Journal of Physics: Conference Series

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To cite this article: B Yong 2019 *J. Phys.: Conf. Ser.* **1320** 012024

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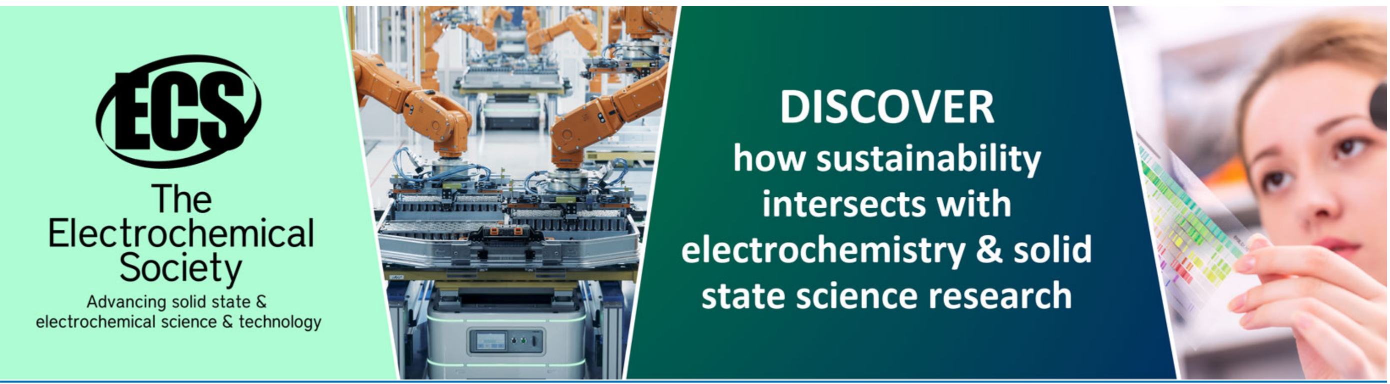

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Application of Analytic Approximation Method Using HAM for Solving The Democratic Elections Model

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Abstract. This article presents the solutions of the democratic elections model by homotopy analysis method (HAM). We proposed a simple democratic elections model in the form non-linear differential equation in a closed voters population. Voters population divided into three sub-population, i.e neutral sub-population, supportive sub-population, and aphatetic sub-population. HAM is applied to compute the solutions of the model. HAM is an analytic approximation method in the form of power series to solve the non-linear differential equation. HAM contains the auxiliary parameter for controlling the convergent region of power series solutions. Numerical simulations on the model in the form graphical results are presented and discussed quantitatively to describe the dynamics of each sub-population. It is shown that HAM performs well in terms of efficiency which converge rapidly and results obtained require only a few iterations.

1. Introduction

Democratic elections is one of the electoral systems used around the world. In this system, political parties and presidential candidates play an important role for the elections. Presidential candidates as individuals represent a party platform and with the help and support of their affiliated political parties, they carry out campaigns and try to convince voters to cast their ballot for them [1]. Many dynamical models have been introduced to study the dynamics of political parties and political figures in response to the spread of voters, such as in [2], [3], and [4]. In this article, a non-linear model of democratic elections will be constructed based on simple epidemiological model approach.

Many non-linear problems in applied mathematics are difficult to be solved analytically. The homotopy analysis method (HAM) is an analytical technique used to solve non-linear differential equations by using homotopy to generate a series of convergent linear equations from a non-linear one [5]. There are two operators used in this method, linear and non-linear operators. Non-linear operator is stated based on the form of the non-linear differential equation. In order to apply HAM, we need to construct the homotopy equation. This homotopy equation needs the auxiliary parameter for controlling the convergence of solution. The HAM gives the solution in the form of power series.

The HAM has been applied to solve many types of non-linear problems, such as in fluid mechanics [6], oscillations [7], Blasius viscous flow problems [8], and American put options [9]. In this paper, this method will be applied in finding the analytic approximate solutions of the democratic elections model. In the next section, the model construction will be discussed, followed by basic idea of HAM in the third section and application of HAM to find the solutions of the democratic elections model in

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the fourth section. Some numerical simulation with varies auxiliary parameter will be performed in the fifth section and the paper will close with conclusions in the last section.

2. Model Construction

In this section, the democratic elections model construction to understand how voters spread will be described. Let T be the population of all voters considered in the system, separated into three classes (sub-populations), i.e neutral sub-population $V(t)$, supportive sub-population $S(t)$, and apathetic (undecided) sub-population $A(t)$. We assume there are no natural exits from all classes in the system as a result of death or moving. Also, in each sub-population all voters are the same, i.e. there is homogeneity within each class of voters.

It is considered that the individuals of voters are neutral to political figure or political parties. The voters of supportive sub-population S contact the individuals of sub-population V and try to convince them to support their figure or parties with constant rate β. The supportive individuals move into the apathetic individuals with boredom rate γ. Thus, the number of individuals in V class may decide to support the figure or parties is βVS and the number of individuals enter to A class is γS. The transition diagram of the democratic elections model is illustrated in Figure 1.

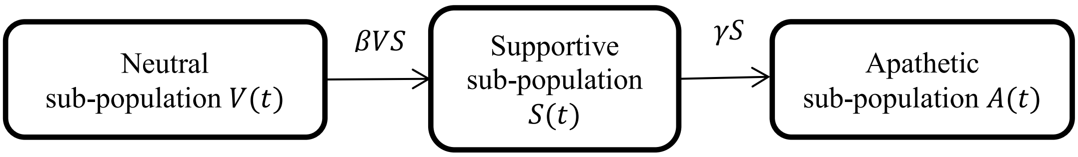


Figure 1. Transition diagram of the democratic elections model.

Based on the above assumptions, the democratic elections model to explain the spread of voters is mathematically described by:

$$\left.\begin{aligned}\frac{dV(t)}{dt} &= -\beta V(t)S(t)\\ \frac{dS(t)}{dt} &= \beta V(t)S(t) - \gamma S(t)\\ \frac{dA(t)}{dt} &= \gamma S(t)\end{aligned}\right\} \quad (1)$$

with initial value condition of neutral sub-population, supportive sub-population, and apathetic sub-population are T_V, T_S, and T_A respectively. It can be seen from model (1) that $\frac{dT}{dt} = 0$, it means total voters population is constant.

Table 1. Variables and parameters of the democratic elections model.

Symbol	Description
$V(t)$	Neutral sub-population
$S(t)$	Supportive sub-population
$A(t)$	Apathetic sub-population
T	Total voters population, $T = V(t) + S(t) + A(t)$
β	Transition rate of $V(t)$ into $S(t)$
γ	Boredom rate of $S(t)$ into $A(t)$

All variables and parameters in Table 1 are non-negative.

3. Method

Derivation of HAM for solving non-linear differential equation will be discussed here briefly. The method was proposed first time by Liao in 1992 and modified in 1997 to include the auxiliary parameter. HAM is independent of any small/large physical parameters at all. Unlike all other analytical techniques, HAM provides us a convenient way to guarantee the convergence of solution series. Also, HAM provides us extremely large freedom to choose the auxiliary linear operator and base functions [10].

Consider the following general non-linear equation:

$$\aleph[f(t)] = 0 \tag{2}$$

where $\aleph$ is any non-linear operator, $f(t)$ is unknown function, and t is the independent variable with $f_0(t)$ denote an initial guess of the exact solution $f(t)$. Construct the homotopy equation $\widehat{H}$:

$$\widehat{H}[\phi(t;r); f_0(t), H(t), h, r] = (1-r)\mathcal{L}[\phi(t;r) - f_0(t)] - rhH(t)\aleph[\phi(t;r)] \tag{3}$$

where $r \in [0,1]$ is the embedding parameter, h is a non-zero auxiliary parameter, $H(t)$ is a non-zero auxiliary function, $f_0(t)$ is an initial approximation of $f(t)$ which satisfies the initial conditions, $\phi(t;r)$ is a function which also must satisfy the initial conditions, and $\mathcal{L}$ is an auxiliary linear operator. The parameter h allows control over the convergence of the series. Because HAM is based off the concept of homotopy, we have freedom to choose $f_0(t)$, $\mathcal{L}$, h, and $H(t)$. Setting the homotopy equation (3) to zero, we have:

$$(1-r)\mathcal{L}[\phi(t;r) - f_0(t)] = rhH(t)\aleph[\phi(t;r)] \tag{4}$$

Equation (4) is often called the deformation equation zero-order.
When $r = 0$, it follows that:

$$\mathcal{L}[\phi(t;0) - f_0(t)] = 0$$

We know $\mathcal{L}[f(t)] = 0$ when $f(t) = 0$, then:

$$\phi(t;0) = f_0(t) \tag{5}$$

Similarly when $r = 1$, the equation (4) becomes:

$$\aleph[\phi(t;1)] = 0$$

Since $\phi(t;r)$ must satisfy the initial conditions of the equation, it follows that:

$$\phi(t;1) = f(t) \tag{6}$$

According to equations (5) and (6), $\phi(t;r)$ varies continuously from $f_0(t)$ to $f(t)$ as parameter r increases from 0 to 1. Define linear approximation k^{th}-order $f_k(t)$ as follows:

$$f_k(t) = \frac{1}{k!}\frac{\partial^k \phi(t;r)}{\partial r^k}\bigg|_{r=0} \tag{7}$$

Using Taylor power series with respect to r, $\phi(t;r)$ can be written as:

$$\phi(t;r) = \phi(t;0) + \sum_{k=1}^{\infty}\left(\frac{1}{k!}\frac{\partial^k \phi(t;r)}{\partial r^k}\bigg|_{r=0}\right) r^k \tag{8}$$

The equation (8) exists for all $r \in [0,1]$ and the k^{th}-derivative $\frac{\partial^k \phi(t;r)}{\partial r^k}$ exists for $k = 1,2,3, \ldots$
If $\mathcal{L}$, $f_0(t)$, h, and H are properly chosen such that the series (8) converges at $r = 1$,

$$\phi(t;1) = f_0(t) + \sum_{k=1}^{\infty} f_k(t)$$

To find an expression for $f_k(t)$, differentiating the equation (4) k times with respect to r, dividing by $k!$, and setting $r = 0$, we have the deformation equation k^{th}-order:

$$\mathcal{L}\big(f_k(t) - \chi_k f_{k-1}(t)\big) = hH(t)\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph[\phi(t;r)]}{\partial r^{k-1}}\bigg|_{r=0} \tag{9}$$

where

$$\chi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases}$$

Equation (9) can be rearranged as:

$$\mathcal{L}\big(f_k(t)\big) = \chi_k f_{k-1}(t) + \mathcal{L}^{-1}\left\{hH(t)\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph[\phi(t;r)]}{\partial r^{k-1}}\bigg|_{r=0}\right\} \tag{10}$$

where $\mathcal{L}^{-1}$ is the inverse of the linear operator. Using the equation (10), the analytic approximate solution can be found for any non-linear operator $\aleph$.

IOP Conf. Series: Journal of Physics: Conf. Series **1320** (2019) 012024 doi:10.1088/1742-6596/1320/1/012024

4. Application of HAM in Finding The Solutions of The Democratic Elections Model

In this section, application of HAM in finding the solutions of the democratic elections model will be performed. To apply the method to the model (1), we choose T_V, T_S, and T_A as initial value approximations of $V(t), S(t),$ and $A(t)$ respectively. The method is based on a kind of continuous mappings:

$$V(t) \to \psi_1(t;r), \qquad S(t) \to \psi_2(t;r), \qquad A(t) \to \psi_3(t;r)$$

such that, as r increases from 0 to 1, $\psi_i(t;r)$ varies from the initial value approximation to the exact solution. Choose auxiliary linear operators as:

$$\mathcal{L}_i[\psi_i(t;r)] = \frac{\partial \psi_i(t;r)}{\partial t}, i = 1,2,3$$

with the property

$$\mathcal{L}_i[C_i] = 0$$

where C_i are integral constants. Define the non-linear operators:

$$\aleph_1[\psi_i(t;r)] = \frac{\partial \psi_i(t;r)}{\partial t} + \beta\psi_1(t;r)\psi_2(t;r)$$

$$\aleph_2[\psi_i(t;r)] = \frac{\partial \psi_i(t;r)}{\partial t} - \beta\psi_1(t;r)\psi_2(t;r) + \gamma\psi_2(t;r)$$

$$\aleph_3[\psi_i(t;r)] = \frac{\partial \psi_i(t;r)}{\partial t} - \gamma\psi_2(t;r)$$

Using the parameter r, we construct equations:

$$(1-r)\mathcal{L}[\psi_1(t;r) - T_V] = rh_1H_1(t)\aleph_1[\psi_1(t;r)]$$
$$(1-r)\mathcal{L}[\psi_2(t;r) - T_S] = rh_2H_2(t)\aleph_2[\psi_2(t;r)]$$
$$(1-r)\mathcal{L}[\psi_3(t;r) - T_A] = rh_3H_3(t)\aleph_3[\psi_3(t;r)]$$

subject to the initial conditions:

$$\psi_1(0;r) = T_V, \psi_2(0;r) = T_S, \psi_3(0;r) = T_A$$

Expand $\psi_i(t;r)$ by Taylor series of the parameter r:

$$\psi_1(t;r) = T_V + \sum_{k=1}^{\infty} V_k(t)\, r^k,$$

$$\psi_2(t;r) = T_S + \sum_{k=1}^{\infty} S_k(t)\, r^k,$$

$$\psi_3(t;r) = T_A + \sum_{k=1}^{\infty} A_k(t)\, r^k$$

where

$$V_k(t) = \frac{1}{k!}\frac{\partial^k \psi_1(t;r)}{\partial r^k}\bigg|_{r=0},$$

$$S_k(t) = \frac{1}{k!}\frac{\partial^k \psi_2(t;r)}{\partial r^k}\bigg|_{r=0},$$

$$A_k(t) = \frac{1}{k!}\frac{\partial^k \psi_3(t;r)}{\partial r^k}\bigg|_{r=0}$$

From the deformation equations k^{th}-order, we have:

$$\mathcal{L}[V_k(t) - \chi_k V_{k-1}(t)] = h_1H_1(t)\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_1[\psi_1(t;r)]}{\partial r^{k-1}},$$

$$\mathcal{L}[S_k(t) - \chi_k S_{k-1}(t)] = h_2H_2(t)\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_2[\psi_2(t;r)]}{\partial r^{k-1}},$$

$$\mathcal{L}[A_k(t) - \chi_k A_{k-1}(t)] = h_3H_3(t)\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_3[\psi_3(t;r)]}{\partial r^{k-1}}$$

where

$$V_k(0) = 0, S_k(0) = 0, A_k(0) = 0$$

Choosing $h_i = -1$ and $H_i(t) = -1$, for $i = 1,2,3$. The deformation equations k^{th}-order of the model for $k \geq 1$ are as follows.

For $V_k(t)$:

$$\mathcal{L}[V_k(t) - \chi_k V_{k-1}(t)] = -\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_1[\psi_1(t;r)]}{\partial r^{k-1}}$$

$$V_k(t) - \chi_k V_{k-1}(t) = -\int_0^t \frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_1[\psi_1(t;r)]}{\partial r^{k-1}} d\tau$$

$$V_k(t) = \chi_k V_{k-1}(t) - \int_0^t \left[V'_{k-1}(\tau) + \beta \sum_{l=0}^{k-1} V_l(\tau) S_{k-1-l}(\tau)\right] d\tau$$

For $S_k(t)$:

$$\mathcal{L}[S_k(t) - \chi_k S_{k-1}(t)] = -\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_2[\psi_2(t;r)]}{\partial r^{k-1}}$$

$$S_k(t) - \chi_k S_{k-1}(t) = -\int_0^t \frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_2[\psi_2(t;r)]}{\partial r^{k-1}} d\tau$$

$$S_k(t) = \chi_k S_{k-1}(t) - \int_0^t \left[S'_{k-1}(\tau) - \beta \sum_{l=0}^{k-1} V_l(\tau) S_{k-1-l}(\tau) + \gamma S_{k-1}(\tau)\right] d\tau$$

For $A_k(t)$:

$$\mathcal{L}[A_k(t) - \chi_k A_{k-1}(t)] = -\frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_3[\psi_3(t;r)]}{\partial r^{k-1}}$$

$$A_k(t) - \chi_k A_{k-1}(t) = -\int_0^t \frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph_3[\psi_3(t;r)]}{\partial r^{k-1}} d\tau$$

$$A_k(t) = \chi_k A_{k-1}(t) - \int_0^t [A'_{k-1}(\tau) - \gamma S_{k-1}(\tau)] d\tau$$

The solutions of the model can be expressed as:

$$V(t) = \sum_{k=0}^{\infty} V_k(t),$$

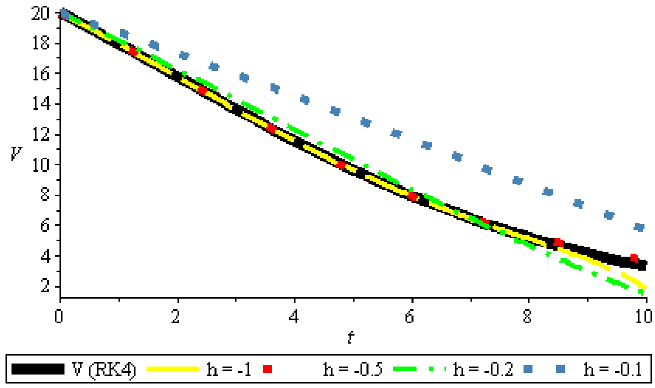
$$S(t) = \sum_{k=0}^{\infty} S_k(t),$$

$$A(t) = \sum_{k=0}^{\infty} A_k(t)$$

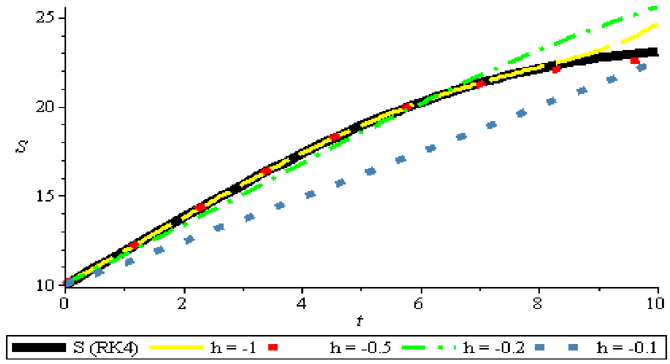
which are valid where ever the solutions converges.

5. Comparison The Solutions of Model by HAM with Numerical Solutions

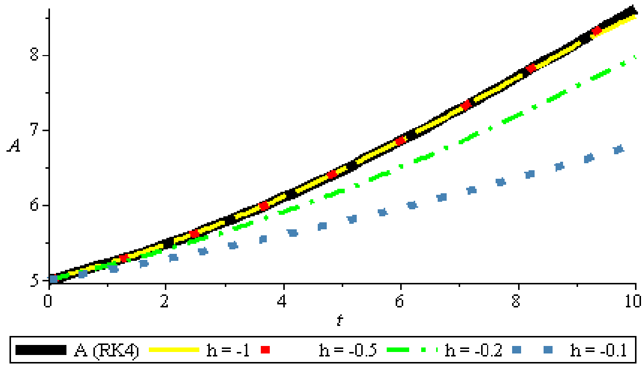
In this section, the democratic elections model is solved numerically using Runge-Kutta 4th order method to investigate the accuracy and effectiveness of HAM. The method is used for calculating numerical solutions for nonlinear democratic elections model. The plots show the dynamical behavior of each sub-population.



(a)



(b)



(c)

Figure 2. Solutions of the democratic elections model for $H(t) = -1$ and different $h(t)$; (a) $V(t)$, (b) $S(t)$, and (c) $A(t)$ with initial conditions $T_V = 20, T_S = 10, T_A = 5$ and parameter $\beta = 0.01, \gamma = 0.02$.

The solutions of tenth terms approximations produced by HAM for $V(t), S(t)$, and $A(t)$ with varies auxiliary parameter are compared with the numerical solutions in Figures 2(a)-2(c). From Figure 2, the number of neutral individuals always decreases, while the number of supportive individuals and the number of apathetic individuals always increases.

Table 2. The solutions of the democratic elections model based on HAM ($h = -1$) and the absolute error compared with numerical method.

t	V_{HAM}	$\lvert V_{RK4}-V_{HAM}\rvert$	S_{HAM}	$\lvert S_{RK4}-S_{HAM}\rvert$	A_{HAM}	$\lvert A_{RK4}-A_{HAM}\rvert$
1	17.9314	0	11.8502	0	5.2184	0
2	15.7772	0	13.7485	0	5.4743	0
3	13.6223	0	15.6096	0	5.7680	0
4	11.5511	0.0001	17.3510	0.0001	6.0979	0
5	9.6344	0.0002	18.9049	0.0001	6.4608	0
6	7.9201	0.0004	20.2278	0.0008	6.8521	0.0005
7	6.4235	0.0106	21.3105	0.0129	7.2660	0.0022
8	5.0998	0.0777	22.2064	0.0867	7.6938	0.0090
9	3.7581	0.3790	23.1214	0.4100	8.1205	0.0310
10	1.8389	1.4507	24.6440	1.5435	8.5171	0.0928

From Table 2, we can see the absolute error for solutions of the democratic elections model given by both methods in four decimal places from $t = 1$ to $t = 10$. The convergence control parameter is chosen as $h = -1$ to show the accuracy of the solutions and it gives better results than the other value in $[-1,1] - \{0\}$.

6. Conclusions

In this paper, a simple mathematical model to demonstrate the spread of voters population in a closed population is constructed. The analytic approximation technique based on HAM has been applied for solving the democratic elections non-linear model. From the numerical simulation, it is shown that convergence of the solutions based on HAM depend on choosing the appropriate auxiliary parameter. It can be seen that these solutions are convergent rapidly. The solutions of the model obtained from HAM are good as compared to numerical method.

Acknowledgements

I would like to thank all the reviewers for all suggestions and comments for this paper. I would also like to thank UNPAR Monodisiplin Research Grant 2018.

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