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On continuous symmetries of second-order homogeneous linear ordinary differential equations

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Abstract. In this work, a method to extract continuous symmetries of general second-order linear ordinary differential equation is presented. The formalism is illustrated by two examples.

1. Introduction

Ordinary differential equations (ODEs) appear in many fields of Physics [1] and the analysis of their symmetries plays an important role to extract information about the solutions of those equations [2-4]. In this work, we have applied the Anderson-Kumei-Wulfman method [5-8] to extract continuous symmetries of general second-order linear ordinary differential equation.

In general, we consider general homogeneous linear ODEs represented by the action of the differential operator $\hat{A}(x)$ on a function $f(x)$

$$\hat{A}(x)f(x) = 0, \tag{1}$$

where the differential operator $\hat{A}(x)$ is give by

$$\hat{A}(x) = \alpha_0(x) + \alpha_1(x)\partial_x + \alpha_2(x)\partial_{xx} + ... \tag{2}$$

In this work we are interested in second order ODEs, then $\alpha_i = 0$, for $i \geq 3$.

A symmetry operator $\hat{Q}$ of equation (1) is defined as a differential operator that maps solution of equation (1) into solution of the same equation, i.e.,

$$\hat{A}(x)g(x) = 0, \tag{3}$$

where

$$g(x) = \hat{Q}(x)f(x). \tag{4}$$

After assuming a particular form for the operators $\hat{Q}$, condition (3) under constrain (1) defines the symmetries of the ODE (1).

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2. Symmetry extraction

Let us start by considering a general second-order homogeneous linear ODE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0. \tag{5}$$

After substituting

$$y(x) = f(x)\exp\left[-\frac{1}{2}\int^x p(\xi)d\xi\right] \tag{6}$$

in equation (5), it is easy to see that the function $f(x)$ satisfies the following differential equation:

$$\frac{d^2f}{dx^2} + v(x)f(x) = 0, \tag{7}$$

where

$$v(x) = q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}\frac{dp}{dx}. \tag{8}$$

We are interested in continuous symmetry generators of the form

$$\hat{Q}(x) = \alpha(x) + \beta(x)\frac{d}{dx} \tag{9}$$

for the equation (7). In other words, if $f(x)$ is a solution of equation (7) then we need to find the functions $\alpha(x)$ and $\beta(x)$ so that:

$$g(x) = \hat{Q}(x)f(x) \tag{10}$$

is also a solution of equation (7), i.e.

$$\frac{d^2g}{dx^2} + v(x)g(x) = 0. \tag{11}$$

Then differentiating equation (10), replacing in (11) and using that $f(x)$ is lineally independent, we obtain the following equation system:

$$\alpha'' - v'\beta - 2v\beta' = 0, \tag{12}$$

$$2\alpha' + \beta'' = 0. \tag{13}$$

The above system determines the conditions for $\alpha(x)$ and $\beta(x)$ function.
Working on equations (12) and (13), we obtain:

$$\beta''' + 4v\beta' + 2v'\beta = 0. \tag{14}$$

Equation (14) may be solved if $v(x)$ function is known.
Let us assume that we know two particular solutions $u_1(x)$ and $u_2(x)$ for equation (7), i.e.

$$u_1'' + vu_1 = 0 \quad \texttt{and} \quad u_2'' + vu_2 = 0. \tag{15}$$

Now, if we consider the following definition

$$\phi(x) = C_1u_1^2 + C_2u_2^2 + C_3u_1u_2, \tag{16}$$

It is easy to show that $\phi(x)$ function satisfies the equation:

$$\phi''' + 4v\phi' + 2v'\phi = 0. \tag{17}$$

Therefore, equation (14) defines the $\beta(x)$ function as

$$\beta(x) = C_1 u_1^2 + C_2 u_2^2 + C_3 u_1 u_2, \tag{18}$$

where the functions $u_1(x)$ and $u_2(x)$ are two lineally independent particular solutions of equation (7):

$$u_1'' + v u_1 = 0 \quad \texttt{and} \quad u_2'' + v u_2 = 0.$$

However, it is only necessary to know one solution since the second linear independent solution of equation (7) is obtained by the relation

$$u_2(x) = u_1(x) \int^x \frac{d\xi}{[u_1(\xi)]^2}. \tag{19}$$

Finally, having $\beta(x)$ it is possible to obtain $\alpha(x)$ expansion coefficient by:

$$\alpha'' = -\frac{1}{2}\beta'''. \tag{20}$$

3. Examples

The symmetry extaction method will be exemplified by two simple equations:

3.1. Example 1

Let us consider the simplest second order ODE:

$$f_{xx} = 0. \tag{21}$$

Step 1: Find a particular solution of (21)

$$u_1 = C_1 x. \tag{22}$$

Step 2: Use equation (19), to build a second independent solution

$$u_2 = C_1 x \int^x \frac{d\xi}{(C_1 \xi)^2} = C_2. \tag{23}$$

Step 3: Use equation (18) to obtain $\beta(x)$ function

$$\beta(x) = B_1 x^2 + B_2 + B_3 x. \tag{24}$$

Step 4: Use equation (13) to obtain $\alpha(x)$ function

$$\alpha(x) = -\frac{1}{2}\int^x \beta''(\xi) d\xi = -B_1 x. \tag{25}$$

Step 5: Build the symmetry generators

$$\hat{Q}_1 = \frac{d}{dx}, \quad \hat{Q}_2 = x\frac{d}{dx}, \quad \hat{Q}_3 = x - x^2\frac{d}{dx}. \tag{26}$$

Step 6: Check the symmetry property

$$\hat{A}\hat{Q}_1 f = 0, \quad \hat{A}\hat{Q}_2 f = 0, \quad \hat{A}\hat{Q}._3 f = 0 \tag{27}$$

Step 7: Find the algebra

$$[\hat{Q}_1, \hat{Q}_2] = \hat{Q}_1, \quad [\hat{Q}_1, \hat{Q}_3] = I - 2\hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = \hat{Q}_3. \tag{28}$$

And, introducing the new definition:

$$\hat{A}_0 = \hat{Q}_2 - 1/2, \quad \hat{A}_- = \hat{Q}_1, \quad \hat{A}_+ = \hat{Q}_3, \tag{29}$$

the following commutation relations are obtained:

$$[\hat{A}_0, \hat{A}_\pm] = \pm\hat{A}_\pm, \quad [\hat{A}_+, \hat{A}_-] = -2\hat{A}_0. \tag{30}$$

3.1.1. ***Symmetry Visualization:*** With the symmetry generators we can obtain the action of this generators on the solution of the original ODE.

The general solution of ODE (21) is:

$$y = a + bx, \tag{31}$$

where a and b are arbitrary constants. Then the actions of $\hat{A}_+$, $\hat{A}_-$ and $\hat{A}_0$ over (31) are

$$\begin{aligned} \exp(\theta\hat{A}_0)(a+bx) &= \bar{a}+\bar{b}x, & \bar{a} &= e^{-\theta/2}a, \quad \bar{b} = e^{\theta/2}b \\ \exp(\theta\hat{A}_-)(a+bx) &= \bar{a}+bx, & \bar{a} &= a+\theta b \\ \exp(\theta\hat{A}_+)(a+bx) &= a+\bar{b}x, & \bar{b} &= b+\theta a \end{aligned}$$

Figure 1 shows the plot of (31) for particular values of a and b constants. Figures 2, 3 and 4 show the action of $\hat{A}_+$, $\hat{A}_-$ and $\hat{A}_0$ symmetry operators on solution (31) respectively.

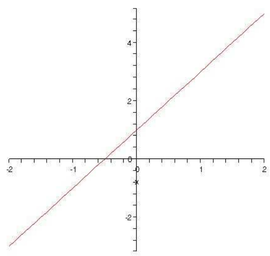

Figure 1

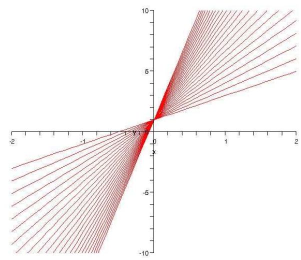

Figure 2

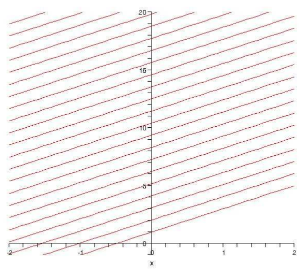

Figure 3

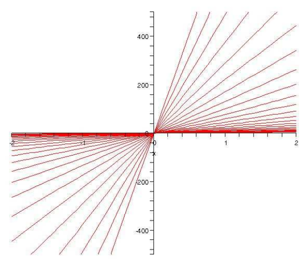

Figure 4

3.2. Example 2

Let us consider the second order ODE:

$$f_{xx} + k^2 f = 0. \tag{32}$$

Step 1: Find a particular solution of (32)

$$u_1 = \cos(kx). \tag{33}$$

Step 2: Use equation (19) to build a second independent solution

$$u_2 = \cos(kx)\int^x \frac{d\xi}{[\cos(kx)]^2} = \frac{\sin(kx)}{k}. \tag{34}$$

Step 3: Use equation (18) to obtain $\beta(x)$ function

$$\beta(x) = C_1 \cos^2(kx) + C_2 \frac{\sin^2(kx)}{k^2} + C_3 \frac{\sin(kx)}{k} \cos(kx). \tag{35}$$

Step 4: Use equation (13) to obtain $\alpha(x)$ function

$$\alpha(x) = C_1 \frac{k\sin(2kx)}{2} - C_2 \frac{\sin(2kx)}{2k} + C_3 \frac{1}{2}(1-\cos(2kx)) + C_4. \tag{36}$$

Step 5: Build the symmetry generators

$$\hat{Q}_1 = \tfrac{k}{2}\sin(2kx) + \tfrac{1}{2}(1+\cos(2kx))\tfrac{d}{dx}, \tag{37}$$
$$\hat{Q}_2 = \tfrac{1}{2} - \tfrac{1}{2}\cos(2kx) + \tfrac{1}{2k}\sin(2kx)\tfrac{d}{dx}, \tag{38}$$
$$\hat{Q}_3 = \tfrac{1}{2k}\sin(2kx) + \tfrac{1}{2k^2}(\cos(2kx)-1)\tfrac{d}{dx}. \tag{39}$$

Step 6: Check the symmetry property

$$\hat{A}\hat{Q}_1 f = 0 \quad \hat{A}\hat{Q}_2 f = 0, \quad \hat{A}\hat{Q}_3 f = 0. \tag{40}$$

Step 7: Find the algebra

$$[\hat{Q}_1, \hat{Q}_2] = \hat{Q}_1, \quad [\hat{Q}_1, \hat{Q}_3] = I - 2\hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = \hat{Q}_3. \tag{41}$$

And, introducing the new definitions:

$$\hat{A}_0 = \hat{Q}_2 - 1/2, \quad \hat{A}_- = \hat{Q}_1, \quad \hat{A}_+ = \hat{Q}_3, \tag{42}$$

the following commutation relations are obtained:

$$[\hat{A}_0, \hat{A}_\pm] = \pm\hat{A}_\pm, \quad [\hat{A}_+, \hat{A}_-] = -2\hat{A}_0. \tag{43}$$

3.2.1. ***Matching examples 1 and 2:*** From the equation (33) and (34) we have the general solution of EDO (32)

$$y = C_1\cos(kx) + \frac{C_2}{k}\sin(kx). \tag{44}$$

Taking the limit $k \to 0$ on equation (32), we obtain equation (21). In same way, this limit reduces solution (44) to the solution of equation (21).

$$y = C_1 + C_2 x. \tag{45}$$

On the other hand, taking the limit $k \to 0$ on symmetry generators(37), (38) and (39), we obtain

$$\lim_{k\to 0}(\hat{Q}_1) = \lim_{k\to 0}\left(\frac{k}{2}\sin(2kx) + \frac{1}{2}(1+\cos(2kx))\frac{d}{dx}\right) = \frac{d}{dx},$$
$$\lim_{k\to 0}(\hat{Q}_2) = \lim_{k\to 0}\left(\frac{1}{2} - \frac{1}{2}\cos(2kx) + \frac{1}{2k}\sin(2kx)\frac{d}{dx}\right) = x\frac{d}{dx},$$
$$\lim_{k\to 0}(\hat{Q}_3) = \lim_{k\to 0}\left(\frac{1}{2k}\sin(2kx) + \frac{1}{2k^2}(\cos(2kx)-1)\frac{d}{dx}\right) = x - x^2\frac{d}{dx}.$$

Showing the compatibility between the symmetry operators obtained in example 1 and 2 respectively. It is noteworthy that commutations relations (28) remain invariant under the limit $k \to 0$.

4. Conclusion

A general method to find continuous symmetries of second-order linear ODEs has been presented. To obtain the symmetry generators a particular solution of the ODE under study is required. The method has been illustrated by two examples.

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