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Algebraic generalization of quantum statistics

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Abstract. Generalized quantum statistics such as para-Bose and para-Fermi statistics are related to the basic classical Lie superalgebras $B(0|n)$ and B_n. We give a quite general definition of "a generalized quantum statistics associated to a Lie superalgebra G". This definition is closely related to a certain $\mathbb{Z}$-grading of G. The generalized quantum statistics is determined by a set of root vectors (the creation and annihilation operators of the statistics) and the set of algebraic relations for these operators. Then we give a complete classification of all generalized quantum statistics associated to the Lie superalgebras A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8, $A(m|n)$, $B(m|n)$, $C(n)$, $D(m|n)$, $G(3)$, $F(4)$ and $D(2,1;\alpha)$.

1. Introduction

Green [1] extended the ordinary Bose and Fermi statistics to parastatistics. Both for para-Bose and para-Fermi operators, the bilinear commutators or anticommutators for bosons and fermions are replaced by trilinear relations. These trilinear relations are closely related to a Lie algebra or Lie superalgebra. For example, a set of $2n$ para-Fermi operators f_i^ξ ($\xi = \pm$, $i = 1, \ldots, n$)

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = \frac{1}{2}(\epsilon-\eta)^2\delta_{kl}f_j^\xi - \frac{1}{2}(\epsilon-\xi)^2\delta_{jl}f_k^\eta,\ \xi,\eta,\epsilon = \pm \text{ or } \pm 1;\ j,k,l = 1,\ldots,n \quad (1)$$

generates the Lie algebra $B_n = so(2n+1)$ [2], whereas a set of $2n$ para-Bose operators b_i^ξ ($\xi = \pm$, $i = 1, \ldots, n$)

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon-\xi)\delta_{jl}b_k^\eta + (\epsilon-\eta)\delta_{kl}b_j^\xi,\ \xi,\eta,\epsilon = \pm \text{ or } \pm 1;\ j,k,l = 1,\ldots,n \quad (2)$$

considered as odd elements, generates the orthosymplectic Lie superalgebra $B(0|n) = osp(1|2n)$ [3]. Therefore parastatistics can be associated with representations of the Lie superalgebras of class B [4]. Then it is natural to expect that new types of generalized quantum statistics (GQS) can be associated with other Lie superalgebras. Note that, for conveniece, we often consider Lie algebras as a subset of Lie superalgebras (namely as Lie superalgebras with even elements only).

Examples of alternative types of GQS have been considered by Palev [5]-[13]. Inspired by the examples of parastatistics and Palev's examples we give a classification of all GQS associated

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with the Lie superalgebras A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8, $A(m|n)$, $B(m|n)$, $C(n)$, $D(m|n)$, $G(3)$, $F(4)$ and $D(2,1;\alpha)$. More details of the classification presented here can be found in the series of papers [14]-[16].

GQS associated to a Lie superalgebra G is related to a $\mathbb{Z}$-grading of G of the type $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$. Such a $\mathbb{Z}$-grading is said to be of length 5 if $G_{\pm 2} \neq 0$; if $G_{\pm 2} = 0$, but $G_{\pm 1} \neq 0$, then the $\mathbb{Z}$-grading is of length 3. These $\mathbb{Z}$-gradings imply that one is dealing with Lie (super)triple systems (in the case of Lie algebras, see [17] and in the case of Lie superalgebras, see [18]). Such systems have also been referred to as "(super)ternary algebras" in [19]. Another mathematical structure, F-Lie algebra, also called ternary algebra for $F = 3$ was introduced in [20]. However the systems considered here are in the sense of [19]. In the last paper, examples and explicit constructions of (super)ternary algebras are given for many simple Lie algebras and Lie superalgebras. In section VII of [19] one notes: "It would be interesting to embark on a complete classification of ternary algebras and superternary algebras and provide a list of all possible constructions of a given Lie (super)algebra from (super)ternary algebras." The results of the present paper provide such a complete classification.

In the next section we give a definition of generalized quantum statistics associated with a Lie superalgebra G and the corresponding creation and annihilation operators. This notion is closely related to gradings of G, and to regular subalgebras of G. Following the definition, we describe the classification method. In the process of investigations Dynkin diagram techniques [21, 22] play a crucial role. In the remaining sections of the paper, the classification results are summarised.

2. Definition and classification method

Let G be a Lie superalgebra [4], with bracket $[\![x,y]\!]$, where (in $U(G)$)

$$[\![x,y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx,$$

if x and y are homogeneous (for a Lie algebra, all elements are even and have degree 0; for a Lie superalgebra, homogeneous elements are even or odd, having degree 0 or 1).

Definition 1 *Let G be a Lie superalgebra, with antilinear anti-involutive mapping ω. A set of $2N$ root vectors $x_i^\pm$ $(i = 1, \ldots, N)$ is called a set of creation and annihilation operators (CAOs) for G if:*

- $\omega(x_i^\pm) = x_i^\mp$,
- $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ *is a $\mathbb{Z}$-grading of G, with $G_{\pm 1} = span\{x_i^\pm,\ i = 1\ldots, N\}$ and $G_{j+k} = [\![G_j, G_k]\!]$.*

The algebraic relations $\mathcal{R}$ satisfied by the operators $x_i^\pm$ are the relations of a generalized quantum statistics (GQS) associated with G.

A consequence of this definition is that G is generated by G_{-1} and G_{+1}, i.e. by the set of CAOs, and since $G_{j+k} = [\![G_j, G_k]\!]$, it follows that

$$G = \mathrm{span}\{x_i^\xi,\ [\![x_i^\xi, x_j^\eta]\!];\quad i,j = 1,\ldots,N,\ \xi,\eta = \pm\}. \tag{3}$$

The second condition in Definition 1 implies that G_0 itself is a subalgebra of G spanned by root vectors of G. So G_0 is a regular subalgebra containing the Cartan subalgebra H of G. By the adjoint action, the remaining G_i's are G_0-modules. This implies the following technique in order to obtain a complete classification of all GQS associated with G:

(i) Determine all regular subalgebras G_0 of G. If not yet contained in G_0, replace G_0 by $G_0 + H$.

(ii) For each regular subalgebra G_0, determine the decomposition of G into simple G_0-modules g_k $(k = 1, 2, \ldots)$.

(iii) Investigate whether there exists a $\mathbb{Z}$-grading of G of the form

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}, \tag{4}$$

where each G_i is either directly a module g_k or else a sum of such modules $g_1 \oplus g_2 \oplus \cdots$, such that $\omega(G_{+i}) = G_{-i}$.

To find regular subalgebras one can use the method of deleting nodes from (extended) Dynkin diagrams [21, 22]. The second stage is straightforward by means of representation theoretical techniques. The third stage requires most of the work: one must try out all possible combinations of the G_0-modules g_k, and see whether it is possible to obtain a grading of the type (4). In this process, if one of the simple G_0-modules g_k is such that $\omega(g_k) = g_k$, then it follows that this module should be part of G_0. In other words, such a case reduces essentially to another case with a larger regular subalgebra.

In the following sections we shall give a summary of the classification process for the simple Lie algebras and basic classical Lie superalgebras.

3. Simple Lie algebras

3.1. The Lie algebra A_n

Let G be the special linear Lie algebra $sl(n+1) \equiv A_n$, consisting of traceless $(n+1) \times (n+1)$ matrices. The Cartan subalgebra H of G is the subspace of diagonal matrices. The root vectors of G are the elements e_{jk} $(j \neq k = 1, \ldots, n+1)$, where e_{jk} is a matrix with zeros everywhere except a 1 on the intersection of row j and column k. The corresponding root is $\epsilon_j - \epsilon_k$, in the usual basis. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. The simple roots, the Dynkin diagram and the extended Dynkin diagram of A_n are given in Table 1.

In order to find regular subalgebras of $G = A_n$, one should delete nodes from the Dynkin diagram of G or from its extended Dynkin diagram.

Step 1. Delete node i from the Dynkin diagram. The corresponding diagram is the Dynkin diagram of $sl(i) \oplus sl(n-i+1)$, so $G_0 = H + sl(i) \oplus sl(n-i+1)$. In this case, there are only two G_0 modules and we can put

$$G_{-1} = \mathrm{span}\{e_{kl};\ k = 1, \ldots, i,\ l = i+1, \ldots, n+1\}, \quad G_{+1} = \omega(G_{-1}). \tag{5}$$

Therefore $sl(n+1)$ has the following grading:

$$sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}, \tag{6}$$

the number of creation and annihilation operators is $N = i(n-i+1)$. Note that the cases i and $n+1-i$ are isomorphic.

The most interesting cases are those with $i = 1$ and $i = 2$, for which we shall explicitly give the relations $\mathcal{R}$ between the CAOs.

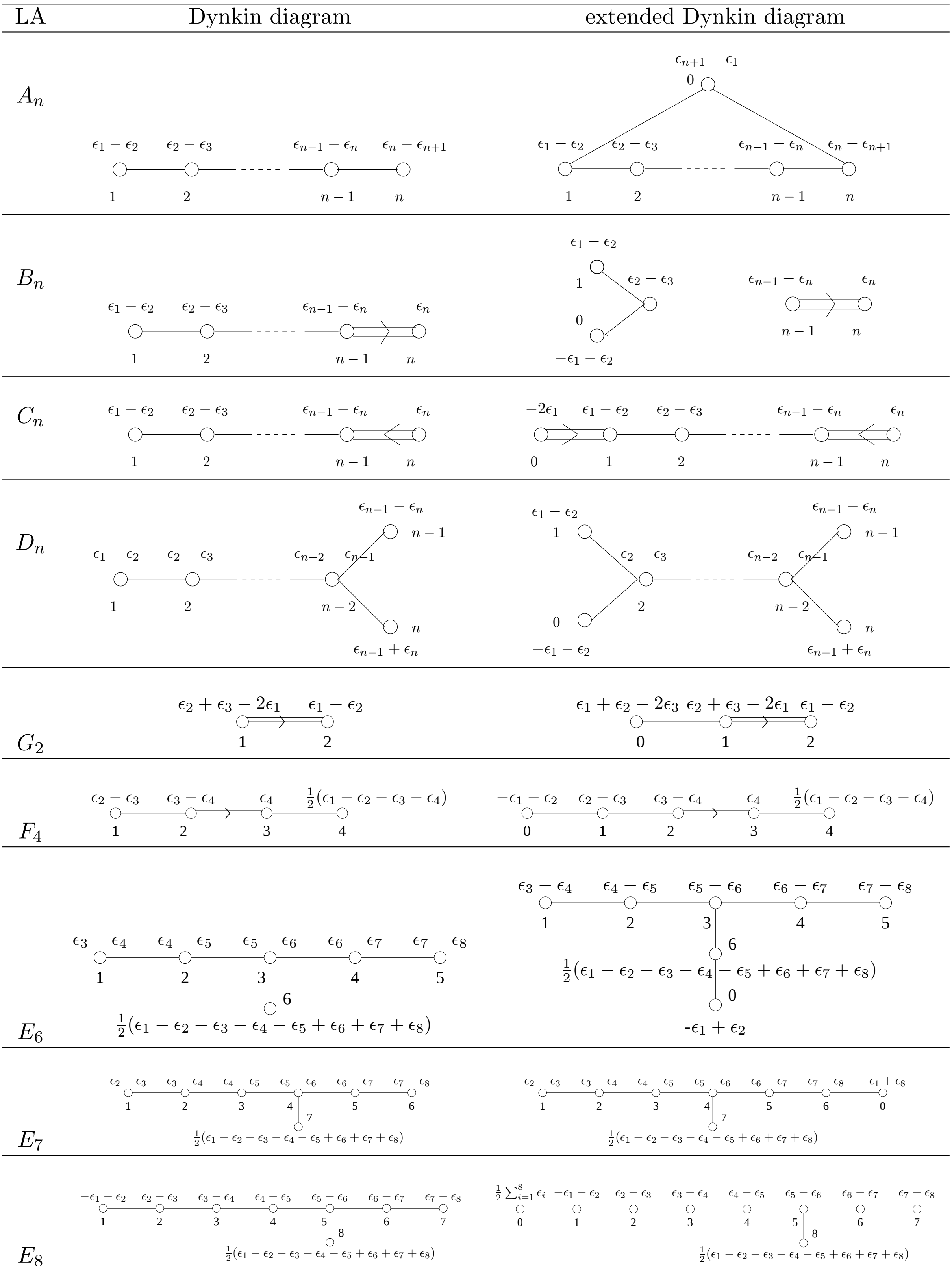
For $i = 1$, $N = n$, the rank of A_n. Putting

$$a_j^- = e_{1,j+1}, \qquad a_j^+ = e_{j+1,1},\ j = 1, \ldots, n, \tag{7}$$

the corresponding relations $\mathcal{R}$ read $(j, k, l = 1, \ldots, n)$:

$$\begin{aligned} &[a_j^+, a_k^+] = [a_j^-, a_k^-] = 0, \\ &[[a_j^+, a_k^-], a_l^+] = \delta_{jk} a_l^+ + \delta_{kl} a_j^+, \\ &[[a_j^+, a_k^-], a_l^-] = -\delta_{jk} a_l^- - \delta_{jl} a_k^-. \end{aligned} \tag{8}$$

Table 1. Simple Lie algebras, their (extended) Dynkin diagrams with a labelling of the nodes and the corresponding simple roots.



These are the relations of the so called A-statistics [5]-[6], [9]-[11].

For $i=2$, $N=2(n-1)$, let

$$\begin{aligned} &a^-_{-j}=e_{1,j+2}, \quad a^-_{+j}=e_{2,j+2}, \qquad j=1,\ldots,n-1,\\ &a^+_{-j}=e_{j+2,1}, \quad a^+_{+j}=e_{j+2,2}, \qquad j=1,\ldots,n-1. \end{aligned} \tag{9}$$

Now the corresponding relations are ($\xi,\eta,\epsilon=\pm;\ j,k,l=1,\ldots,n-1$):

$$\begin{aligned} &[a^+_{\xi j},a^+_{\eta k}]=[a^-_{\xi j},a^-_{\eta k}]=0,\\ &[a^+_{\xi j},a^-_{-\xi k}]=0, \qquad j\neq k,\\ &[a^+_{-j},a^-_{-k}]=[a^+_{+j},a^-_{+k}], \qquad j\neq k,\\ &[a^+_{+j},a^-_{-j}]=[a^+_{+k},a^-_{-k}],\\ &[a^+_{-j},a^-_{+j}]=[a^+_{-k},a^-_{+k}],\\ &[[a^+_{\xi j},a^-_{\eta k}],a^+_{\epsilon l}]=\delta_{\eta\epsilon}\delta_{jk}a^+_{\xi l}+\delta_{\xi\eta}\delta_{kl}a^+_{\epsilon j},\\ &[[a^+_{\xi j},a^-_{\eta k}],a^-_{\epsilon l}]=-\delta_{\xi\epsilon}\delta_{jk}a^-_{\eta l}-\delta_{\xi\eta}\delta_{jl}a^-_{\epsilon k}. \end{aligned} \tag{10}$$

These relations are already more complicated than (8). But they are still defining relations for the Lie algebra A_n.

Step 2. Delete node i and j from the Dynkin diagram. By the symmetry of the Dynkin diagram, it is sufficient to consider $1\leq i\leq\lfloor\frac{n}{2}\rfloor$ and $i<j<n+1-i$. We have $G_0=H+sl(i)\oplus sl(j-i)\oplus sl(n+1-j)$. In this case, there are six simple G_0-modules. All the possible combinations of these modules give rise to gradings of the form

$$sl(n+1)=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}.$$

There are essentially three different ways in which these G_0-modules can be combined. To characterize these three cases, it is sufficient to give only G_{-1}:

$$\begin{aligned} G_{-1} &= \text{span}\{e_{kl},e_{lp};\ k=1,\ldots,i,\ l=i+1,\ldots,j,\ p=j+1,\ldots,n+1\}, \qquad (11)\\ &\quad\text{with } N=(j-i)(n+1-j+i);\\ G_{-1} &= \text{span}\{e_{kl},e_{pk};\ k=1,\ldots,i,\ l=i+1,\ldots,j,\ p=j+1,\ldots,n+1\}, \qquad (12)\\ &\quad\text{with } N=i(n+1-i);\\ G_{-1} &= \text{span}\{e_{kl},e_{lp};\ k=1,\ldots,i,\ p=i+1,\ldots,j,\ l=j+1,\ldots,n+1\}, \qquad (13)\\ &\quad\text{with } N=j(n+1-j). \end{aligned}$$

It turns out that the sets of CAOs corresponding to (12) and (13) are isomorphic to (11), so it is sufficient to consider only (11). Each case of (11) with $1\leq i\leq\lfloor\frac{n}{2}\rfloor$ and $i<j<n+1-i$ gives rise to a distinct GQS. For reasons explained earlier, we shall give the corresponding set of relations explicitly only for small N. In this case, it is interesting to give $\mathcal{R}$ for $j-i=1$, because then the number of creation or annihilation operators is $N=n$. One can label the CAOs as follows:

$$\begin{aligned} &a^-_k=e_{k,i+1}, \quad a^+_k=e_{i+1,k}, \qquad k=1,\ldots,i;\\ &a^-_k=e_{i+1,k+1}, \quad a^+_k=e_{k+1,i+1}, \qquad k=i+1,\ldots,n. \end{aligned} \tag{14}$$

Using

$$\langle k\rangle=\begin{cases} 0 & \text{if}\quad k=1,\ldots,i\\ 1 & \text{if}\quad k=i+1,\ldots,n \end{cases} \tag{15}$$

the quadratic and triple relations read:

$$\begin{aligned}
&[a_k^+, a_l^+] = [a_k^-, a_l^-] = 0, \qquad k,l = 1,\ldots,i \text{ or } k,l = i+1,\ldots,n,\\
&[a_k^-, a_l^+] = [a_k^+, a_l^-] = 0, \qquad k = 1,\ldots,i,\ l = i+1,\ldots,n, \\
&[[a_k^+, a_l^-], a_m^+] = (-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^+ + (-1)^{\langle l\rangle+\langle m\rangle}\delta_{lm}a_k^+,\ k,l = 1,\ldots,i \text{ or } k,l = i+1,\ldots,n,\\
&[[a_k^+, a_l^-], a_m^-] = -(-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^- - (-1)^{\langle l\rangle+\langle m\rangle}\delta_{km}a_l^-,\ k,l = 1,\ldots,i \text{ or } k,l = i+1,\ldots,n,\\
&[[a_k^\xi, a_l^\xi], a_m^{-\xi}] = -\delta_{km}a_l^\xi + \delta_{lm}a_k^\xi,\ k = 1,\ldots,i,\ l = i+1,\ldots,n,\\
&[[a_k^\xi, a_l^\xi], a_m^\xi] = 0, \qquad (\xi = \pm;\ k,l,m = 1,\ldots,n).
\end{aligned} \tag{16}$$

The existence of the set of CAOs (14) is pointed out in [5] as a possible example. The relations (16) with $n = 2m$ and $i = m$ are the commutation relations of the so called causal A-statistics investigated in [8].

Step 3. If we delete 3 or more nodes from the Dynkin diagram, the resulting $\mathbb{Z}$-gradings of $sl(n+1)$ are no longer of the form $sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, but there would be non-zero G_i with $|i| > 2$, so these cases are not relevant for our classification.

Step 4. Next, we move on to the extended Dynkin diagram of G. If we delete node i from the extended Dynkin diagram, then remaining diagram is again of type A_n, so $G_0 = G$, and there are no CAOs.

Step 5. If we delete node i and j from the extended Dynkin diagram ($0 \le i < j \le n+1$), then $sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}$ with $G_0 = H + sl(j-i) \oplus sl(n-j+i+1)$, and

$$G_{-1} = \text{span}\{e_{kl};\ k = i+1\ldots,j,\ l \neq i+1,\ldots,j\}.$$

The number of annihilation operators is $N = (j-i)(n+1-j+i)$. It is not difficult to see that all these cases are isomorphic to those of Step 1. This can also be deduced from the symmetry of the Dynkin diagram.

Step 6. If we delete nodes i, j and k from the extended Dynkin diagram ($i < j < k$), then the corresponding $\mathbb{Z}$-gradings are of the form

$$sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}.$$

All the corresponding CAOs, however, are isomorphic to those of Step 2 (which can again be seen from the remaining Dynkin diagram).

Step 7. If we delete 4 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $sl(n+1)$ has no longer the required properties (i.e. there are non-zero subspaces G_i with $|i| > 2$).

3.2. The Lie algebras B_n, C_n and D_n

$G = so(2n+1) \equiv B_n$ is the subalgebra of $sl(2n+1)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b & c \\ d & -a^t & e \\ -e^t & -c^t & 0 \end{pmatrix}, \tag{17}$$

where a is any $(n \times n)$-matrix, b and d are antisymmetric $(n \times n)$-matrices, and c and e are $(n \times 1)$-matrices. The Cartan subalgebra H of G is again the subspace of diagonal matrices.

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The root vectors and corresponding roots of G are given by:

$$\begin{array}{rcll} e_{jk} - e_{k+n,j+n} & \leftrightarrow & \epsilon_j - \epsilon_k, & j \neq k = 1,\ldots,n, \\ e_{j,k+n} - e_{k,j+n} & \leftrightarrow & \epsilon_j + \epsilon_k, & j < k = 1,\ldots,n, \\ e_{j+n,k} - e_{k+n,j} & \leftrightarrow & -\epsilon_j - \epsilon_k, & j < k = 1,\ldots,n, \\ e_{j,2n+1} - e_{2n+1,j+n} & \leftrightarrow & \epsilon_j, & j = 1,\ldots,n, \\ e_{n+j,2n+1} - e_{2n+1,j} & \leftrightarrow & -\epsilon_j, & j = 1,\ldots,n. \end{array}$$

$G = sp(2n) \equiv C_n$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{18}$$

where a is any $(n\times n)$-matrix, and b and c are symmetric $(n\times n)$-matrices. The Cartan subalgebra H consist of the diagonal matrices, and the root vectors and corresponding roots of G are:

$$\begin{array}{rcll} e_{jk} - e_{k+n,j+n} & \leftrightarrow & \epsilon_j - \epsilon_k, & j \neq k = 1,\ldots,n, \\ e_{j,k+n} + e_{k,j+n} & \leftrightarrow & \epsilon_j + \epsilon_k, & j \leq k = 1,\ldots,n, \\ e_{j+n,k} + e_{k+n,j} & \leftrightarrow & -\epsilon_j - \epsilon_k, & j \leq k = 1,\ldots,n. \end{array}$$

$G = so(2n) \equiv D_n$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{19}$$

where a is any $(n \times n)$-matrix, and b and c are antisymmetric $(n \times n)$-matrices. The Cartan subalgebra H consist of the diagonal matrices, and the root vectors and corresponding roots of G are:

$$\begin{array}{rcll} e_{jk} - e_{k+n,j+n} & \leftrightarrow & \epsilon_j - \epsilon_k, & j \neq k = 1,\ldots,n, \\ e_{j,k+n} - e_{k,j+n} & \leftrightarrow & \epsilon_j + \epsilon_k, & j < k = 1,\ldots,n, \\ e_{j+n,k} - e_{k+n,j} & \leftrightarrow & -\epsilon_j - \epsilon_k, & j < k = 1,\ldots,n. \end{array}$$

The simple roots, Dynkin diagrams and extended Dynkin diagrams of B_n, C_n and D_n are given in Table 1. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. Just as for A_n, we have performed the process of deleting nodes from the Dynkin diagrams and from the extended Dynkin diagrams of B_n, C_n and D_n. The results are summarized in Table 2.

3.3. The exceptional Lie algebras

The Lie algebra G_2 of rank 2 has dimension 14. In terms of the orthonormal vectors $\epsilon_1, \epsilon_2, \epsilon_3$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, the root system is given by

$$\Delta = \{\epsilon_i - \epsilon_j,\ \epsilon_i + \epsilon_j - 2\epsilon_k\ \ (1 \leq i \neq j \neq k \leq 3)\}. \tag{20}$$

The simple root system is

$$\Pi = \{\alpha_1 = \epsilon_2 + \epsilon_3 - 2\epsilon_1,\ \alpha_2 = \epsilon_1 - \epsilon_2\} \tag{21}$$

and the corresponding Dynkin diagram and extended Dynkin diagram are given in Table 1. The process described in the previous section, deleting nodes from the (extended) Dynkin diagram, leads to the results in Table 2.

Table 2. Summary of the classification for simple Lie algebras: all non-isomorphic GQS are given. For each GQS, we list: the subalgebra G_0 (each G_0 contains the complete Cartan subalgebra H, so we only list the remaining part of $G_0 = H + \cdots$); the length l of the $\mathbb{Z}$-grading (3 or 5), $\dim G_1 = \dim G_{-1}$, which is also the number N of creation and annihilation operators.

Lie algebra	$G_0 = H+$	ℓ	dim $G_1 = N$
A_n	$sl(i) \oplus sl(n-i+1), i \leq \lfloor \frac{n+1}{2} \rfloor$	3	$i(n+1-i)$
	$sl(i) \oplus sl(j-i) \oplus sl(n+1-j), (i \leq \lfloor \frac{n}{2} \rfloor, i < j < n+1-i)$	5	$(j-i)(n+1-j+i)$
B_n	$so(2n-1)$	3	$2n-1$
	$sl(i) \oplus so(2(n-i)+1), (2 \leq i \leq n)$	5	$2i(n-i)+i$
C_n	$sl(i) \oplus sp(2(n-i)), (1 \leq i \leq n-1)$	3	$2i(n-i)$
	$sl(n)$	3	$\frac{n(n+1)}{2}$
D_n	$so(2n-2)$	3	$2(n-1)$
	$sl(i) \oplus so(2(n-i)), (2 \leq i \leq n-3)$	5	$2i(n-i)$
	$sl(n-2) \oplus sl(2) \oplus sl(2)$	5	$4(n-2)$
	$sl(n-1)$	5	$2(n-1)$ and $\frac{n(n-1)}{2}$
	$sl(n)$	3	$\frac{n(n-1)}{2}$
G_2	$sl(2)$	5	4
F_4	$sp(6)$	5	14
	$so(7)$	5	8
E_6	$so(10)$	3	16
	$sl(2) \oplus sl(5)$	5	20
	$sl(6)$	5	20
	$so(8)$	5	16
E_7	E_6	3	27
	$sl(2) \oplus so(10)$	5	32
	$so(12)$	5	32
	$sl(7)$	5	35
E_8	E_7	5	56
	$so(14)$	5	64

Let us next consider the Lie algebra F_4, of rank 4 and dimension 52. In terms of the orthonormal vectors $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ the root system is given by

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \ (1 \leq i \neq j \leq 4);\ \pm\epsilon_j \ (1 \leq j \leq 4);\ \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}. \tag{22}$$

The simple root system is

$$\Pi = \{\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\ \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\} \tag{23}$$

and the corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1. We have considered now the various ways of deleting nodes from these diagrams, and investigate whether they give rise to $\mathbb{Z}$-gradings of the type (4) giving the result in Table 2.

For the remaining exceptional Lie algebras E_6, E_7 and E_8, our labeling of simple roots is again the usual one. But our choice of (simple) roots in terms of vectors ϵ_i is slightly different. For E_8, our choice is essentially the same as in [23, Table 1], except that we work with an independent basis ϵ_i in R^8 (and not a redundant basis in R^9). The roots for E_8 are the same as in [24] (but in this last textbook the choice of simple roots is different). For E_7 and E_6 it is convenient for us to take the same root space as for E_8. The simple roots of E_7 are then those of E_8 with the first one deleted, and the simple roots of E_6 are then those of E_7 with the first one deleted.

The Lie algebra E_6 of rank 6 has dimension 78. We will use the following root system of E_6. Consider the 8-dimensional real vector space R^8 with orthonormal basis vectors ϵ_i $(i = 1, \dots, 8)$. The roots of E_6 are elements of the 6-dimensional subspace V of R^8 consisting of those elements $\sum_{i=1}^{8} c_i\epsilon_i$ with $c_1 + c_2 = 0$ and $\sum_{i=3}^{8} c_i = 0$. A set of simple roots of E_6 is then given by the elements

$$\alpha_i = \epsilon_{i+2} - \epsilon_{i+3} \ \ (i = 1, \dots, 5), \ \ \alpha_6 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8). \tag{24}$$

All 72 nonzero roots are given by

$$\begin{aligned} &\pm(\epsilon_i - \epsilon_j), \ \ (1 \le i \ne j \le 2 \text{ or } 3 \le i \ne j \le 8) \\ &\frac{1}{2}(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i), \ \ (a_i \in \{0,1\}; \ \sum_{i=1}^{2} a_i = 1, \ \ \sum_{i=3}^{8} a_i = 3). \end{aligned} \tag{25}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1.

The roots of E_7 are elements of the 7-dimensional subspace V' consisting of elements $\sum_{i=1}^{8} c_i\epsilon_i$ with $\sum_{i=1}^{8} c_i = 0$. A set of simple roots of E_7 consists of the six simple roots α_i $(i = 1, \dots, 6)$ of E_6 plus the extra root

$$\epsilon_2 - \epsilon_3. \tag{26}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1.

By construction, the E_6 subsystem of E_7 is evident. The nonzero roots of E_7 consist of

$$\begin{aligned} &\pm(\epsilon_i - \epsilon_j), \ \ (1 \le i < j \le 8); \\ &\frac{1}{2}(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i), \ \ (a_i \in \{0,1\}; \ \sum_{i=1}^{8} a_i = 4). \end{aligned} \tag{27}$$

Note that the 72 nonzero roots of E_6 are indeed part of the 126 nonzero roots (27).

In terms of the orthonormal vectors ϵ_i, $i = 1, \dots, 8$ the root system of E_8 is given by

$$\begin{aligned} &\pm\epsilon_i \pm \epsilon_j, \ \ (1 \le i < j \le 8); \\ &\frac{1}{2}\sum_{i=1}^{8}\xi_i\epsilon_i, \ \ \xi_i = \pm 1 \text{ and the number of } \xi_i = +1 \text{ is even.} \end{aligned} \tag{28}$$

A set of simple roots of E_8 consists of the seven simple roots α_i $(i = 1, \dots, 7)$ of E_7 plus the extra root

$$-\epsilon_1 - \epsilon_2. \tag{29}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1, whereas summary of the GQSs associated with the Lie algebras E_6, E_7 and E_8 are presented in Table 2.

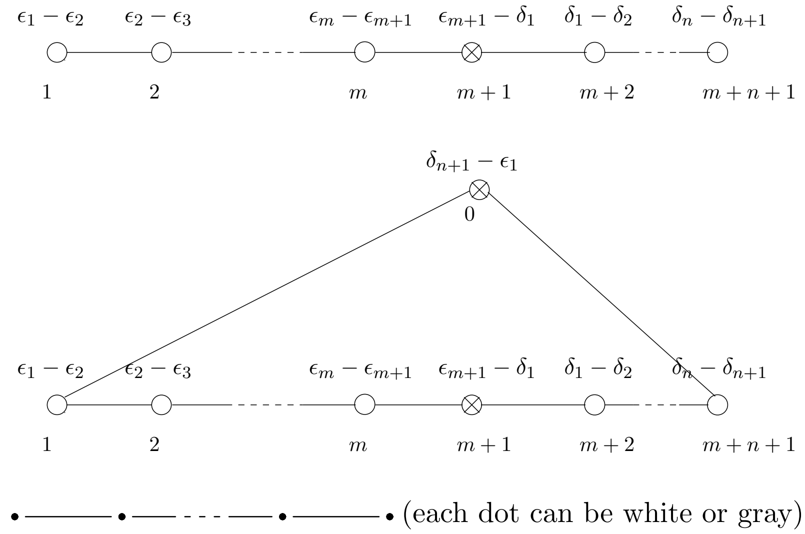
4. The basic classical Lie superalgebras

Let us note first that for the basic classical Lie superalgebras, the description by means of a Dynkin diagram is not unique: besides the so-called distinguished Dynkin diagram, other non-equivalent Dynkin diagrams exist [4], [25]. This feature makes it harder to obtain a complete classification of all generalized quantum systems since one should repeat the process of investigation for all non distinguished Dynkin diagrams and their extensions. In Table 3 we give all (extended) Dynkin diagrams of basic classical Lie superalgebras, relevant for the classification process. All GQS associated with the basic classical Lie superalgebras are listed in Table 4.

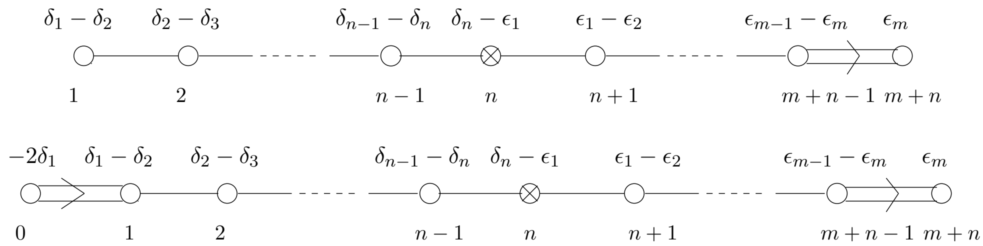
Table 3. Basic classical Lie superalgebras, the (extended) Dynkin diagrams, relevant for the classification process, with a labeling of the nodes and the corresponding simple roots.

LSA	Dynkin diagram and extended Dynkin diagram

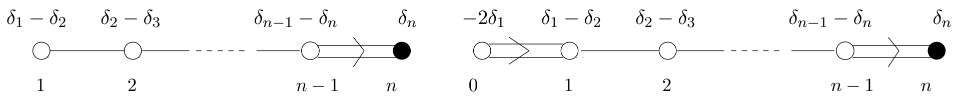
$A(m|n)$



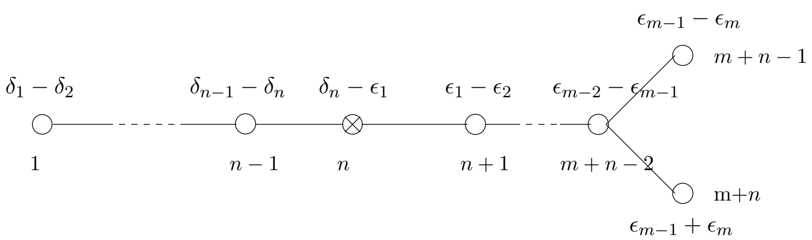
$B(m|n)$

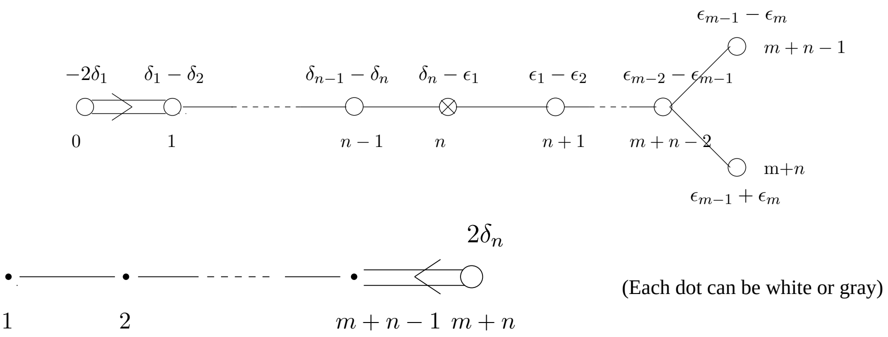


$B(0|n)$

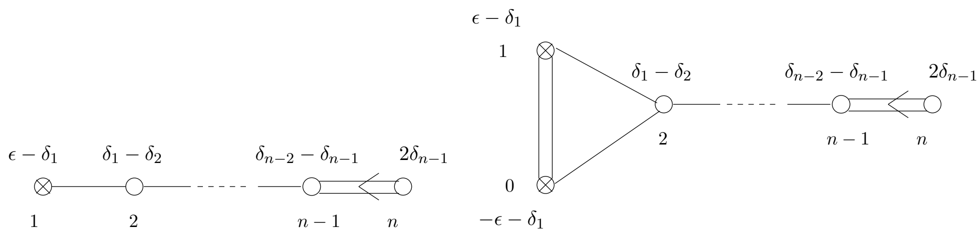


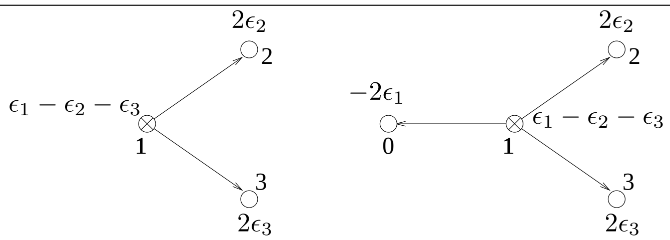
$D(m|n)$





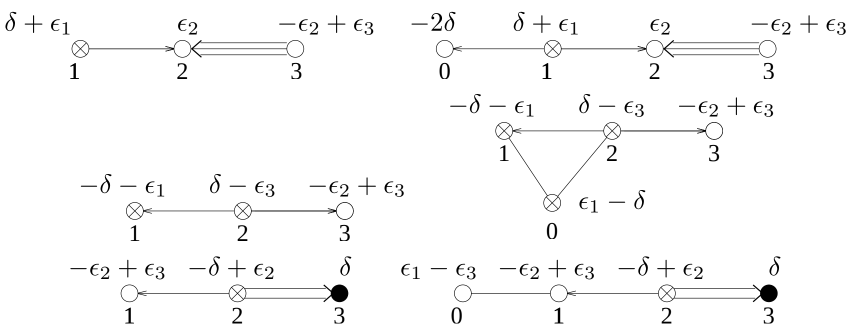
$C(n)$





$D(2,1;\alpha)$

$G(3)$



$F(4)$

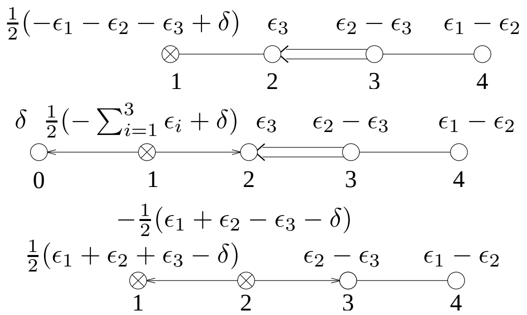


Table 4. Summary of the classification for basic classical Lie superalgebras: all non-isomorphic GQS are given. For each GQS, we list: the subalgebra G_0 (each G_0 contains the complete Cartan subalgebra H, so we only list the remaining part of $G_0 = H + \cdots$); the length ℓ of the $\mathbb{Z}$-grading; $\dim G_{-1} = \dim G_{+1}$, which is also the number N of creation or annihilation operators

LSA	$G_0 = H + \cdots$	ℓ	dim $G_1 = N$
$A(m\vert n)$	$sl(k\vert l) \oplus sl(p\vert q)$	3	$(k+l)(p+q)$
	$(k+p = m+1,\ l+q = n+1,$		
	$k+l \neq 0,\ p+q \neq 0)$		
	$sl(k\vert l) \oplus sl(p\vert q) \oplus sl(r\vert s)$	5	$(k+l)(p+q+r+s)$
	$(k+p+r = m+1,$	5	$(p+q)(k+l+r+s)$
	$l+q+s = n+1,$	5	$(r+s)(k+l+p+q)$
	$k+l \neq 0,\ p+q \neq 0,\ r+s \neq 0)$		
$B(m\vert n)$	$sl(k\vert l) \oplus B(m-k\vert n-l)$	5	$(k+l)(2m-2k+2n-2l+1)$,
	$(k = 0, \ldots, m;\ l = 0, \ldots, n;$		
	$(k,l) \notin \{(0,0),(1,0)\})$		
	$B(m-1\vert n)$ $[(k,l) = (1,0)]$	3	$2m+2n-1$
$B(0\vert n)$	$sl(i) \oplus B(0\vert n-i)$	5	$i(2n-2i+1)$
	$(i = 1, \ldots, n)$		
$D(m\vert n)$	$sl(k\vert l) \oplus D(m-k\vert n-l)$	5	$2(k+l)(m+n-k-l)$,
	$(k = 0, 1, \ldots, m;$		
	$l = 0, 1, \ldots, n;$		
	$(k,l) \notin \{(0,0),(1,0),(m-1,n),(m,n)\})$		
	$D(m-1\vert n)$ $[(k,l) = (1,0)]$	3	$2(m+n-1)$
	$sl(m\vert n)$ $[(k,l) = [m,n)]$	3	$(m+n)(m+n+1)/2-m$
	$sl(m-1\vert n)$ $[(k,l) = (m-1,n)]$	5	$(m+n)(m+n+1)/2-m$
	$sl(m-1\vert n)$ $[(k,l) = (m-1,n)]$	5	$2(m+n-1)$
$C(n)$	$sl(k\vert l) \oplus D(1-k\vert n-1-l)$	5	$2(k+l)(n-k-l)$
	$(k = 0,1;\ l = 1, \ldots, n-2)$		
	C_{n-1} $[(k,l) = (1,0)]$	3	$2(n-1)$
	$sl(1\vert n-1)$ $[(k,l) = (1,n-1)]$	3	$n(n+1)/2-1$
	$sl(n-1)$ $[(k,l) = (0,n-1)]$	5	$n(n+1)/2-1$
	$sl(n-1)$ $[(k,l) = (0,n-1)]$	5	$2(n-1)$
$D(2,1;\alpha)$	$sl(2) \oplus sl(2)$	5	4
	$sl(1\vert 2)$	3	4
	$sl(1\vert 1)$	5	4
$G(3)$	G_2	5	7
	$sl(1\vert 2)$	5	7
	$sl(3\vert 1)$	3	8
	$osp(3\vert 2)$	5	8
	$sl(3) \oplus osp(1\vert 2)$	3	9
$F(4)$	$so(7)$	5	8
	$sl(1\vert 2) \oplus sl(2)$	5	10
	$osp(2\vert 4)$	3	10
	$sl(2) \oplus so(5)$	5	8
	$D(2,1;-1/3)$	5	10
	$sl(3\vert 1)$	5	8

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