansky theorem.

In fact, two vector spaces are isomorphic if and only if their endomorphism rings are isomorphic. Also in case module over a simple Artin ring, i.e. two modules over a simple Artin ring isomorphic if and only if their endomorphism rings are isomorphic. In fact that matrix ring $n \times n$ over a division ring is a simple Artin ring and every a simple Artin ring isomorphic to a matrix ring over a division ring (see [9] page 152-153). Moreover, a ring R and the $n \times n$ matrix ring over R are equivalent ring (see [9] page 262-265). Then division ring and a simple Artin are equivalent rings.

Let R and S be a pair of equivalent rings. In this paper we will prove that if R has property that a pair of modules over R isomorphic if and only if their endomorphism rings are isomorphic then S has same property.

In this paper a ring will be a ring with unity and all modules will be nonzero unital right module except if in special case. Definition of ring, module, category, and others which are used in this paper refer to [9].

2. IP-isomorphism

In this section we will discuss about an IP-isomorphism and it's relation with module isomorphism. We will discuss about primitive idempotent and property of indecomposability of direct summand.

An element e in a ring R is called idempotent if $e^2 = e$. Two idempotents e_1 and e_2 in R are called orthogonal if $e_1 e_2 = 0 = e_2 e_1$. An idempotent $e \neq 0$ in R is called primitive idempotent if e can not be a sum of two nonzero orthogonal idempotent.

Let M be a module over a ring R and e an idempotent in $\mathrm{End}_R(M)$. Then $1 - e$ is also idempotent in $\mathrm{End}_R(M)$. Moreover, e and $1 - e$ orthogonal and M has decomposition $M = eM \oplus (1 - e)M$. The following lemma will give characterization of primitiveness of e.

Lemma 1. Let M be a module over a ring R and $e \neq 0$ be an idempotent in $\mathrm{End}_R(M)$. Then the direct summand eM is indecomposable if and only if e is a primitive idempotent in $\mathrm{End}_R(M)$.

Proof: Let eM is indecomposable. We will prove that e is a primitive idempotent. Let $e = e_1 + e_2$, where e_1 and e_2 are orthogonal idempotents in $\mathrm{End}_R(M)$. Since $e = e_1 + e_2$ then $eM = e_1 M + e_2 M$. Let $x \in e_1 M \cap e_2 M$. Then $x = e_1 m_1 = e_2 m_2$, for some $m_1, m_2 \in M$. So

$$x = e_1 m_1 = {e_1}^2 m_1 = e_1(e_1 m_1) = e_1(e_2 m_2) = (e_1 e_2) m_2 = 0 m_2 = 0.$$

So $e_1 M \cap e_2 M = 0$. Therefore $eM = e_1 M \oplus e_2 M$. Since eM is indecomposable then $e_1 M = 0$ or $e_2 M = 0$. So $e_1 = 0$ or $e_2 = 0$. Then e is a primitive idempotent in $\mathrm{End}_R(M)$.

Conversely, let e is a primitive idempotent in $\mathrm{End}_R(M)$. We will prove that eM is indecomposable. Let $eM = K \oplus L$. Then

$$M = eM \oplus (1 - e)M = K \oplus L \oplus (1 - e)M.$$

Let e_K be a projection on K along $L \oplus (1 - e)M$ and e_L be a projection on L along $K \oplus (1 - e)M$. Since $e_K M = K \subseteq \mathrm{Ker}(e_L)$ and $e_L M = L \subseteq \mathrm{Ker}(e_K)$ then $(e_K e_L)M = 0 = (e_L e_K)M$. Then e_K and e_L are orthogonal idempotents in $\mathrm{End}_R(M)$. Furthermore, for each $x \in M = K \oplus L \oplus (1 - e)M$ can be written $x = x_K + x_L + x'$ for some $x_K \in K,\ x_L \in L,\ x' \in (1 - e)M$. So

$$ex = x_K + x_L = e_K x + e_L x = (e_K + e_L)x.$$

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Therefore $e = e_K + e_L$, where e_K and e_L are orthogonal idempotents in $\mathrm{End}_R(M)$. Since e is a primitive idempotent in $\mathrm{End}_R(M)$ then $e_K = 0$ or $e_L = 0$. Therefore $K = e_K M = 0$ or $L = e_L M = 0$. So eM is indecomposable. □

Let given M and N be modules over a ring R, and $\varphi: M \to N$ any R-isomorphism. The mapping α which defined

$$\alpha: \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(N), \qquad \alpha(\psi) = \varphi\psi\varphi^{-1}$$

is a ring isomorphism. Furthermore, for any idempotent e in $\mathrm{End}_R(M)$ satisfies

$$\alpha(e)N = \varphi e\varphi^{-1}N = (\varphi e)(\varphi^{-1}N) = (\varphi e)M = \varphi(eM).$$

Therefore

$$\varphi|_{eM}: eM \longrightarrow \alpha(e)N$$

is a R-isomorphism. So $eM \cong \alpha(e)N$, for all e idempotent in $\mathrm{End}_R(M)$.

In general, if $\beta: \mathrm{End}_R(M) \to \mathrm{End}_R(N)$ is any ring isomorphism and e is any primitive idempotent in $\mathrm{End}_R(M)$ then $\beta(e)$ is also a primitive idempotent in $\mathrm{End}_R(N)$ but it must not $eM \cong \beta(e)N$. Ring isomorphism

$$\phi: \mathrm{End}_R(M) \to \mathrm{End}_R(N)$$

is called IP-isomorphism if $\phi(e)N \cong eM$, for all primitive idempotents e in $\mathrm{End}_R(M)$. From discussion above, if $M \cong N$ then there is an IP-isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$. The following theorem gives property that an IP-isomorphism between endomorphism rings will give module isomorphism.

Proposition 1. ([7], Proposition 1) Let M and N be modules over a ring R where M has the finite imbedding property with respect to a decomposition into indecomposable direct summands and N be generated by indecomposable direct summands. Then M and N isomorphic if and only if there is an IP-isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$.

Proof: Let $M = \bigoplus_{i\in I} M_i$ where M_i indecomposable and this decomposition has the finite imbedding property and $N = \sum_{j\in J} N_j$ where N_j indecomposable direct summand of N. Let there is an IP-isomorphism

$$\phi: \mathrm{End}_R(M) \to \mathrm{End}_R(N)$$

We will prove that there is a module isomorphism between M and N. Let

$$e_i: M \to M$$

is a projection on M_i. Then e_i is a primitive idempotent in $\mathrm{End}_R(M)$, for all $i \in I$. Let $f_i = \phi(e_i) \in \mathrm{End}_R(N)$. Since ϕ is an IP-isomorphism then

$$e_i M \cong \phi(e_i)N = f_i N, \qquad \forall i \in I.$$

Let this isomorphic is given by

$$\psi_i: e_i M \to f_i N, \qquad \forall i \in I.$$

For each $x \in M$ can be written $x = \sum_{i\in I} x_i$ for some $x_i \in M_i$ uniquely and almost all $x_i = 0$. Therefore $S(x) = \{ i \in I \mid x_i \neq 0 \}$ is a finite set. We define

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$$\psi: M \to N, \qquad \psi(x) = \sum_{i \in S(x)} \psi_i(x_i).$$

Then ψ is an one-one R-homomorphism with

$$\mathrm{Im}(\psi) = \sum_{i \in \mathrm{I}} \mathrm{Im}(\psi_i) = \sum_{i \in \mathrm{I}} f_i N.$$

We will prove that ψ is onto. Let N_j is any indecomposable direct summand of N which generate N. Let $f_j: N \to N_j$ be a projection and $e_j = \phi^{-1}(f_j)$. Furthermore, since f_j is a primitive idempotent in $\mathrm{End}_R(N)$ then e_j is also a primitive idempotent in $\mathrm{End}_R(M)$. Since M has the finite imbedding property then $e_j M$ is contained in a sum of finite number of the M_i. Let

$$e_j M \subseteq M_{i_1} \oplus \dots \oplus M_{i_n}.$$

We will prove that $(1 - e_{i_1} - \cdots - e_{i_n})e_j = 0$. For any $m \in M$ we have

$$e_j m \in e_j M \subseteq M_{i_1} \oplus \dots \oplus M_{i_n}.$$

We write

$$e_j m = m_1 + \cdots + m_n, \qquad \text{for some } m_k \in M_{i_k}, \qquad k = 1, \dots, n.$$

Therefore

$$\begin{aligned}\left((1 - e_{i_1} - \cdots - e_{i_n})e_j\right) m &= (1 - e_{i_1} - \cdots - e_{i_n})(e_j m) \\ &= (1 - e_{i_1} - \cdots - e_{i_n})(m_1 + \cdots + m_n) \\ &= (m_1 + \cdots + m_n) - m_1 - \cdots - m_n \\ &= 0.\end{aligned}$$

So $(1 - e_{i_1} - \cdots - e_{i_n})e_j = 0$. Then

$$(1 - f_{i_1} - \cdots - f_{i_n})f_j = \phi\left((1 - e_{i_1} - \cdots - e_{i_n})e_j\right) = \phi(0) = 0.$$

Therefore $f_j = (f_{i_1} + \cdots + f_{i_n})f_j$. So

$$\begin{aligned}N_j = f_j N &= \left((f_{i_1} + \cdots + f_{i_n})f_j\right) N = (f_{i_1} + \cdots + f_{i_n})(f_j N) = (f_{i_1} + \cdots + f_{i_n})N_j \\ &\subseteq (f_{i_1} + \cdots + f_{i_n})N = f_{i_1}N + \cdots + f_{i_n}N \subseteq \sum_{i \in I} f_i N = \mathrm{Im}(\psi).\end{aligned}$$

Then $N_j \subseteq \mathrm{Im}(\psi)$ for all $j \in J$. Therefore $N = \mathrm{Im}(\psi)$. So ψ is onto. Thus $\psi: M \to N$ is a R-isomorphism. □

3. Case on Vector Space

We will discuss a case on vector space. Firstly we will prove that every isomorphism between endomorphism rings of two vector spaces is an IP-isomorphism.

Proposition 2. Let V and W be vector spaces over a field F. Then every ring isomorphism between $\mathrm{End}_F(V)$ and $\mathrm{End}_F(W)$ is an IP-isomorphism.

Proof: Let $\varphi\colon \mathrm{End}_F(V) \to \mathrm{End}_F(W)$ be a ring isomorphism and e be a primitive idempotent in $\mathrm{End}_F(V)$. We will proof that $eV \cong \varphi(e)W$. Since e is a primitive idempotent in $\mathrm{End}_F(V)$ then $\varphi(e)$ is a primitive idempotent in $\mathrm{End}_F(W)$. According to Lemma 1, eV and $\varphi(e)W$ are indecomposable. Thus dimension of eV and $\varphi(e)W$ are one. Since two vector spaces are isomorphic if and only if have same dimension then $eV \cong \varphi(e)W$. Thus φ is an IP-isomorphism. □

Corollary 1. Let V and W be vector spaces over a field F. Then $V \cong W$ if and only if $\mathrm{End}_F(V) \cong \mathrm{End}_F(W)$.

Proof: Since every ring isomorphism between $\mathrm{End}_F(V)$ and $\mathrm{End}_F(W)$ is IP-isomorphism and every decomposition of a vector space satisfies the finite embedding property then according to Proposition 1 we have $V \cong W$. □

4. Case on Module Over Simple Artin ring

Matrix ring $n \times n$ over division ring is a simple Artin ring. Furthermore, every simple Artin ring is isomorphic to matrix ring over a division ring ([9], 152-153). Therefore, every modules over a simple Artin ring is semi simple modules and all simple modules over a simple Artin ring are isomorphic.

We will prove that every isomorphism between endomorphism rings of module over a simple Artin ring is an IP-isomorphism. We use similar method with method which used in vector space case. It is possible because a module over a simple Artin ring has similar property with a vector space, i.e. every indecomposable direct summand of a module over a simple Artin ring is a simple module and all these simple modules are isomorphic.

Proposition 3. Let M and N be modules over a simple Artin ring R. Then every ring isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ is an IP-isomorphism.

Proof: Let $\varphi\colon \mathrm{End}_R(M) \to \mathrm{End}_R(N)$ be a ring isomorphism and e be a primitive idempotent in $\mathrm{End}_R(M)$. We will prove that $eM \cong \varphi(e)N$. Since e is a primitive idempotent in $\mathrm{End}_R(M)$ then $\varphi(e)$ is also a primitive idempotent in $\mathrm{End}_R(N)$. According to Lemma 1, eM and $\varphi(e)N$ are indecomposable. Therefore eM and $\varphi(e)N$ are simple module. So $eM \cong \varphi(e)N$. Then φ is an IP-isomorphism. □

Corollary 2. Let M and N be modules over a simple Artin ring R. Then $M \cong N$ if and only if $\mathrm{End}_R(M) \cong \mathrm{End}_R(N)$.

Proof: Since every ring isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ is an IP-isomorphism and every decomposition of semi simple module satisfies the finite imbedding property then according to Proposition 1, we have $M \cong N$. □

5. Equivalent Functor

Let MOD-R and MOD-S be categories of modules over a ring R and S respectively. A functor (covariant and additive)

$$F\colon \mathrm{MOD}\text{-}R \longrightarrow \mathrm{MOD}\text{-}S$$

can viewed as "homomorphism of categories", i.e.

$$F \quad : \quad \begin{array}{rcl} M & \longmapsto & F(M) \\ f\colon M \to N & \longmapsto & F(f)\colon F(M) \to F(N) \end{array}$$

with properties

1. $F(id_M) = id_{F(M)}$ for every M in MOD-R;
2. $F(gf) = F(g)F(f)$ for all K, M, N di MOD-R and all morphisms $f\colon K \to M$ dan $g\colon M \to N$;
3. $F(g+f) = F(g) + F(f)$ for all M, N di MOD-R and all morphisms $f\colon M \to N$ dan $g\colon M \to N$.

Two functors F:MOD-$R \longrightarrow$ MOD-S and G:MOD-$R \longrightarrow$ MOD-S are said isomorphic, written $F \cong G$, if there exist an indexed class of isomorphism

$$(\eta_M\colon F(M) \to G(M))_{M \in \text{MOD-}R}$$

such that for each pair M, N in MOD-R and each morphism $f\colon M \to N$ the diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ \downarrow \eta_M & & \downarrow \eta_N \\ G(M) & \xrightarrow{G(f)} & G(N) \end{array}$$

commutes, i.e.

$$\eta_N \circ F(f) = G(f) \circ \eta_M.$$

A functor F:MOD-$R \longrightarrow$ MOD-S is called an equivalent functor if there is a functor G:MOD-$R \longrightarrow$ MOD-S such that $GF \cong id_{\text{MOD}-R}$ and $FG \cong id_{\text{MOD}-S}$. In this case, ring R and S are said equivalent, written $R \approx S$, and G is called an inverse equivalence of F.

Proposition 4. Let MOD-R and MOD-S be categories of modules over a ring R and S respectively. If

$$F\text{:MOD-}R \longrightarrow \text{MOD-}S$$

be an equivalent functor and M, N in MOD-R then

$$F_{M,N} \quad : \quad \begin{array}{rcl} \text{Hom}_R(M,N) & \longmapsto & \text{Hom}_S(F(M),F(N)) \\ f & \longmapsto & F(f) \end{array}$$

is a group isomorphism with property that $F(f)$ is an epimorphism (resp. monomorphism) if and only if f is an epimorphism (resp. monomorphism). Moreover, if $M \neq 0$ then

$$F_M \quad : \quad \begin{array}{rcl} \text{End}_R(M) & \longmapsto & \text{End}_S(F(M)) \\ f & \longmapsto & F(f) \end{array}$$

is a ring isomorphism.

Proof: Since F is additive, we have $F_{M,N}$ is an abelian group homomorphism. Let G is an inverse equivalence of F. Then for each f in $\text{Hom}_R(M,N)$ the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \eta_M & & \downarrow \eta_N \\ GF(M) & \xrightarrow{GF(f)} & GF(N) \end{array}$$

commutes. Let f in $\mathrm{Hom}_R(M,N)$ and $F_{M,N}(f) = 0$. Then

$$f = \eta_N^{-1} GF(f) \eta_M = \eta_N^{-1} G(0) \eta_M = 0.$$

Thus we have $F_{M,N}$ is monic. We will prove that $F_{M,N}$ is epic. Let g in $\mathrm{Hom}_S(F(M), F(N))$. Then $G(g)$ in $\mathrm{Hom}_R(GF(M), GF(N))$. We define

$$h = \eta_N G(g) \eta_M^{-1}.$$

Then $GF(h) = G(g)$. So $F(h) = g$. Thus $F_{M,N}$ is epic. We have $F_{M,N}$ is an abelian group isomorphism. Since F preserve composition and identity, we have F_M is a ring isomorphism. From diagram above we have f is an epimorphism (resp. monomorphism) if and only if $GF(f)$ is an epimorphism (resp. monomorphism). Assume f is a monomorphism. We will prove that $F(f)$ is a monomorphism. Let g be a morphism in MOD-S such that $gF(f) = 0$. Then

$$G(g)GF(f) = G\big(gF(f)\big) = G(0) = 0.$$

Since $GF(f)$ monic, we have $G(g) = 0$. Thus

$$FG(g) = F(0) = 0.$$

Since $FG \cong id_{\mathrm{MOD}-S}$, we have $g = 0$. Therefore $F(f)$ is a monomorphism. The remainder of proof is entirely similar. □

Let $F : \mathrm{MOD}\text{-}R \longrightarrow \mathrm{MOD}\text{-}S$ be an equivalent functor and M, N in MOD-R. Use proposition above, f in $\mathrm{Hom}_R(M,N)$ is a module isomorphism if and only if $F(f)$ in $\mathrm{Hom}_R(F(M), F(N))$ is a module isomorphism. Furthermore, if $\Phi : \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(N)$ is a ring isomorphism than $F_N \Phi F_{\mathrm{M}}^{-1}$ is a ring isomorphism from $\mathrm{End}_S\big(F(M)\big)$ to $\mathrm{End}_S\big(F(N)\big)$. This result is written in two following corollaries.

Corollary 3. Let $F : \mathrm{MOD}\text{-}R \longrightarrow \mathrm{MOD}\text{-}S$ be an equivalent functor and M, N in MOD-R. Then $F(M) \cong F(N)$ if and only if $M \cong N$.

Corollary 4. Let $F : \mathrm{MOD}\text{-}R \longrightarrow \mathrm{MOD}\text{-}S$ be an equivalent functor and M, N in MOD-R. Then $\mathrm{End}_R(M) \cong \mathrm{End}_R(N)$ if and only if $\mathrm{End}_S\big(F(M)\big) \cong \mathrm{End}_S\big(F(N)\big)$.

Let $F : \mathrm{MOD}\text{-}R \longrightarrow \mathrm{MOD}\text{-}S$ be an equivalent functor and M, N in MOD-R. Use Corollary 3 and Corollary 4 we have following diagram.

$$\begin{array}{ccc} M \cong N & \Longrightarrow & \mathrm{End}_R(M) \cong \mathrm{End}_R(N) \\ \Updownarrow & & \Updownarrow \\ F(M) \cong F(N) & \Longrightarrow & \mathrm{End}_S\big(F(M)\big) \cong \mathrm{End}_S\big(F(N)\big) \end{array}$$

If we assume that $M \cong N$ if and only if $\mathrm{End}_R(M) \cong \mathrm{End}_R(N)$, then we have following diagram.

$$\begin{array}{ccc} M \cong N & \Longleftrightarrow & \mathrm{End}_R(M) \cong \mathrm{End}_R(N) \\ \Updownarrow & & \Updownarrow \\ F(M) \cong F(N) & \Longleftrightarrow & \mathrm{End}_S\big(F(M)\big) \cong \mathrm{End}_S\big(F(N)\big) \end{array}$$

This result is written in following proposition.

Proposition 5. Let R and S be a pair of equivalent rings. If R has property that a pair of modules over R are isomorphic if and only if their endomorphism rings are isomorphic than S has same property, i.e. a pair of modules over S are isomorphic if and only if their endomorphism rings are isomorphic.

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