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To cite this article: Zainab Esa Abdulnaby and Ahmed E Abdul-Nabi 2019 *J. Phys.: Conf. Ser.* **1294** 032008

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Integral transforms defined by a new fractional class of analytic function in a complex Banach space

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Abstract.In this effort, we define a new class of fractional analytic functions containing functional parameters in the open unit disk. By employing this class, we introduce two types of fractional operators, differential and integral. The fractional differential operator is considered to be in the sense of Ruscheweyh differential operator, while the fractional integral operator is in the sense of Noor integral. The boundedness and compactness in a complex Banach space are discussed. Other studies are illustrated in the sequel.

Keywords: Analytic functions, Hadamard product, Fox- Wright function, norm Banach space.

1. Introduction

Fractional calculus is a major branch of analysis (real and complex) that deals with the possibility of captivating real number powers or complex number powers of operators (differential and integral). It has increased substantial admiration and significance throughout the past four decades, in line for mainly to its established requests and applications in various apparently different and extensive fields and areas of science, medicine and engineering. It does certainly deliver several possibly advantageous apparatuses for explaining and solving differential, integral and differ-integral equations, and numerous other difficulties and problems connecting special functions of mathematical physics as well as their generalizations, modification and extensions in one and more variables (real and complex) (see [2], [13]). The utensils employed contain numerous standard and contemporary nonlinear analysis methods in real and complex, such as fixed point theory, boundedness and compactness techniques. It is beneficial to investigators and researchers, in pure and applied mathematics. The classical integral and derivative are understanding and employing with normal, ordinary and simulated methods. Fractional Calculus is a field of mathematical studies that produces out of the classical definitions of the calculus integral and derivative operators in considerable the similar technique fractional advocates is an extension of advocates with integer value. The theory of geometric function concerns with a special class of analytic functions, which are defined in the open unit disk; such as see [10], the Koebe function of first order

$$f(\mathbf{z}) = \frac{\mathbf{z}}{(1-\mathbf{z})'}$$

and of second order



$$f(z) = \frac{z}{(1-z)^2}.$$

The fractional type of analytic functions is suggested and studied in [12] as follows:

$$f(z) = \frac{z^{\alpha}}{(1-z)^{\alpha'}}$$

with $\alpha = \frac{n+m}{m}$, $n, m \in \mathbb{N}$, where $\alpha = 1$, in the case n = 1.

In this effort, we define a new class of fractional analytic functions \mathbf{F} in the unit disk \mathbb{U} : = $\{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\}$, with two functional parametric power as follows:

$$F(z) = \frac{z^{\mu}}{(1 - z^{\mu})^{\alpha}} = z^{\mu} (\sum_{\substack{n=0\\ i \in z}}^{\infty} \frac{(\alpha)_n}{(n)!} z^{\mu}), \quad z \in \mathbb{U}$$
$$= z + \sum_{\substack{n=2\\ n=2}}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{\mu}$$
(1)

where $0 < \alpha \le 1$ and $\mu \ge 1$, note that the latter takes it value from the following relation; $n + \alpha$

$$\mu := \frac{n+\alpha}{\alpha} \quad n \in \mathbb{N}$$

if $\mu = 1$ in the case n = 0 and $\alpha = 1$, which leads to return to Koebe's function of the first order. Hence, we obtain the formal of fractional analytic function as well as

$$F(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{\mu} .$$

Let \mathscr{A}_{μ} be the class of all analytic functions F in the unit disk \mathbb{U} and take the form

$$F(z) := z + \sum_{n=2}^{\infty} \alpha_n z^{\mu}$$
 (2)

$$(|\alpha_n| \le \frac{(\alpha)_{n-1}}{(n-1)!}; \ 0 < \alpha \le 1; \ n \in \mathbb{N} \setminus \{0,1\}).$$

For $0 < \beta \leq 1$, let $G \in \mathcal{A}_{\mu}$ be a function give by

$$G(z) = z + \sum_{n=2}^{\infty} \beta_n z^{\mu}$$

then the convolution (or Hadamard product) F * G defined by

$$(F*G)(z) = z + \sum_{n=2}^{\infty} \alpha_n \beta_n z^{\mu} ,$$

and

$$(F * G)'(z) = F(z) * G'(z), \quad (|z| < 1)$$

where F given by (1). Let S_{μ} denoted the class of all univalent functions F given by (2). Next, we proceed to define a new operator $\mathcal{D}^{\beta,\mu}: \mathcal{A}_{\mu} \to \mathcal{A}_{\mu}$ by the convolution product of two functions

$$\mathcal{D}^{\beta,\mu}F(z) = \frac{z^{\mu}}{(1-z^{\mu})^{\beta+1}} * F(z)$$

= $z + \sum_{n=2}^{\infty} \frac{(\beta+1)_{n-1}}{(n-1)!} \alpha_n z^{\mu}$. (3)

2nd International Science Conference

IOP Conf. Series: Journal of Physics: Conf. Series 1294 (2019) 032008 doi:10.1088/1742-6596/1294/3/032008

Note that

$$\mathcal{D}^{\mathbb{C},\mu}F(z) = F(z), \quad (z \in \mathbb{U}).$$

We expected that the operator $\mathcal{D}^{\beta,\mu}$ is closer to similarities than the Ruscheweyh differential operator (see [11]).

Next, by using the product of two univalent functions, we aim to define a new integral operator denote by $\mathcal{I}_{\beta,\mu}: \mathcal{A}_{\mu} \to \mathcal{A}_{\mu}$. Let define the functional

$$F_{\beta} = \frac{z^{\mu}}{(1-z^{\mu})^{\beta+1}}, \quad (z \in \mathbb{U}, \mu \ge 1, \beta \ge 1)$$
(4)

such that

$$F_{\beta}(z) * F_{\beta}^{-1}(z) = \frac{z^{\mu}}{1 - z^{\mu}}.$$

Consequently, we receive the integral operator $\mathcal{I}_{\beta,\mu}$ defined by

$$\begin{aligned} \mathcal{I}_{\beta,\mu}F(z) &= F_{\beta}^{-1}(z) * F(z) \\ &= \left[\frac{z^{\mu}}{(1-z^{\mu})^{\beta+1}}\right]^{-1} * F(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu} . \end{aligned}$$
(5)

Clearly, we can note that

$$\mathcal{I}_{\underline{u},\mu}F(\underline{z}) = F(\underline{z}), \quad |\underline{z}| < 1.$$

For $a \ge 1$ and $\mu \ge 1$, then the integral operator $\mathcal{I}_{\mathbb{L},\mu}$ is closed to the Noor integral (see [6]) of the *n*-th order of function $F \in \mathcal{A}_{\mu}$. Corresponding to equation in (5), we have the following conclusion:

$$z\mathcal{I}_{\beta,\mu}F'(z) = z + \sum_{n=2}^{\infty} \frac{\mu(n)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu}$$
(6)

or

$$z\mathcal{I}_{\beta,\mu}F'(z) = z + \sum_{n=2}^{\infty} \mu \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta)} \alpha_n z^{\mu} .$$
⁽⁷⁾

Since we are dealing with the analytic and univalent functions in the open unit disk, so we are concerned to study remarkable geometrical and topological properties under the norm of the complex plane as in the following sections.

2. Geometric properties of $J_{\mu,\mu}$

In this section, we study the geometrical properties of the integral operator $\mathcal{I}_{\beta,\mu}$ of analytic functions F in the class \mathcal{A}_{μ} . First of all, we need the following lemma, stated by Duren.

Lemma 1. (Duren [1]) Let the function $f(z) = \mathcal{A}$ be a starlike, then $|a_n| = n$ for all n = 2. And if the function $f(z) = \mathcal{A}$ is convex, then $|a_n| = 1$ for all n = 2.

Theorem 1. If α 1 and μ 1 achieving the inequality

$$\alpha(\alpha+1)\dots(\alpha+n-2) < n!$$

then $F = S_{\mu}$, and

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$$|\mathcal{I}_{\beta,\mu}F(z)| = (\beta + 1)r^{2\mu} {}_{2-1}[r^{\mu}|^{(3,1)(1,1)}_{(\beta+2,1)}],$$

for all r < 1.

Proof. Directly by the assumption of the theorem, we obtain $|a_n| < n$ consequently, in view of Lemma 1, this implies that **F** is starlike, where

$$a_1 = 1, a_2 = a, a_3 = \frac{a(a+1)}{2!}, \dots, a_n = \frac{a(a+1)\dots(a+n-2)}{(n-1)!}.$$

We proceed to show that the integral operator $\mathcal{I}_{\beta,\mu}$ is bounded by a special function.

$$\begin{aligned} |\mathcal{I}_{\beta,\mu}F(z)| &= |z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu} |\\ &(\beta+1)r^{2\mu} \sum_{n=1}^{\infty} \frac{\Gamma(n+3)\Gamma(n+1)}{\Gamma(\beta+n+2)} \frac{r^{\mu}}{n!}, \quad |z| < r, |\alpha_n| < n\\ &= (\beta+1)r^{2\mu} \sum_{|z|=1}^{\infty} [r^{\mu}|_{(\beta+2,1)}^{(3,1)(1,1)}], \end{aligned}$$

whe \mathbb{Z}_{1} is the well known Fox-Wright function.

Similarly, we have the following result:

Theorem 2. If α 1 and μ 1 achieving the inequality

$$a(a + 1)...(a + n - 2) < (n - 1)!$$

then $F = \mathcal{C}_{\mu}$, and

$$|\mathcal{I}_{\beta,\mu}F(z)| = (\beta + 1)r^{2\mu} \sum_{z=1} [r^{\mu}|^{(2,1)(1,1)}_{(\beta+2,1)}]$$

for all r < 1.

Proof. By the hypotheses of the theorem, we obtain

 $|\alpha_n| < 1$

consequently, in view of Lemma 1, this yields F is convex. We proceed to show that the integral operator $\mathcal{I}_{\beta,\mu}$ is bounded by a special function.

$$\begin{aligned} |\mathcal{I}_{\beta,\mu}F(z)| &= |z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu} |\\ &(\beta+1)r^{2\mu} \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(\beta+n+2)} \frac{r^{\mu}}{n!}, \quad |z| < r, |\alpha_n| < n \\ &= (\beta+1)r^{2\mu} \sum_{|z|=1}^{\infty} [r^{\mu}|_{(\beta+2,1)}^{(2,1)(1,1)}], \quad r < 1. \end{aligned}$$

3. Applications

In this section, we introduce some applications dealing with a complex norm of locally univalent functions of finite order, where the order of a function $f(z) = z + a_1 z + a_2 z^2 + ...$ is known as [13-14]

$$||f||_T = \sup_{z \in U} (1 - |z|^2) |T_f| < , \quad (|z| < 1).$$

where $T_f = f''/f'$ denotes by the pre-Schwarzian derivative of function f. From Becker's theorem, we recall that, if f is an analytic function then $||f||_T = 1$ impales f is a univalent function.

In the following theorem, we consider the class of positive linear operator $\mathcal{I}_{\mu,\mu}$ of analytic and univalent functions defined on the unit disk |z| < 1, endowed with the supremum norm \cdot_T .

Theorem 3. (Univalence Boundedness) Let $F = \mathcal{A}_{\mu}$, if

$$||\mathcal{I}_{\beta,\mu}F||_T = ||F||_T$$

then $\mathcal{I}_{\beta,\mu}F = S_{\mu}$.

Proof. Supposing $F = \mathcal{A}_{\mu}$, we get

$$\begin{aligned} \|\mathcal{I}_{\beta,\mu}F(z)\|_{T} &= \sup_{z \in \mathbb{U}} (1 - |z|^{2}) \left| \frac{z\mathcal{I}_{\beta,\mu}F''(z)}{\mathcal{I}_{\alpha,\mu}F'(z)} \right| \\ &= \sup_{z \in \mathbb{U}} (1 - |z|^{2}) \left| \frac{F_{\beta}^{-1}(z) - zF''(z)}{F_{\beta}^{-1}(z) - F'(z)} \right| \\ &= \sup_{z \in \mathbb{U}} (1 - |z|^{2}) \left| \frac{zF''(z)}{F'(z)} \right|, \end{aligned}$$

then, we have

$$\sup_{z \in \mathbb{U}} (1 - |z|^2) \frac{|z\mathcal{I}_{\beta,\mu}F''(z)|}{\mathcal{I}_{\alpha,\mu}F'(z)} = \frac{|zF''(z)|}{F'(z)}.$$
(8)

By setting the supremum for the last assertion over the unit disk \mathbb{U} , the boundedness of the operator $\mathcal{I}_{B,\mu}F_{\alpha}(\mathbb{Z})$ is satisfied.

Theorem 4. (Compactness) For $F = \mathcal{A}_{\mu}$, then the integral operator $\mathcal{I}_{\vec{p},\mu}F$ is compact in complex norm space.

proof: If $\mathcal{I}_{\beta,\mu}F$ a is compact, then the function F is bounded and by Theorem 3, it is follow that F = B the integral operator $\mathcal{I}_{\beta,\mu}F$ is compact. Let suppose that $\mathcal{I}_{\beta,\mu}F = S$, that $F_m, m = n$ is a sequence in Banach space, and $F_m \to 0$ uniformly on \overline{U} as $m \to \infty$. For every $\varepsilon > 0$, there is $\delta = (0,1)$ such that

$$\frac{1}{(1-|z|^2)} < \varepsilon$$

where $\delta < |\mathbf{z}| < 1$, since δ arbitrary, then

$$\begin{aligned} \left\| \mathcal{I}_{\beta,\mu} F_m \right\|_T &= \sup_{z \in \mathbb{U}} (1 - |z|^2) \left| \frac{z \mathcal{I}_{\beta,\mu} F_m^{\ \prime \prime}}{\mathcal{I}_{\alpha,\mu} F_m^{\ \prime \prime}} \right| \\ &= \sup_{z \in \mathbb{U}} (1 - |z|^2) \left| \frac{F_m^{-1} \quad z F_m^{\ \prime \prime}}{F_m^{-1} \quad F_m^{\ \prime \prime}} \right| \end{aligned}$$

$$= \sup_{z \in \mathbb{U}} (1 - |z|^2) \left| \frac{z F_m^{\ r}}{F_m^{\ r}} \right|$$

$$\in F_m^{\ T}$$
(9)

Since for F_m ! on \overline{U} we get F_m T 0, and that ε is a arbitrary number, by setting m in (9), we have that

$$\left\| \mathcal{I}_{\beta,\mu} F_m \right\|_T = 0$$

Therefore, $\mathcal{I}_{B,\mu}F_{m}$ is compact.

4. Conclusion

We generalized a class of analytic functions (Koebe type), by utilizing the concept of fractional calculus. This class involves the well known geometric functions in the open unit disk. Moreover, by employing the above class, we defined two types of fractional operators, differential and integral. The fractional differential operator is supposed to be in the sense of Ruscheweyh differential operator, while the fractional integral operator is assumed to be in the sense of Noor integral. Some geometrical properties are illustrated for the integral operator such as the starlikeness and convexity. Topological properties are investigated in a complex Banach space, such as the boundedness and compactness.

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