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Total Edge Irregularity Strength of Arithmetic Book Graphs

Lucia Ratnasari[1,2], **Sri Wahyuni**[1], **Yeni Susanti**[1], **Diah Junia Eksi Palupi**[1] **and Budi Surodjo**[1]

[1] Department of Mathematics Universitas Gadjah Mada, Yogyakarta Indonesia
[2] Department of Mathematics Universitas Diponegoro, Semarang Indonesia

E-mail: `ratnasari.lucia@gmail.com`

Abstract. For any simple undirected graph $G(V,E)$, a map $f : V \cup E \to \{1, 2, \ldots, k\}$ such that for any different edges xy and $x'y'$ their weights are distinct is called an edge irregular total k-labeling. The weight of edge xy is defined as the sum of edge label of xy, vertex label of x and vertex label of y. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G and is denoted by $tes(G)$. In this paper, we determine the exact value of the total edge irregularity strength of odd arithmetic book graph $B_n(C_{3,5,7,\ldots 2n+1})$ and even arithmetic book graph $B_n(C_{4,6,8,\ldots,2n+2})$ of n sheets. We found that the tes of odd arithmetic book graph $B_n(C_{3,5,7,\ldots,2n+1})$ of n sheets is equal to the ceiling function of $\frac{n^2+n+3}{3}$ and the tes of even arithmetic book graph $B_n(C_{4,6,8,\ldots,2n+2})$ is equal to the ceiling function of $\frac{n^2+2n+3}{3}$.

1. Introduction

Let $G = G(V,E)$ be a connected, simple and undirected graph with vertex set V and edge set E. A labeling of a graph is any mapping that sends some sets of graph elements to a set of numbers (usually positive integers). If the domain is the vertex set V or the edge set E, the labeling is called respectively vertex labeling or edge labeling. If the domain is $V \cup E$ then we call the labeling as total labeling.

In [1], Chartrand et al. introduced edge irregular k-labeling δ of a graph $G(V,E)$. An edge irregular k-labeling is a map $\delta : E \to \{1, 2, \ldots, k\}$ such that $\omega_\delta(x) \neq \omega_\delta(y)$ for every two distinct vertices $x, y \in V$ where $\omega_\delta(x)$ is defined as $\omega_\delta(x) = \Sigma\delta(xy)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G and is denoted by $s(G)$.

Motivated by that labeling, Bača et al. in [2] introduced an edge irregular total labeling of graphs. An edge irregular total k-labeling on graph $G(V,E)$ is a map $f : V \cup E \to \{1, 2, \ldots, k\}$ such that for any two different edges xy and $x'y'$, their weights $f(x) + f(xy) + f(y)$ and $f(x') + f(x'y') + f(y')$ are distinct. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G and is denoted by $tes(G)$. In [2] it is given a lower bound on the total edge irregularity strength by the form $tes(G) \geq max\{\lceil\frac{|E|+2}{3}\rceil, \lceil\frac{\Delta(G)+1}{2}\rceil\}$ where $\Delta(G)$ is the maximum vertex degree of G. The authors of [2] also found the total edge irregularity strength of path graphs, cycle graphs, star graphs, complete graphs, wheel graphs and friendship graphs.

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Some results on the exact value of total edge irregularity strength for some classes of graphs have been also investigated, for instance, in [3] for complete graphs and complete bipartite graphs, in [4] for generalized petersen graphs, in [5] for generalized prism, in [6] for generalized web graphs and related graphs, in [7] for helm and sun graphs, in [8] for disjoint union of wheel graphs, in [9] for corona product of cycles with isolated vertices, in [10] for centralized uniform theta graphs, in [11] for uniform theta graphs and in [12] for ladder related graphs.

In this paper we investigate the total edge irregularity strength of odd arithmetic book graphs, $B_n(C_3, 5, 7, ..., 2n+1)$ and even arithmetic book graphs $B_n(C_{4,6,8,...,2n+2})$ of n sheets. We prove the exact values of the total edge irregularity strength of these graphs.

2. Main Result

In this section we focus to determine the total edge irregularity strength of odd arithmetic book graphs and even arithmetic book graphs.

2.1. Odd Arithmetic Book Graphs

We start this section with the definition of odd arithmetic book graphs as follows:

Definition 2.1 Given cycles C_{2p+1}, $p = 1, 2, ..., n$ with

$$V(C_{2p+1}) = \{u, v, x_{ij} | i = p,\ j = 1, 2, ..., 2p-1\}.$$

A graph obtained from C_{2p+1}, $p = 1, 2, ..., n$ by joining edge uv is called odd arithmetic book graph of n sheets denoted by $B_n(C_3, 5, ..., 2n+1)$. Thus the vertex set and the edge set are

$$V(B_n(C_{3,5,7,...,2n+1})) = \{u, x_{i,j}, v :\ i = 1, 2, ..., n, j = 1, 2, ...2n-1\}$$

and

$$E(B_n(C_{3,5,7,...,2n+1})) = \{uv, ux_{i,1}, x_{i,j}x_{i+1,j+1}, x_{i,2n-1}v :\ i = 1, 2, ...n, j = 1, 2, ..., 2n-2\}.$$

Example 2.2 *Figure 1 shows an odd arithmetic book graph with three sheets i.e. C_3, C_5 and C_7 denoted by $B_3(C_{3,5,7})$.*

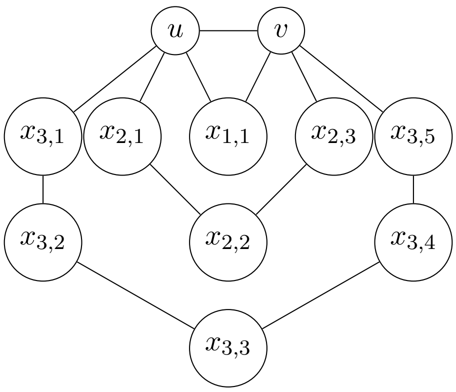

Figure 1. An Odd Arithmetic Book Graph $B_3(C_{3,5,7})$

The first main result of this paper is the following theorem.

Theorem 2.3 *Let $B_n(C_{3,5,7,\ldots,2n+1})$ be an odd arithmetic book graph with n sheets. Then $tes(B_n(C_{3,5,7,\ldots,2n+1})) = \lceil \frac{n^2+n+3}{3} \rceil$.*

Proof: Any odd arithmetic book graph $B_n(C_{3,5,7,\ldots,2n+1})$ has maximum degree $\Delta(B_n(C_{3,5,7,\ldots,2n+1})) = n+1$. Therefore, by $tes(G) \geq max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ [2] we have $tes(B_n(C_{3,5,7,\ldots,2n+1})) \geq \lceil \frac{n^2+n+3}{3} \rceil$. To prove that $k = \lceil \frac{n^2+n+3}{3} \rceil$ is the tes, it is sufficient to show the existence of an edge irregular total k-labeling.

For $n = 3$**,** we have $k = \lceil \frac{3^2+3+3}{3} \rceil = 5$. We can label vertices and edges as shown in the Figure 2.

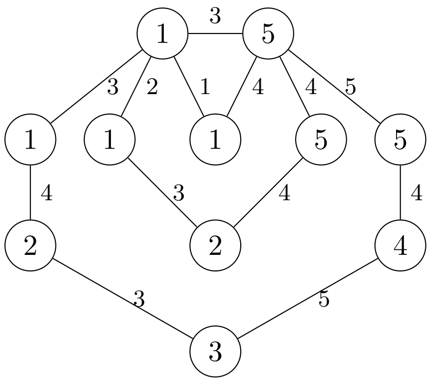

Figure 2. Total edge irregular labeling for $B_3(C_{3,5,7})$

The labeling gives weights of the edges as follows:

$\omega(ux_{1,1}) = 3$, $\omega(ux_{2,1}) = 4$, $\omega(ux_{3,1}) = 5$, $\omega(x_{2,1}, x_{2,2}) = 6$,
$\omega(x_{3,1}, x_{3,2}) = 7$, $\omega(x_{3,2}, x_{3,3}) = 8$, $\omega(uv) = 9$, $\omega(x_{1,1}, v) = 10$,
$\omega(x_{2,2}, x_{2,3}) = 11$, $\omega(x_{3,3}, x_{3,4}) = 12$, $\omega(x_{3,4}, x_{3,5}) = 13$, $\omega(x_{2,3}, v) = 14$,
$\omega(x_{3,5}, v) = 15$.

Thus we have a total edge irregular k-labeling for $B_3(C_{3,5,7})$ with $k = \lceil \frac{3^2+3+3}{3} \rceil = 5$.

For $n = 2$ **and** $n \geq 4$**,** let $k = \lceil \frac{n^2+n+3}{3} \rceil$, we construct the function f as follows:

$$
\begin{array}{lcll}
f(u) & = & 1 & \\
f(v)=f(x_{i,j}) & = & k & 2 \leq i \leq n, \;\; j = 2i-1 \\
f(x_{i,j}) & = & \lceil \frac{j^2+j+3}{6} \rceil, & 1 \leq i \leq n, \;\; 1 \leq j \leq \lceil \frac{2i-1}{2} \rceil \\
f(x_{i,j}) & = & k - \lceil \frac{(2i-j)^2+(2i-j)+3}{6} \rceil, & 3 \leq i \leq n, \;\; \lceil \frac{2i-1}{2} \rceil + 1 \leq j \leq 2i-2 \\
f(uv) & = & \lceil \frac{k}{2} \rceil, & \\
f(x_{1,1}v) & = & \lceil \frac{k}{2} \rceil + 1, & \\
f(ux_{i,1}) & = & i, & 1 \leq i \leq n \\
f(x_{i,1}x_{i,2}) & = & n+i-2, & 2 \leq i \leq n \\
f(x_{i,j-1}x_{i,j}) & = & (j-1)n + i - (j-1)^2 - 2, & 3 \leq i \leq n, \;\; 3 \leq j \leq i \\
f(x_{2,2}x_{2,3}) & = & \lceil \frac{k}{2} \rceil + 1 &
\end{array}
$$

$$
\begin{array}{llll}
f(x_{i,i}x_{i,i+1}) & = & \lceil\frac{k}{2}\rceil + \lceil\frac{2(i+1)}{3}\rceil, & 3 \le i \le n \\
f(x_{i,j-1}x_{i,j}) & = & 2\lceil\frac{k}{2}\rceil - (2i-j)n + i + (j-1)^2 + 3, & 3 \le i \le n, \quad i+1 \le j \le 2i-2 \\
f(x_{i,2i-1}v) & = & 2\lceil\frac{k}{2}\rceil - n + i - 1, & 2 \le i \le n
\end{array}
$$

This labeling gives weights of the edges as follows:

$$
\begin{array}{llll}
\omega(uv) & = & 1 + k + \lceil\frac{k}{2}\rceil, & \\
\omega(ux_{i,1}) & = & i + 2, & 1 \le i \le n \\
\omega(x_{2,2}x_{2,3}) & = & \lceil\frac{k}{2}\rceil + 6 & \\
\omega(x_{i,1}x_{i,2}) & = & n + i + 1, & 2 \le i \le n \\
\omega(x_{i,j-1}x_{i,j}) & = & \lceil\frac{j^2-j+3}{6}\rceil + \lceil\frac{j^2+j+3}{6}\rceil + (j-1)n + i - j^2 + & 3 \le i \le n, \quad 3 \le j \le i \\
 & & 2j - 3, & \\
\omega(x_{i,j-1}x_{i,j}) & = & 2k - \lceil\frac{4i^2-4ij+j^2+6i-3j+5}{6}\rceil + i + (j-1)^2 + & 3 \le i \le n, \quad i+1 \le j \le 2i-1 \\
 & & 3 - \lceil\frac{4i^2-4ij+j^2+2i-j+3}{6}\rceil + 2\lceil\frac{k}{2}\rceil - (2i-j)n, & \\
\omega(x_{i,2i-1}v) & = & 2k + 2\lceil\frac{k}{2}\rceil n - n + i - 2, & 1 \le i \le n
\end{array}
$$

The total labeling f as constructed above is indeed an edge irregular total k-labeling with $k = \lceil\frac{n^2+n+3}{3}\rceil$. This completes the proof ■

Example 2.4 *Figure 3 shows the total edge irregular k-labeling for an odd arithmetic book graph $B_4(C_{3,5,7,9})$ with $k = \lceil\frac{4^2+4+3}{3}\rceil = 8$. This labeling gives weights of the edges as follows:*

$$
\begin{array}{llll}
\omega(ux_{1,1}) = 3, & \omega(ux_{2,1}) = 4, & \omega(ux_{3,1}) = 5, & \omega(ux_{4,1}) = 6, \\
\omega(x_{2,1}, x_{2,2}) = 7, & \omega(x_{3,1}, x_{3,2}) = 8, & \omega(x_{4,1}, x_{4,2}) = 9, & \omega(x_{3,2}, x_{3,3}) = 10, \\
\omega(x_{4,2}, x_{4,3}) = 11, & \omega(x_{4,3}, x_{4,4}) = 12, & \omega(uv) = 13, & \omega(x_{1,1}, v) = 14, \\
\omega(x_{2,2}, x_{2,3}) = 15, & \omega(x_{3,3}, x_{3,4}) = 16, & \omega(x_{4,4}, x_{4,5}) = 17, & \omega(x_{4,5}, x_{4,6}) = 18, \\
\omega(x_{3,4}, x_{3,5}) = 19, & \omega(x_{4,6}, x_{4,7}) = 20, & \omega(x_{2,3}, v) = 21, & \\
\omega(x_{3,5}, v) = 22, & \omega(x_{4,7}, v) = 23. & &
\end{array}
$$

We have a total edge irregular k-labeling for $B_4(C_{3,5,7,9})$ with $k = 8$.

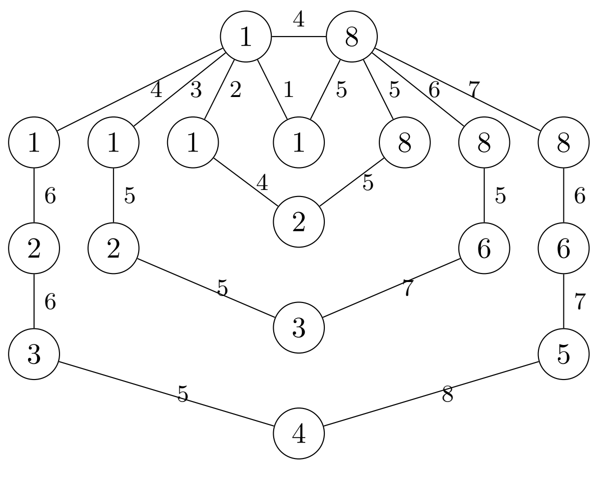

Figure 3. Total edge irregular labeling for $B_4(C_{3,5,7,9})$

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2.2. Even Arithmetic Book Graphs

In this subsection we will define an even arithmetic book graphs and determine the total edge irregularity strength for these graphs.

Definition 2.5 Given cycles C_{2p+2}, $p = 1, 2, ..., n$ with

$$V(C_{2p+2}) = \{u, v, x_{ij} | i = p,\ j = 1, 2, ..., 2p\}.$$

A graph obtained from C_{2p+2}, $p = 1, 2, ..., n$ by joining edge uv is called even arithmetic book graph of n sheets denoted by $B_n(C_{4,6,...,2n+2})$. Thus the vertex set and the edge set are

$$V(B_n(C_{4,6,8,...,2n+2})) = \{u, x_{i,j}, v :\ i = 1, 2, ..., n, j = 1, 2, ...2n\}$$

and

$$E(B_n(C_{4,6,8,...,2n+2})) = \{uv, ux_{i,1}, x_{i,j}x_{i+1,j+1}, x_{i,2n}v :\ i = 1, 2, ...n, j = 1, 2, ..., 2n-1\}.$$

Example 2.6 *Figure 4 shows an even arithmetic book graph with three sheets i.e. C_4, C_6 and C_8 denoted by $B_3(C_{4,6,8})$.*

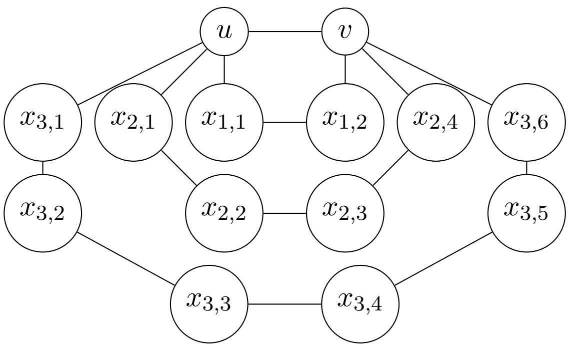

Figure 4. An Even Arithmetic Book Graph $B_3(C_{4,6,8})$

The following theorem determines the exact value of the edge irregularity strength for arbitrary even arithmetic book graph.

Theorem 2.7 *Let $B_n(C_{4,6,8,...,2n+2})$ be an even arithmetic book graph with n sheets. Then $tes(B_n(C_{4,6,8,...,2n+2})) = \lceil \frac{n^2+2n+3}{3} \rceil$.*

Proof: Any even arithmetic book graph $B_n(C_{4,6,8,...,2n+2})$ has maximum degree $\Delta(B_n(C_{4,6,8,...,2n+2})) = n + 1$. Therefore, by $tes(G) \geq max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ we have $tes(B_n(C_{4,6,8,...,m})) \geq \lceil \frac{n^2+2n+3}{3} \rceil$. To prove that $k = \lceil \frac{n^2+2n+3}{3} \rceil$ is the tes, it is sufficient to show the existence of an edge irregular total k-labeling.

For $n = 3$, we have $k = \lceil \frac{3^2+2.3+3}{3} \rceil = 6$. We define the total labeling f for the vertices and edges of $B_3(C_{4,6,8})$ as shown in Figure 5.
The labeling gives weights of the edges as follows:

$\omega(ux_{1,1}) = 3,$ $\omega(ux_{2,1}) = 4,$ $\omega(ux_{3,1}) = 5,$ $\omega(x_{2,1}, x_{2,2}) = 6,$
$\omega(x_{3,1}, x_{3,2}) = 7,$ $\omega(x_{3,2}, x_{3,3}) = 8,$ $\omega(x_{1,1}, x_{1,2}) = 9,$ $\omega(x_{2,2}, x_{2,3}) = 10,$
$\omega(x_{3,3}, x_{3,4}) = 11,$ $\omega(uv) =$ ‘12, $\omega(x_{1,2}, v) = 13,$ $\omega(x_{2,3}, x_{2,4}) = 14,$
$\omega(x_{3,4}, x_{3,5}) = 15,$ $\omega(x_{3,5}, x_{3,6}) = 16,$ $\omega(x_{2,4}, v) = 17,$ $\omega(x_{3,6}, v) = 18.$

We have a total edge irregular k-labeling for $B_3(C_{4,6,8})$ with $k = \lceil \frac{3^2+2.3+3}{3} \rceil = 6$.

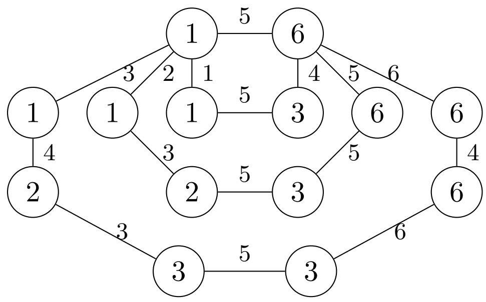

Figure 5. Total edge irregular labeling for $B_3(C_{4,6,8})$

For $n = 4$, we have $k = \lceil \frac{4^2+2.4+3}{3} \rceil = 9$. Then we define the labeling f for the vertices and edges of $B_4(C_{4,6,8,10})$ as shown in the Figure 6.

Total labeling f as defined in Figure 6. gives weights of the edges as follows:
$\omega(ux_{1,1}) = 3,$ $\omega(ux_{2,1}) = 4,$ $\omega(ux_{3,1}) = 5,$ $\omega(u, x_{4,1}) = 6,$
$\omega(x_{2,1}, x_{2,2}) = 7,$ $\omega(x_{3,1}, x_{3,2}) = 8,$ $\omega(x_{4,1}, x_{4,2}) = 9,$ $\omega(x_{3,2}, x_{3,3}) = 10,$
$\omega(x_{4,2}, x_{4,3}) = 11,$ $\omega(x_{4,3}, x_{4,4}) = 12,$ $\omega(x_{1,1}, x_{1,2}) = 13,$ $\omega(x_{2,2}, x_{2,3}) = 14,$
$\omega(x_{3,3}, x_{3,4}) = 15,$ $\omega(x_{4,4}, x_{4,5}) = 16,$ $\omega(uv) = 17,$ $\omega(x_{1,2}, v) = 18,$
$\omega(x_{2,3}, x_{2,4}) = 19,$ $\omega(x_{3,4}, x_{3,5}) = 20,$ $\omega(x_{4,5}, x_{4,6}) = 21,$ $\omega(x_{4,6}, x_{4,7}) = 22,$
$\omega(x_{3,5}, x_{3,6}) = 23,$ $\omega(x_{4,7}, x_{4,8}) = 24,$ $\omega(x_{2,4}, v) = 25,$ $\omega(x_{3,6}, v) = 26,$
$\omega(x_{4,8}, v) = 27.$

Thus we have a total edge irregular k-labeling for $B_4(C_{4,6,8,10})$ with $k = \lceil \frac{4^2+2.4+3}{3} \rceil = 9$.

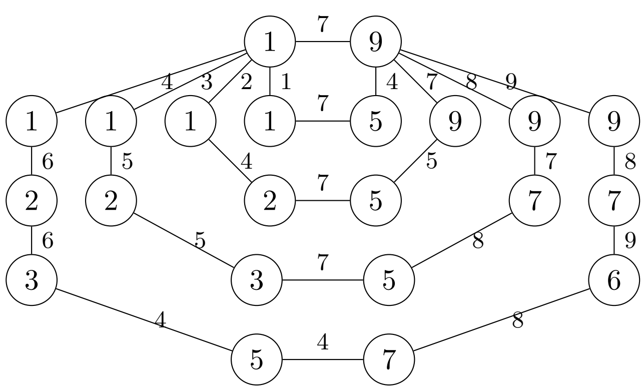

Figure 6. Total edge irregular labeling for $B_4(C_{4,6,8,10})$

For $n = 2$**,** $n \geq 5$.
Let $k = \lceil \frac{n^2+2n+3}{3} \rceil$, we define a total labeling f in the following way:
We distinguish the proof into 2 cases.

Case 1 $n = 0(mod3)$ **and** $n = 1(mod3)$

$$
\begin{array}{lcll}
f(u) & = & 1 & \\
f(v){=}f(x_{i,j}) & = & k & 2 \leq i \leq n, \quad j = 2i \\
 & & & \\
f(x_{i,j}) & = & \lceil \frac{j^2+2j+3}{6} \rceil, & 1 \leq i \leq n, \quad 1 \leq j \leq i \\
f(x_{i,j}) & = & \lceil \frac{k}{2} \rceil, & 1 \leq i \leq n, \quad j = i+1 \\
f(x_{i,j}) & = & k - \lceil \frac{(2i-j+1)^2+2(2i-j+1)+3}{6} \rceil, & 3 \leq i \leq n, \quad i+2 \leq j \leq 2i-1 \\
f(uv) & = & \lceil \frac{k}{2} \rceil + \lceil \frac{n}{2} \rceil, & \\
f(ux_{i,1}) & = & i, & 1 \leq i \leq n \\
f(x_{i,1}x_{i,2}) & = & n+i-2, & 2 \leq i \leq n \\
f(x_{i,j}x_{i,j+1}) & = & (\lfloor \frac{i}{6} \rfloor (2(i(mod6)+1)-1)+(n-1)j+i- & 3 \leq i \leq n, \quad 2 \leq j \leq i-1 \\
 & & j^2 + 6\lfloor \frac{i}{6} \rfloor (\lfloor \frac{i}{6} \rfloor - 1) & \\
f(x_{i,i}x_{i,i+1}) & = & k - \lfloor \frac{n}{2} \rfloor & \\
f(x_{i,i}x_{i,i+1}) & = & \frac{n^2+n}{2} + 2 + i - \lceil \frac{k}{2} \rceil - \lceil \frac{i^2+2i+3}{6} \rceil & 3 \leq i \leq n \\
f(x_{i,i+1}x_{i,i+2}) & = & \frac{n^2+3n}{2} - (k - \lceil \frac{(i-1)^2+2(i-1)+3}{6} \rceil) + 3 + i - \lceil \frac{k}{2} \rceil & 3 \leq i \leq n \\
f(x_{i,j}x_{i,j+1}) & = & n^2 + i - 4 - 2k + \frac{(2i-j+1)^2+4i-2j+5}{6} & 3 \leq i \leq n, \quad j = 2i-1 \\
f(x_{i,j}x_{i,j+1}) & = & 2n^2 + 2(2i-j-1)n + i - (2i-j+1) + & 4 \leq i \leq n, \quad i+2 \leq j \leq 2i-2 \\
 & & \frac{(2i-j+2)^2+(2i-j+1)^2+4(2i-j+3)}{6} - 2k & \\
f(x_{i,j}v) & = & k - n + i & 2 \leq i \leq n, \quad j = 2i
\end{array}
$$

This labeling gives weights of the edges as follows:

$$
\begin{array}{lcll}
\omega(uv) & = & 1 + k + \lceil \frac{k}{2} \rceil + \lceil \frac{n}{2} \rceil, & \\
\omega(ux_{i,1}) & = & 2 + i, & 1 \leq i \leq n \\
\omega(x_{i,1}x_{i,2}) & = & n + i + 1 & 2 \leq i \leq n \\
\omega(x_{i,j}x_{i,j+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \lceil \frac{j^2+4j+6}{6} \rceil + (n-1)j + i - j^2 + & 3 \leq i \leq n, \quad 2 \leq j \leq (i-1) \\
 & & \lfloor \frac{i}{6} \rfloor (2i(mod\ 6)+1) + 6\lfloor \frac{i}{6} \rfloor^2 - 6\lfloor \frac{i}{6} \rfloor, & \\
\omega(x_{i,i}x_{i,i+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \lceil \frac{k}{2} \rceil + k - \lfloor \frac{n}{2} \rfloor, & i = 1,2 \\
\omega(x_{i,i}x_{i,i+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \frac{n^2+2}{2} + 2 + i + \lceil \frac{i^2+2i+3}{6} \rceil, & 3 \leq i \leq n, \quad j = i \\
\omega(x_{i,i+1}x_{i,i+2}) & = & \frac{n^2+3n}{3} + 3 + i - \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil + & 3 \leq i \leq n, \quad j = i+1 \\
 & & \lceil \frac{(i-1)^2+2(i-1)+3}{6} \rceil, & \\
\omega(x_{i,i+2}x_{i,j}) & = & 2n^2 + i + \frac{(2i-j+2)^2+(2i-j+1)^2+4(2i-j+3)}{6} - & 4 \leq i \leq n, \quad i+2 \leq j \leq 2i-2 \\
 & & \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil + 2(2i-j-1)n - (2i- & \\
 & & j+1) - \lceil \frac{2i-(j+1)^2+2((2i-(j+1))+3}{6} \rceil, & \\
\omega(x_{i,2i-1}x_{i,2i}) & = & n^2 + +i - 4 + \frac{(2i-j+1)^2+4i-2j+5}{6} - & 3 \leq i \leq n, j = 2i-1 \\
 & & \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil, & \\
\omega(x_{i,2i}v) & = & 3k - n + i &
\end{array}
$$

case 2 $n = 2(mod3)$

$$
\begin{array}{lcll}
f(u) & = & 1 & \\
f(v)=f(x_{i,j}) & = & k & 2 \le i \le n, \quad j = 2i \\
f(x_{i,j}) & = & \lceil \frac{j^2+2j+3}{6} \rceil, & 1 \le i \le n, \quad 1 \le j \le i \\
f(x_{i,j}) & = & \lceil \frac{k}{2} \rceil, & 1 \le i \le n, \quad j = i+1 \\
f(x_{i,j}) & = & k - \lceil \frac{(2i-j+1)^2+2(2i-j+1)+3}{6} \rceil, & 3 \le i \le n, \quad i+2 \le j \le 2i-1 \\
f(uv) & = & \lceil \frac{k}{2} \rceil + \lfloor \frac{n}{2} \rfloor, & \\
f(ux_{i,1}) & = & i, & 1 \le i \le n \\
f(x_{i,1}x_{i,2}) & = & n+i-2, & 2 \le i \le n \\
f(x_{i,j}x_{i,j+1}) & = & \lfloor \frac{i}{6} \rfloor (2(i(mod6)+1)-1) + (n-1)j + i - & 3 \le i \le n, \quad 2 \le j \le i-1 \\
 & & j^2 + 6\lfloor \frac{i}{6} \rfloor (\lfloor \frac{i}{6} \rfloor - 1) & \\
f(x_{i,i}x_{i,i+1}) & = & k - \lceil \frac{n}{2} \rceil & i = 1,2 \\
f(x_{i,i}x_{i,i+1}) & = & \frac{n^2+n}{2} + 2 + i - \lceil \frac{k}{2} \rceil - \lceil \frac{i^2+2i+3}{6} \rceil & 3 \le i \le n \\
f(x_{i,i+1}x_{i,i+2}) & = & \frac{n^2+3n}{2} - (k - \lceil \frac{(i-1)^2+2(i-1)+3}{6} \rceil) + 3 + i - \lceil \frac{k}{2} \rceil & 3 \le i \le n \\
f(x_{i,j}x_{i,j+1}) & = & n^2 + i - 4 - 2k + \frac{(2i-j+1)^2+4i-2j+5}{6} & 3 \le i \le n, \quad j = 2i-1 \\
f(x_{i,j}x_{i,j+1}) & = & 2n^2 + 2(2i-j-1)n + i - (2i-j+1) + & 4 \le i \le n, \quad i+2 \le j \le 2i-2 \\
 & & \frac{(2i-j+2)^2+(2i-j+1)^2+4(2i-j+3)}{6} - 2k & \\
f(x_{i,j}v) & = & k - n + i - 1 & 2 \le i \le n, \quad j = 2i
\end{array}
$$

This labeling gives weights of the edges as follows:

$$
\begin{array}{lcll}
\omega(uv) & = & 1 + k + \lceil \frac{k}{2} \rceil + \lfloor \frac{n}{2} \rfloor, & \\
\omega(ux_{i,1}) & = & 2 + i, & 1 \le i \le n \\
\omega(x_{i,1}x_{i,2}) & = & n + i + 1, & 2 \le i \le n \\
\omega(x_{i,j}x_{i,j+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \lceil \frac{j^2+4j+6}{6} \rceil + (n-1)j + i - j^2 + & 3 \le i \le n, \quad 2 \le j \le (i-1) \\
 & & \lfloor \frac{i}{6} \rfloor (2i(mod\ 6)+1) + 6\lfloor \frac{i}{6} \rfloor^2 - 6\lfloor \frac{i}{6} \rfloor, & \\
\omega(x_{i,i}x_{i,i+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \lceil \frac{k}{2} \rceil + k - \lceil \frac{n}{2} \rceil, & i = 1,2 \\
\omega(x_{i,i}x_{i,i+1}) & = & \lceil \frac{j^2+2j+3}{6} \rceil + \frac{n^2+2}{2} + 2 + i + \lceil \frac{i^2+2i+3}{6} \rceil, & 3 \le i \le n, \quad j = i \\
\omega(x_{i,i+1}x_{i,i+2}) & = & 4\frac{n^2+3n}{3} + 3 + i - \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil + & 3 \le i \le n, \quad j = i+1 \\
 & & \lceil \frac{(i-1)^2+2(i-1)+3}{6} \rceil, & \\
\omega(x_{i,i+2}x_{i,j}) & = & 2(2i-j-1)n - (2i-j+1) + i + & 4 \le i \le n, \quad i+2 \le j \le 2i-2 \\
 & & 2n^2 + \frac{(2i-j+2)^2+(2i-j+1)^2+4(2i-j+3)}{6} - & \\
 & & \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil - & \\
 & & \lceil \frac{2i-(j+1)^2+2(2i-j-1)+3}{6} \rceil, & \\
\omega(x_{i,2i-1}x_{i,2i}) & = & n^2 + +i - 4 + \frac{(2i-j+1)^2+4i-2j+5}{6} - & 3 \le i \le n, \quad j = 2i-1 \\
 & & \lceil \frac{(2i-j)^2+2(2i-j)+3}{6} \rceil, & \\
\omega(x_{i,2i}v) & = & 3k - n + i - 1 &
\end{array}
$$

We conclude that the weights of all edges are distinct. ■

Example 2.8 *Figure 7 shows the total edge irregular k-labeling for an even arithmetic book graph $B_5(C_{4,6,8,10,12})$. We use formula for $n = 2(mod3)$ in the proof of Theorem 2.7.*
This labeling gives weights of the edge as follows:

$\omega(ux_{1,1}) = 3,$ $\omega(ux_{2,1}) = 4,$ $\omega(ux_{3,1}) = 5,$ $\omega(u, x_{4,1}) = 6$
$\omega(u, x_{5,1}) = 7,$ $\omega(x_{2,1}, x_{2,2}) = 8,$ $\omega(x_{3,1}, x_{3,2}) = 9,$ $\omega(x_{4,1}, x_{4,2}) = 10,$
$\omega(x_{5,1}, x_{5,2}) = 11,$ $\omega(x_{3,2}, x_{3,3}) = 12,$ $\omega(x_{4,2}, x_{4,3}) = 13,$ $\omega(x_{5,2}, x_{5,3}) = 14,$
$\omega(x_{4,3}, x_{4,4}) = 15,$ $\omega(x_{5,3}, x_{5,4}) = 16,$ $\omega(x_{5,4}, x_{5,5}) = 17,$ $\omega(x_{1,1}, x_{1,2}) = 18,$
$\omega(x_{2,2}, x_{2,3}) = 19,$ $\omega(x_{3,3}, x_{3,4}) = 20,$ $\omega(x_{4,4}, x_{4,5}) = 21,$ $\omega(x_{5,5}, x_{5,6}) = 22,$
$\omega(u, v) = 23,$ $\omega(v, x_{1,2}) = 24,$ $\omega(x_{2,3}, x_{2,4}) = 25,$ $\omega(x_{3,4}, x_{3,5}) = 26,$
$\omega(x_{4,5}, x_{4,6}) = 27,$ $\omega(x_{5,6}, x_{5,7}) = 28,$ $\omega(x_{5,7}, x_{5,8}) = 28,$ $\omega(x_{4,6}, x_{4,7}) = 30,$
$\omega(x_{5,8}, x_{5,9}) = 31,$ $\omega(x_{3,5}, x_{3,6}) = 32,$ $\omega(x_{4,7}, x_{4,8}) = 33,$ $\omega(x_{5,9}, x_{5,10}) = 34,$
$\omega(x_{2,4}, v) = 35,$ $\omega(x_{3,6}, v) = 36,$ $\omega(x_{4,8}, v) = 37,$ $\omega(x_{5,10}, v) = 38.$

We have a total edge irregular k-labeling for $B_5(C_{4,6,8,10,12})$ with $k = \lceil \frac{5^2+2.5+3}{3} \rceil = 13.$

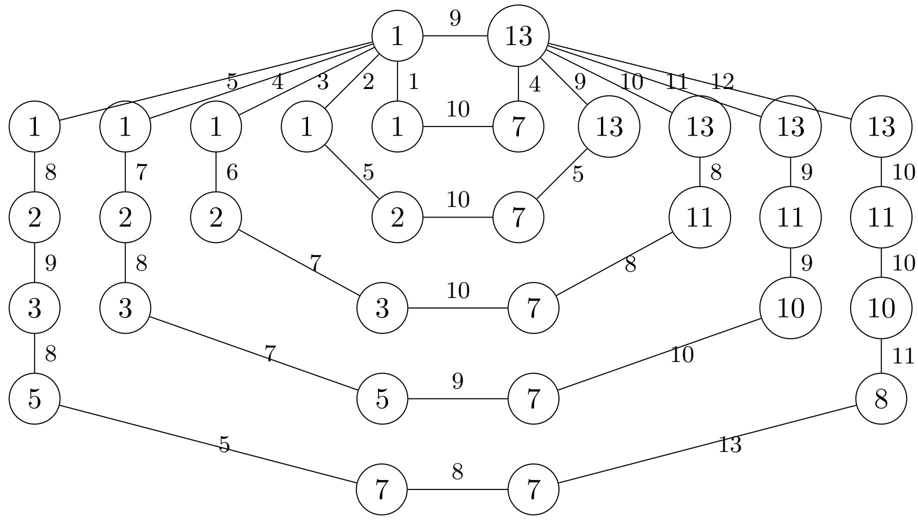

Figure 7. Total edge irregular labeling for $B_5(C_{4,6,8,10,12})$

3. Conclusion

Based on the previous discussion we conclude that the total edge irregularity strength of an odd arithmetic book graph $B_n(C_{3,5,7,\ldots,2n+1})$ of n sheets is $tes(B_n(C_{3,5,7,\ldots,2n+1})) = \lceil \frac{n^2+n+3}{3} \rceil$ and of an even arithmetic book graph $B_n(C_{4,6,8,\ldots,2n+2})$ of n sheets is $tes(B_n(C_{4,6,8,\ldots,2n+2})) = \lceil \frac{n^2+2n+3}{3} \rceil$.

In this research the arithmetic sequence used, is very specific. For the future work, we will use the more general arithmetic sequence for the graphs.

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